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Research Article

Notes on Interpolation Inequalities

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An easy proof of the John-Nirenberg inequality is provided by merely using the Calderón-Zygmund decomposition. Moreover, an interpolation inequality is presented with the help of the John-Nirenberg inequality.

1. Introduction

It is well known that various interpolation inequalities play an important role in the study of operational equations, partial differential equations, and variation problems (see, e.g., [1–6]). So, it is an issue worthy of deep investigation.

Let Q_0 be either R^n or a fixed cube in R^n. For $f \in L^1_{\mathrm{loc}}(Q_0)$, write

$$\|f\|_{\mathrm{BMO}} := \sup_{Q \subset Q_0} \frac{1}{|Q|} \int_Q |f - f_Q|\, dx, \tag{1.1}$$

where the supremum is taken over all cubes $Q \subset Q_0$ and $f_Q := (1/|Q|) \int_Q f\, dx$.

Recall that $\mathrm{BMO}(Q_0)$ is the set consisting of all locally integrable functions on Q_0 such that $\|f\|_{\mathrm{BMO}} < \infty$, which is a Banach space endowed with the norm $\|\cdot\|_{\mathrm{BMO}}$. It is clear that any bounded function on Q_0 is in $\mathrm{BMO}(Q_0)$, but the converse is not true. On the other hand, the BMO space is regarded as a natural substitute for L^∞ in many studies. One of the important features of the space is the John-Nirenberg inequality. There are several versions of its proof; see, for example, [2, 7–9]. Stimulated by these works, we give, in this paper, an easy proof of the John-Nirenberg inequality by using the Calderón-Zygmund decomposition only. Moreover, with the help of this inequality, an interpolation inequality is showed for L^p and BMO norms.

2. Results and Proofs

Lemma 2.1 (John-Nirenberg inequality). *If $f \in BMO(Q_0)$, then there exist positive constants c_1, c_2 such that, for each cube $Q \subset Q_0$,*

$$\left|\left\{x \in Q : \left|f(x) - f_Q\right| > t\right\}\right| \le c_1 \exp\left\{-\frac{c_2}{\|f\|_{BMO}} t\right\} |Q|, \quad t > 0. \tag{2.1}$$

Proof. Without loss of generality, we can and do assume that $\|f\|_{\mathrm{BMO}} = 1$.

For each $t > 0$, let $F(t)$ denote the least number for which we have

$$\left|\left\{x \in Q : \left|f(x) - