# ASYMPTOTIC ESTIMATES AND EXPONENTIAL STABILITY FOR HIGHER-ORDER MONOTONE DIFFERENCE EQUATIONS

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Asymptotic estimates are established for higher-order scalar difference equations and inequalities the right-hand sides of which generate a monotone system with respect to the discrete exponential ordering. It is shown that in some cases the exponential estimates can be replaced with a more precise limit relation. As corollaries, a generalization of discrete Halanay-type inequalities and explicit sufficient conditions for the global exponential stability of the zero solution are given.

## 1. Introduction

Consider the higher-order scalar difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n \in \mathbb{N} = \{0, 1, 2, \dots\}, \tag{1.1}$$

where $k$ is a positive integer and $f : \mathbb{R}^{k+1} \to \mathbb{R}$. With (1.1), we can associate the discrete dynamical system $(T^n)_{n\geq 0}$ on $\mathbb{R}^{k+1}$, where $T : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is defined by

$$T(x) = (f(x), x_0, x_1, \dots, x_{k-1}), \quad x = (x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1}. \tag{1.2}$$

As usual, $T^n$ denotes the $n$th iterate of $T$ for $n \geq 1$ and $T^0 = I$, the identity on $\mathbb{R}^{k+1}$. It follows by easy induction on $n$ that if $(x_n)_{n\geq -k}$ is a solution of (1.1), then

$$(x_n, x_{n-1}, \dots, x_{n-k}) = T^n(x_0, x_{-1}, \dots, x_{-k}), \quad n \geq 0. \tag{1.3}$$

Therefore, the dynamical system $(T^n)_{n\geq 0}$ contains all information about the behavior of the solutions of (1.1).

In a recent paper [7], motivated by earlier results for delay differential equations due to Smith and Thieme [13] (see also [12, Chapter 6]), Krause and the second author have introduced the discrete exponential ordering on $\mathbb{R}^{k+1}$, the partial ordering induced by the convex closed cone

$$C_\mu = \{x = (x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} \mid x_k \geq 0,\ x_i \geq \mu x_{i+1},\ i = 0, 1, \dots, k-1\}, \tag{1.4}$$


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where $\mu \geq 0$ is a parameter. In [7], it has been shown that $T$ is monotone (order preserving) under appropriate conditions on $f$. As a consequence of monotonicity, necessary and sufficient conditions have been given for the boundedness of all solutions and for the local and global stability of an equilibrium of (1.1) (see [7, Section 4]).

In this paper, we give further consequences of the monotonicity of $T$ for (1.1) and for the corresponding difference inequality

$$y_{n+1} \leq f(y_n, y_{n-1}, \dots, y_{n-k}), \quad n \geq 0, \tag{1.5}$$

under the additional assumption that the nonlinearity $f$ is positively homogeneous (of degree one) on the generating cone $C_\mu$, that is,

$$f(\lambda x) = \lambda f(x) \quad \text{for } \lambda \geq 0,\ x \in C_\mu. \tag{1.6}$$

An example of (1.1) with property (1.6) is the max type difference equation

$$x_{n+1} = \sum_{i=0}^{k} K_i x_{n-i} + b \max\{x_n, x_{n-1}, \dots, x_{n-r}\}, \tag{1.7}$$

where $k$ and $r$ are positive integers and the coefficients $K_i$ and $b$ are constants. For other examples of higher-order difference equations with a positively homogeneous right-hand side, see, for example, [6].

Using the monotonicity of $T$ and a simple comparison theorem, we give upper exponential estimates for the solutions of (1.5) in terms of the largest positive root of the characteristic equation

$$\lambda^{k+1} = f(\lambda^k, \lambda^{k-1}, \dots, 1). \tag{1.8}$$

As a corollary for the difference inequality

$$y_{n+1} \leq \sum_{i=0}^{k} K_i y_{n-i} + b \max\{y_n, y_{n-1}, \dots, y_{n-r}\}, \tag{1.9}$$

we obtain a generalization of earlier results of Ferreiro and the first author [8] on discrete Halanay-type inequalities (see Theorems 1.1 and 3.1). For other related results, see, for example, [1, 9, 10].

Further, we will show that a mild strengthening of the monotonicity condition in [7] implies that the map $T$ is eventually strongly monotone. As a consequence, a nonlinear version of the Perron-Frobenius theorem [3] applies and we obtain an asymptotic representation of the solutions of (1.1) starting from $C_\mu$ (see Theorems 1.2 and 3.7). For a similar result, using the standard ordering in $\mathbb{R}^{k+1}$ $(\mu = 0)$, see [6].

Finally, we establish an asymptotic exponential estimate for the growth of the solutions of the equation

$$x_{n+1} = \sum_{i=0}^{k} K_i x_{n-i} + g(n, x_n, x_{n-1}, \dots, x_{n-r}), \tag{1.10}$$

under the assumption that its linear part

$$y_{n+1} = \sum_{i=0}^{k} K_i y_{n-i} \tag{1.11}$$

generates a monotone system and the growth of the nonlinearity $g : \mathbb{N} \times \mathbb{R}^{r+1} \to \mathbb{R}$ is controlled by a positively homogeneous function which is nondecreasing in each of its variables (see Theorems 1.3 and 3.10). As a corollary, we obtain explicit sufficient conditions for the global exponential stability of the zero solution of (1.10) (see Theorems 1.4 and 3.11).

The following four theorems give a flavor of our more general results presented in Section 3. Without loss of generality, we assume that in all Theorems 1.1, 1.2, 1.3, and 1.4 below, $k \ge r$. The first theorem offers an upper estimate for the solutions of inequality (1.9).

Theorem 1.1. *Suppose that $b > 0$ and there exists $\mu > 0$ such that*

$$\mu + \sum_{i=1}^{k} K_i^- \mu^{-i} \le K_0, \tag{1.12}$$

*where $K_i^- = \max\{0, -K_i\}$. Then, for every solution $(y_n)_{n \ge -k}$ of (1.9) there exists a positive constant $M = M(y_0, y_{-1}, \ldots, y_{-k})$ such that*

$$y_n \le M\lambda_0^n, \quad n \ge -k, \tag{1.13}$$

*where $\lambda_0$ is the unique root of the equation*

$$\lambda^{k+1} = \sum_{i=0}^{k} K_i \lambda^{k-i} + b \max\{\lambda^k, \lambda^{k-1}, \ldots, \lambda^{k-r}\} \tag{1.14}$$

*in the interval $(\mu, \infty)$.*

The next result shows in case of (1.7) the exponential estimate (1.13) of Theorem 1.1 is sharp.

Theorem 1.2. *Suppose that $b > 0$ and (1.12) holds with a strict inequality for some $\mu > 0$. Then, for every solution $(x_n)_{n \ge -k}$ of (1.7) with initial data $(x_0, x_{-1}, \ldots, x_{-k}) \in C_\mu \setminus \{0\}$, there exists a positive constant $L = L(x_0, x_{-1}, \ldots, x_{-k})$ such that*

$$\lambda_0^{-n} x_n \longrightarrow L \quad \text{as } n \longrightarrow \infty, \tag{1.15}$$

*where $\lambda_0$ has the meaning from Theorem 1.1.*

The following theorem provides an estimate for the growth of the solutions of (1.10).

Theorem 1.3. *Suppose that there exist $b > 0$ and $\mu > 0$ such that (1.12) and*

$$|g(n, x_0, x_1, \ldots, x_r)| \le b \max\{|x_0|, |x_1|, \ldots, |x_r|\}, \quad n \ge 0,\ x \in \mathbb{R}^{r+1} \tag{1.16}$$

*hold. Then, for every solution* $(x_n)_{n\geq -k}$ *of (1.10) there exists a positive constant* $M = M(x_0, x_{-1},\dots,x_{-k})$ *such that*

$$|x_n| \leq M\lambda_0^n, \quad n \geq -k, \tag{1.17}$$

*where* $\lambda_0$ *has the meaning from Theorem 1.1.*

The existence and uniqueness of the solution $\lambda_0$ of (1.14) in $(\mu,\infty)$ is a part of the conclusions of Theorems 1.1, 1.2, and 1.3. This $\lambda_0$ is a root of either

$$\lambda^{k+1} = \sum_{i=0}^{k} K_i\lambda^{k-i} + b\lambda^k \tag{1.18}$$

or

$$\lambda^{k+1} = \sum_{i=0}^{k} K_i\lambda^{k-i} + b\lambda^{k-r}, \tag{1.19}$$

depending on whether $\lambda_0 \geq 1$ or $\lambda_0 < 1$. It will be shown (see Corollary 2.7) that $\lambda_0 < 1$ if and only if, in addition to the hypotheses of Theorem 1.1, $\mu < 1$ and

$$\sum_{i=0}^{k} K_i + b < 1. \tag{1.20}$$

As a consequence of Theorem 1.3, we have the following criterion for the global exponential stability of the zero solution of (1.10).

Theorem 1.4. *Suppose that there exist* $b > 0$ *and* $\mu \in (0,1)$ *such that (1.12), (1.16), and (1.20) hold. Then, the zero solution of (1.10) is globally exponentially stable.*

For the proofs of Theorems 1.1, 1.2, 1.3, and 1.4, see Remarks 3.4, 3.9 and, 3.12.

In the special case $K_0 \geq 0$, $K_i = 0$ for $i = 1,2,\dots,k$ and $0 < b < 1 - K_0$, the conclusion of Theorem 1.1, a discrete analogue of Halanay's inequality, was obtained by Ferreiro and the first author (see [8, Theorem 1]). The same remark holds for Theorem 1.4 (see [8, Theorem 2]).

Under the hypotheses of Theorem 1.4, the global asymptotic stability of the zero solution of (1.10) was established by the second author using a different approach (see [11, Corollary 2 and Remark 2]).

The paper is organized as follows. In Section 2, we discuss the monotonicity properties of the map $T$ defined by (1.2). The main results on the behavior of the solutions of the above higher-order difference equations and inequalities are given in Section 3.

## 2. Monotonicity

Recall the definition of the discrete exponential ordering from [7]. For every $\mu \geq 0$, the convex closed cone $C_\mu$ defined by (1.4) has nonempty interior $\operatorname{int} C_\mu$ given by

$$\operatorname{int} C_\mu = \{x = (x_0,x_1,\dots,x_k) \in \mathbb{R}^{k+1} \mid x_k > 0,\ x_i > \mu x_{i+1},\ i = 0,1,\dots,k-1\}. \tag{2.1}$$

As a cone in $\mathbb{R}^{k+1}$, each $C_\mu$ induces a partial order $\leq_\mu$ on $\mathbb{R}^{k+1}$ by $x \leq_\mu y$ if and only if $y - x \in C_\mu$. We write $x <_\mu y$ if $x \leq_\mu y$ and $x \neq y$. The strong ordering $\ll_\mu$ is defined by $x \ll_\mu y$ if and only if $y - x \in \operatorname{int} C_\mu$. The ordering $\leq_\mu$ is called the *discrete exponential ordering*. Note that the restriction $\mu < 1$ in [7] is not needed here.

The following result follows immediately from the definition of the ordering $\leq_\mu$ (see also [7, Proposition 1]). It gives a necessary and sufficient condition for the map $T$ defined by (1.2) to be monotone. Recall that $T$ is said to be *monotone (increasing, order preserving)* on $\mathbb{R}^{k+1}$ with respect to $\leq_\mu$ if

$$T(y) \geq_\mu T(x) \quad \text{whenever } x, y \in \mathbb{R}^{k+1} \text{ satisfy } x \leq_\mu y. \tag{2.2}$$

Theorem 2.1. *Let $\mu \geq 0$. The map $T$ defined by (1.2) is monotone with respect to $\leq_\mu$ if and only if*

$$f(y) - f(x) \geq \mu(y_0 - x_0) \quad \text{whenever } x, y \in \mathbb{R}^{k+1} \text{ satisfy } x \leq_\mu y. \tag{2.3}$$

A relatively easily verifiable sufficient condition for (2.3) to hold is given below.

Proposition 2.2 [7, Proposition 2]. *Let $\mu > 0$. Condition (2.3) holds if there exist constants $L_i$, $i = 0, 1, \ldots, k$ such that*

$$f(y) - f(x) \geq \sum_{i=0}^{k} L_i(y_i - x_i) \quad \text{whenever } x_i \leq y_i \text{ for } i = 0, 1, \ldots, k \tag{2.4}$$

*and*

$$\mu + \sum_{i=1}^{k} L_i^- \mu^{-i} \leq L_0, \tag{2.5}$$

*where $L_i^- = \max\{0, -L_i\}$.*

Note that in both previous results the domain $\mathbb{R}^{k+1}$ of $T$ can be replaced with a subset of $\mathbb{R}^{k+1}$.

If $f$ is differentiable, then the constants $L_i$ in (2.4) may be viewed as the infima of the partial derivatives $\partial f / \partial x_i(x)$, where the infimum is taken over all $x \in \mathbb{R}^{k+1}$.

The next theorem shows that a mild strengthening of the monotonicity condition (2.3) implies that $T$ is eventually strongly monotone.

Theorem 2.3. *Let $\mu > 0$ and suppose that*

$$f(y) - f(x) > \mu(y_0 - x_0) \quad \text{whenever } x, y \in \mathbb{R}^{k+1} \text{ satisfy } x <_\mu y. \tag{2.6}$$

*Then, $T^k$ is strongly monotone with respect to $\leq_\mu$, that is,*

$$T^k(y) \gg_\mu T^k(x) \quad \text{whenever } x, y \in \mathbb{R}^{k+1} \text{ satisfy } x <_\mu y. \tag{2.7}$$

*Proof.* Let $x, y \in \mathbb{R}^{k+1}$ satisfy $x <_\mu y$. We must show that $T^k(y) \gg_\mu T^k(x)$. In view of the definition of $\operatorname{int} C_\mu$ and the relation

$$T^k(x) = \big(f(T^{k-1}(x)), f(T^{k-2}(x)), \dots, f(T(x)), f(x), x_0\big), \quad x \in \mathbb{R}^{k+1}, \tag{2.8}$$

the last inequality is equivalent to the system of inequalities

$$f(y) - f(x) > \mu(y_0 - x_0) > 0 \tag{2.9}$$

and

$$f(T^{i+1}(y)) - f(T^{i+1}(x)) > \mu\big(f(T^i(y)) - f(T^i(x))\big) > 0 \tag{2.10}$$

for $i = 0, 1, \dots, k-2$. Since $x <_\mu y$, it follows that $y_0 - x_0 > 0$. (Otherwise, the condition $y - x \in C_\mu$ would imply that $y = x$, a contradiction.) Consequently, (2.6) implies (2.9). Since $T$ is monotone, $T(y) \geq_\mu T(x)$. Further, by virtue of (2.9) and the definition of $T$, we have

$$(T(y))_0 - (T(x))_0 = f(y) - f(x) > 0 \tag{2.11}$$

and hence $T(y) >_\mu T(x)$. Using (2.6) again, we find

$$f(T(y)) - f(T(x)) > \mu\big(f(y) - f(x)\big) > 0. \tag{2.12}$$

Thus, (2.10) holds for $i = 0$. Suppose for induction that (2.10) holds for some $i \geq 0$. By monotonicity, $T^{i+2}(y) \geq_\mu T^{i+2}(x)$. Moreover, in view of (2.10) and the definition of $T$, we have

$$(T^{i+2}(y))_0 - (T^{i+2}(x))_0 = f(T^{i+1}(y)) - f(T^{i+1}(x)) > 0. \tag{2.13}$$

Consequently, $T^{i+2}(y) >_\mu T^{i+2}(x)$ and therefore (2.6) and (2.10) imply that

$$f(T^{i+2}(y)) - f(T^{i+2}(x)) > \mu\big(f(T^{i+1}(y)) - f(T^{i+1}(x))\big) > 0. \tag{2.14}$$

Thus, (2.10) holds for all $i = 0, 1, 2, \dots$. As noted before, (2.9) and (2.10) imply that $T^k(y) \gg_\mu T^k(x)$. □

The next result is similar to Proposition 2.2. It gives a sufficient condition for assumption (2.6) of Theorem 2.3 to hold.

Proposition 2.4. *Let $\mu > 0$. Then, (2.6) holds if (2.4) holds and the inequality in (2.5) is strict,*

$$\mu + \sum_{i=1}^{k} L_i^- \mu^{-i} < L_0. \tag{2.15}$$

The proof of Proposition 2.4 is an obvious modification of the proof of [7, Proposition 2] and thus it is omitted.

In the next theorem, we describe some further properties of $T$ under the additional assumption that $f$ is continuous and positively homogeneous on $C_\mu$. In particular, it can be used to ensure the existence of a strongly positive eigenvector of $T$.

THEOREM 2.5. *Suppose that there exists $\mu \geq 0$ such that $f$ is continuous on $C_\mu$ and (1.6) and (2.3) hold on $C_\mu$. Then, the following hold.*

(i) *$T$ is a continuous, positively homogeneous, and monotone selfmapping of $C_\mu$.*

(ii) *If, in addition, it is assumed that*

$$f(\mu^k,\mu^{k-1},\dots,1) > \mu^{k+1}, \tag{2.16}$$

*then the characteristic equation (1.8) has a unique root $\lambda_0$ in $(\mu,\infty)$. This root $\lambda_0$ is an eigenvalue of $T$ and $u_{\lambda_0} = (\lambda_0^k,\lambda_0^{k-1},\dots,1)$ is a corresponding strongly positive eigenvector, that is,*

$$T(u_{\lambda_0}) = \lambda_0 u_{\lambda_0}, \quad u_{\lambda_0} \gg_\mu 0. \tag{2.17}$$

(iii) *If instead of (2.3) the stronger condition (2.6) is assumed, then (2.16) holds.*

*Proof.* (i) The continuity and the positive homogeneity of $T$ are evident. The monotonicity of $T$ is a consequence of Theorem 2.1. The fact that $T$ maps $C_\mu$ into itself follows from the monotonicity of $T$ and the equality $T(0) = 0$.

(ii) Define

$$h(\lambda) = \lambda^{k+1} - f(\lambda^k,\lambda^{k-1},\dots,1), \quad \lambda \geq \mu. \tag{2.18}$$

Since $(\lambda^k,\lambda^{k-1},\dots,1) \geq_\mu (0,0,\dots,0)$ for $\lambda \geq \mu$ and $f$ is continuous on $C_\mu$, $h$ is continuous on $[\mu,\infty)$. Further, by virtue of (2.16), $h(\mu) < 0$ and, in view of (1.6), we have

$$h(\lambda) = \lambda^k(\lambda - f(1,\lambda^{-1},\dots,\lambda^{-k})) \longrightarrow \infty \quad \text{as } \lambda \longrightarrow \infty. \tag{2.19}$$

This implies the existence of $\lambda_0 > \mu$ such that $h(\lambda_0) = 0$. This $\lambda_0$ is a root of (1.8) and conclusion (2.17) is an immediate consequence of the definitions of $T$ and the strong ordering $\ll_\mu$. It remains to show that (1.8) has no other root in $(\mu,\infty)$. Let $\lambda > \mu$ be a root of (1.8). Define $u_\lambda = (\lambda^k,\lambda^{k-1},\dots,1)$. It is easily seen that

$$T(u_\lambda) = \lambda u_\lambda, \quad u_\lambda \gg_\mu 0. \tag{2.20}$$

Thus, $u_\lambda$ is a strongly positive eigenvector of the continuous, positively homogeneous and monotone selfmapping $T$ of $C_\mu$. According to a result of Kloeden and Rubinov [3, Corollary 3.1], the corresponding eigenvalue $\lambda$ coincides with the spectral radius of $T$ and hence it is uniquely determined.

(iii) Clearly, $(\mu^k,\mu^{k-1},\dots,1) >_\mu (0,0,\dots,0)$. By virtue of (2.6), this together with $f(0,0,\dots,0) = 0$, implies (2.16). □

*Remark 2.6.* The previous proof shows that in case (ii) of Theorem 2.5, $\lambda_0 < 1$ if and only if $\mu < 1$ and $f(1,1,\dots,1) < 1$.

We conclude this section with some corollaries of the previous results for (1.7), a special case of (1.1) when

$$f(x_0,x_1,\dots,x_k) = \sum_{i=0}^{k} K_i x_i + b\max\{x_0,x_1,\dots,x_r\}. \tag{2.21}$$

As in Section 1, we assume that $k \ge r$ in (1.7).

Corollary 2.7. *Suppose that $b \ge 0$ and $\mu > 0$. Then, the following hold.*

(i) *Condition (2.3) holds for (1.7) if (1.12) holds.*
(ii) *Condition (2.6) holds for (1.7) if (1.12) holds with a strict inequality.*
(iii) *Condition (2.16) holds for (1.7) if (1.12) and one of the following hold:*
  (a) *$b > 0$, or*
  (b) *$b = 0$ and $K_i > 0$ for some $i \in \{1,2,\dots,k\}$, or*
  (c) *$b = 0$, $K_i \le 0$ for $i = 1,2,\dots,k$ and the inequality in (1.12) is strict.*

*In that case, (1.14) has a unique root $\lambda_0$ in $(\mu,\infty)$. Furthermore, $\lambda_0 < 1$ if and only if $\mu < 1$ and (1.20) holds.*

*Proof.* Clearly, for $f$ defined by (2.21), condition (2.4) holds with $L_i = K_i$ for $i = 0,1,\dots,k$. Consequently, conclusions (i) and (ii) follow immediately from Propositions 2.2 and 2.4. To prove (iii), observe that, in view of (1.12), we have

$$\begin{aligned} f(\mu^k,\mu^{k-1},\dots,1) &= \mu^k\left(\sum_{i=0}^{k} K_i\mu^{-i} + b\max\{1,\mu^{-1},\dots,\mu^{-r}\}\right) \\ &\ge \mu^k\left(K_0 - \sum_{i=1}^{k} K_i^-\mu^{-i}\right) \ge \mu^{k+1}. \end{aligned} \tag{2.22}$$

If (a), (b), or (c) holds, then one of the above inequalities is strict and thus (2.16) holds. The last two conclusions of (iii) follow from Theorem 2.5(ii) and Remark 2.6. □

## 3. Main results

In the theorems below, we assume that $f$ is positively homogeneous and satisfies either the monotonicity condition (2.3) or (2.6). Sufficient conditions for (2.3) and (2.6) to hold were given in Section 2 (see Propositions 2.2 and 2.4). The first theorem gives an upper estimate for the solutions of inequality (1.5).

Theorem 3.1. *Suppose that there exists $\mu \ge 0$ such that (1.6) and (2.3) hold. If the characteristic equation (1.8) has a root $\lambda_0$ in $(\mu,\infty)$, then for every solution $(y_n)_{n\ge -k}$ of (1.5) there exists a positive constant $M = M(y_0,y_{-1},\dots,y_{-k})$ such that*

$$y_n \le M\lambda_0^n, \quad n \ge -k. \tag{3.1}$$

The existence of a root $\lambda_0$ of (1.8) in $(\mu,\infty)$ can be guaranteed by Theorem 2.5(ii). We have the following corollary of Theorems 2.5 and 3.1.

Corollary 3.2. *Suppose that there exists $\mu \geq 0$ such that $f$ is continuous on $C_\mu$ and conditions (1.6), (2.3), and (2.16) hold. Then, (1.8) has a unique root $\lambda_0$ in $(\mu,\infty)$ and (3.1) holds for every solution $(y_n)_{n\geq -k}$ of (1.5) with a positive constant $M$ depending on the initial data $(y_0,y_{-1},\dots,y_{-k})$.*

*Remark 3.3.* According to Theorem 2.5(iii), condition (2.16) automatically holds if the monotonicity assumption (2.3) in Corollary 3.2 is replaced with the strong monotonicity condition (2.6).

*Remark 3.4.* Theorem 1.1 in Section 1 is a consequence of Corollaries 2.7 and 3.2.

Before we present the proof of Theorem 3.1, we establish a comparison theorem which is interesting in its own right. Note that in this theorem we merely assume the monotonicity condition (2.3).

Theorem 3.5. *Suppose (2.3) holds for some $\mu \geq 0$. Let $(x_n)_{n\geq -k}$ and $(y_n)_{n\geq -k}$ be solutions of (1.1) and (1.5), respectively, such that*

$$(y_0,y_{-1},\dots,y_{-k}) \leq_\mu (x_0,x_{-1},\dots,x_{-k}). \tag{3.2}$$

*Then, for all $n \geq 0$,*

$$(y_n,y_{n-1},\dots,y_{n-k}) \leq_\mu (x_n,x_{n-1},\dots,x_{n-k}). \tag{3.3}$$

*In particular,*

$$y_n \leq x_n, \quad n \geq -k. \tag{3.4}$$

*Proof.* We will prove (3.3) by induction on $n$. By assumption (3.2), (3.3) holds for $n=0$. Suppose for induction that (3.3) holds for some $n \geq 0$. In view of the definition of the ordering $\leq_\mu$, (3.3) implies that

$$x_i - y_i \geq \mu(x_{i-1}-y_{i-1}) \geq 0 \tag{3.5}$$

for $i = n-k+1, n-k+2,\dots,n$. Using (1.1) and (1.5), we find for $n \geq 0$,

$$x_{n+1}-y_{n+1} \geq f(x_n,\dots,x_{n-k}) - f(y_n,\dots,y_{n-k}) \geq \mu(x_n-y_n), \tag{3.6}$$

the last inequality being a consequence of (2.3) and (3.3). Thus, (3.5) also holds for $i = n+1$. Therefore,

$$(y_{n+1},y_n,\dots,y_{n+1-k}) \leq_\mu (x_{n+1},x_n,\dots,x_{n+1-k}). \tag{3.7}$$

Thus, (3.3) is confirmed for all $n \geq 0$. Conclusion (3.4) follows from (3.3) and the definition of $C_\mu$. □

We are in a position to give a proof of Theorem 3.1.

*Proof of Theorem 3.1.* Let $(y_n)_{n\geq -k}$ be a solution of (1.5). Consider the solution $(x_n)_{n\geq -k}$ of (1.1) with initial data

$$(x_0,x_{-1},\dots,x_{-k}) = (y_0,y_{-1},\dots,y_{-k}). \tag{3.8}$$

By Theorem 3.5, $y_n \le x_n$ for $n \ge -k$. Therefore, it is enough to show that

$$x_n \le M\lambda_0^n, \quad n \ge -k, \tag{3.9}$$

for some $M > 0$. Since $\lambda_0 > \mu$, the vector $u_{\lambda_0} = (1, \lambda_0^{-1}, \dots, \lambda_0^{-k})$ is strongly positive, $u_{\lambda_0} \gg_\mu 0$. Consequently,

$$(x_0, x_{-1}, \dots, x_{-k}) \le_\mu Mu_{\lambda_0} = (M, M\lambda_0^{-1}, \dots, M\lambda_0^{-k}) \tag{3.10}$$

for all sufficiently large $M$. Since $\lambda_0$ is a root of (1.8) and $f$ is positively homogeneous, $(M\lambda_0^n)_{n\ge -k}$ is a solution of (1.1). Estimate (3.9) now follows from (3.10) and Theorem 3.5 applied to the solutions $(x_n)_{n\ge -k}$ and $(M\lambda_0^n)_{n\ge -k}$ of (1.1). □

*Remark 3.6.* The constant $M$ in (3.1) of Theorem 3.1 can be computed explicitly from (3.10) (where $x_i = y_i$ for $i = -k, -k+1, \dots, 0$). Writing the system of inequalities corresponding to (3.10) from the definition of the ordering $\le_\mu$, it can be shown that $M$ in (3.1) can be taken as

$$M = K \max\{|y_0|, |y_{-1}|, \dots, |y_{-k}|\}, \tag{3.11}$$

where $K$ is a positive constant independent of the initial data $(y_0, y_{-1}, \dots, y_{-k})$.

Our next aim is to show that for the nontrivial solutions $(x_n)_{n\ge -k}$ of (1.1) starting from $C_\mu$, the exponential estimate (3.1) of Theorem 3.1 can be replaced with the more precise limit relation

$$\lim_{n\to\infty} (\lambda_0^{-n} x_n) = L, \tag{3.12}$$

where $L$ is a positive constant depending on the initial data.

Theorem 3.7. *Suppose that there exists $\mu > 0$ such that $f$ is continuous on $C_\mu$ and (1.6) and (2.6) hold. Then, for every solution $(x_n)_{n\ge -k}$ of (1.1) with initial data $(x_0, x_{-1}, \dots, x_{-k}) \in C_\mu \setminus \{0\}$, there exists a positive constant $L = L(x_0, x_{-1}, \dots, x_{-k})$ such that (3.12) holds, where $\lambda_0$ is the unique root of (1.8) in $(\mu, \infty)$.*

Note that if $f$ in Theorem 3.7 is linear, then the value of the limit (3.12) can be given explicitly in terms of the initial data $(x_0, x_{-1}, \dots, x_{-k})$ (see [2] or [4] for details).

The proof of Theorem 3.7 will be based on a nonlinear version of the Perron-Frobenius theorem due to Kloeden and Rubinov [3] adapted to our situation. For further related results, see [5].

Theorem 3.8. *Let $\mu \ge 0$. Suppose that $T : C_\mu \to \mathbb{R}^{k+1}$ is a continuous, positively homogeneous map with the following properties:*

(i) $T(C_\mu) \subset C_\mu$,
(ii) *there exist $\lambda > 0$ and $u \gg_\mu 0$ such that $T(u) = \lambda u$,*
(iii) *$T$ is monotone on $C_\mu$, that is,*

$$T(y) \ge_\mu T(x) \quad \text{whenever } x, y \in C_\mu \text{ satisfy } x \le_\mu y, \tag{3.13}$$

(iv) *some iterate* $T^s$ $(s \ge 1)$ *of* $T$ *is strongly monotone on* $C_\mu$, *that is,*

$$T^s(y) \gg_\mu T^s(x) \quad \text{whenever } x, y \in C_\mu \text{ satisfy } x <_\mu y, \tag{3.14}$$

*Then, for every* $x \in C_\mu \setminus \{0\}$, *there exists a positive constant* $K = K(x)$ *such that*

$$\lambda^{-n} T^n(x) \longrightarrow Ku \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

Theorem 3.8 is a consequence of [3, Corollary 5.2 and Remark 5.1] applied to the scaled map $\widetilde{T} = \lambda^{-1} T$.

*Proof of Theorem 3.7.* We will prove Theorem 3.7 by applying Theorem 3.8 to the map $T$ defined by (1.2). Theorems 2.3 and 2.5 show that the hypotheses of Theorem 3.8 hold with $\lambda = \lambda_0$ and $u = (\lambda_0^k, \lambda_0^{k-1}, \dots, 1)$, where $\lambda_0$ is the unique root of (1.8) in $(\mu, \infty)$. By the application of Theorem 3.8, we conclude that if $(x_0, x_{-1}, \dots, x_{-k}) \in C_\mu \setminus \{0\}$, then

$$\lambda_0^{-n} T^n(x_0, x_{-1}, \dots, x_{-k}) \longrightarrow K(\lambda_0^k, \lambda_0^{k-1}, \dots, 1) \quad \text{as } n \longrightarrow \infty \tag{3.16}$$

for some $K > 0$. By virtue of (1.3), the last limit relation is equivalent to (3.12) with $L = K\lambda_0^k$. □

*Remark 3.9.* Theorem 1.2 in Section 1 is a consequence of Theorem 3.7 and Corollary 2.7.

Now, we present a theorem concerning the behavior of the solutions of (1.10). We will assume that the linear part of (1.10) generates a monotone system with respect to the ordering $\le_\mu$ and we use the variation-of-constants formula to obtain an exponential estimate for the growth of the solutions. As in Section 1, we assume that $k \ge r$ in (1.10).

Theorem 3.10. *Suppose that there exist* $\mu > 0$ *and a function* $h : \mathbb{R}_+^{r+1} \to \mathbb{R}_+$ *such that for* $n \ge 0$ *and* $x, y \in \mathbb{R}^{r+1}$,

$$|g(n, x_0, x_1, \dots, x_r)| \le h(|x_0|, |x_1|, \dots, |x_r|), \tag{3.17}$$

$$h(y) \ge h(x) \quad \text{whenever } 0 \le x_i \le y_i \text{ for } i = 0, 1, \dots, r, \tag{3.18}$$

$$h \text{ is continuous and positively homogeneous on } C_\mu, \tag{3.19}$$

$$\mu + \sum_{i=1}^{k} K_i^- \mu^{-i} \le K_0, \quad K_i^- = \max\{0, -K_i\} \tag{3.20}$$

*and one of the following holds:*

(a) $h(\mu^r, \mu^{r-1}, \dots, 1) > 0$, *or*
(b) $h(\mu^r, \mu^{r-1}, \dots, 1) = 0$ *and* $K_i > 0$ *for some* $i \in \{1, 2, \dots, k\}$, *or*
(c) $h(\mu^r, \mu^{r-1}, \dots, 1) = 0$, $K_i \le 0$ *for* $i = 1, 2, \dots, k$ *and the inequality in (3.20) is strict.*

*Then, for every solution* $(x_n)_{n \ge -k}$ *of (1.10) there exists a positive constant* $M = M(x_0, x_{-1}, \dots, x_{-k})$ *such that*

$$|x_n| \le M\lambda_0^n, \quad n \ge -k, \tag{3.21}$$

*where $\lambda_0$ is the unique root of the equation*

$$\lambda^{k+1} = \sum_{i=0}^{k} K_i \lambda^{k-i} + h(\lambda^k, \lambda^{k-1}, \ldots, \lambda^{k-r}) \tag{3.22}$$

*in the interval $(\mu, \infty)$.*

*Proof.* First, we show that (3.22) has a unique root in $(\mu, \infty)$. We will apply Theorem 2.5(ii) to the equation

$$x_{n+1} = \sum_{i=0}^{k} K_i x_{n-i} + h(x_n, x_{n-1}, \ldots, x_{n-r}), \quad n \geq 0. \tag{3.23}$$

Equation (3.23) is a special case of (1.1) when

$$f(x_0, x_1, \ldots, x_k) = \sum_{i=0}^{k} K_i x_i + h(x_0, x_1, \ldots, x_r). \tag{3.24}$$

Conditions (3.18) and (3.20) imply that assumptions (2.4) and (2.5) of Proposition 2.2 hold for (3.23) on $C_\mu$ with $L_i = K_i$ for $i = 0, 1, \ldots, k$. By Proposition 2.2, the monotonicity condition (2.3) holds for (3.23) on $C_\mu$. By virtue of (3.19), $f$ is continuous and positively homogeneous on $C_\mu$. Further, by virtue of (3.19) and (3.20), we have

$$\begin{aligned} f(\mu^k, \mu^{k-1}, \ldots, 1) &= \mu^k \left( \sum_{i=0}^{k} K_i \mu^{-i} + \mu^{-r} h(\mu^r, \mu^{r-1}, \ldots, 1) \right) \\ &\geq \mu^k \left( K_0 - \sum_{i=1}^{k} K_i^- \mu^{-i} \right) \geq \mu^{k+1}. \end{aligned} \tag{3.25}$$

Since any of the conditions (a), (b), or (c) implies that one of the last two inequalities is strict, (2.16) holds. The existence and uniqueness of $\lambda_0$ now follows from Theorem 2.5(ii).

Now, we prove (3.21). Let $(x_n)_{n \geq -k}$ be an arbitrary solution of (1.10). Consider the solution $(y_n)_{n \geq -k}$ of the linear equation (1.11) with the same initial data, $(y_0, y_{-1}, \ldots, y_{-k}) = (x_0, x_{-1}, \ldots, x_{-k})$. Since $\lambda_0 > \mu$, we have

$$(1, \lambda_0^{-1}, \ldots, \lambda_0^{-k}) \gg_\mu (0, 0, \ldots, 0). \tag{3.26}$$

Consequently,

$$(y_0, y_{-1}, \ldots, y_{-k}) \leq_\mu M_1 (1, \lambda_0^{-1}, \ldots, \lambda_0^{-k}) \tag{3.27}$$

for all sufficiently large $M_1 > 0$. By Proposition 2.2, (3.20) implies that the monotonicity condition (2.3) holds for the linear equation (1.11). Therefore, we can apply Theorem 3.5 to (1.11) and from (3.27) we obtain

$$y_n \leq M_1 w_n, \quad n \geq -k, \tag{3.28}$$

where $(w_n)_{n\geq -k}$ is the solution of (1.11) with initial data $(w_0, w_{-1}, \ldots, w_{-k}) = (1, \lambda_0^{-1}, \ldots, \lambda_0^{-k})$. The same argument applied to the solution $(-y_n)_{n\geq -k}$ of (1.11) yields the existence of $M_2 > 0$ such that

$$-y_n \leq M_2 w_n, \quad n \geq -k. \tag{3.29}$$

Consequently,

$$|y_n| \leq M_3 w_n, \quad n \geq -k, \tag{3.30}$$

where $M_3 = \max\{M_1, M_2\}$. Here, we have used the fact that $w_n \geq 0$ for $n \geq -k$ which follows from Theorem 3.5 and (3.26). We will show that (3.21) holds with

$$M = \max\{M_3, |x_0|, |x_{-1}|\lambda_0, |x_{-2}|\lambda_0^2, \ldots, |x_{-k}|\lambda_0^k\}. \tag{3.31}$$

By the definition of $M$, we have

$$|x_i| \leq M\lambda_0^i \quad \text{for } i = -k, -k+1, \ldots, 0. \tag{3.32}$$

Suppose that $n \geq 1$ and

$$|x_i| \leq M\lambda_0^i \quad \text{for } i = -k, -k+1, \ldots, n-1. \tag{3.33}$$

By the induction principle, the proof will be complete if we show that (3.33) also holds for $i = n$. By the variation-of-constants formula (see [11, Lemma 1]), the solution $x_n$ of (1.10) can be written in the form

$$x_n = y_n + \sum_{i=0}^{n-1} v_{n-i-1} g(i, x_i, x_{i-1}, \ldots, x_{i-r}), \quad n \geq 0, \tag{3.34}$$

where $y_n$ has the meaning as before and $(v_n)_{n\geq -k}$ is the (fundamental) solution of the linear equation (1.11) with initial data $(v_0, v_{-1}, \ldots, v_{-k}) = (1, 0, \ldots, 0)$. Since $(1, 0, \ldots, 0) \geq_\mu (0, 0, \ldots, 0)$, Theorem 3.5 implies that $v_n \geq 0$ for $n \geq 0$. Using (3.17), (3.18), (3.30), and (3.33) in (3.34), we find

$$|x_n| \leq M w_n + \sum_{i=0}^{n-1} v_{n-i-1} h(M\lambda_0^i, M\lambda_0^{i-1}, \ldots, M\lambda_0^{i-r}). \tag{3.35}$$

Writing the variation-of-constants formula for the solution $(\lambda_0^n)_{n\geq -k}$ of (3.23), we obtain for $n \geq 0$,

$$\lambda_0^n = w_n + \sum_{i=0}^{n-1} v_{n-i-1} h(\lambda_0^i, \lambda_0^{i-1}, \ldots, \lambda_0^{i-r}), \tag{3.36}$$

where $w_n$ and $v_n$ are the solutions of (1.11) defined as before. This and the positive homogeneity of $h$ imply that the right-hand side of (3.35) is equal to $M\lambda_0^n$. Thus, we have shown that (3.33) implies that $|x_n| \leq M\lambda_0^n$. □

The same argument as in Remark 2.6 shows that the constants $M_1$ and $M_2$ in the previous proof and hence $M$ in (3.21) can be written in the form (3.11) (with $y$ replaced with $x$). Consequently, Theorem 3.10 combined with Remark 2.6 yields the following stability criterion.

Theorem 3.11. *In addition to the hypotheses of Theorem 3.10, suppose that $\mu < 1$ and*

$$\sum_{i=0}^{k} K_i + h(1,1,\dots,1) < 1. \tag{3.37}$$

*Then, the zero solution of (1.10) is globally exponentially stable.*

*Remark 3.12.* Theorems 1.3 and 1.4 in Section 1 follow from Theorems 3.10 and 3.11, respectively, when $h(x_0,x_1,\dots,x_r) = b\max\{x_0,x_1,\dots,x_r\}$.

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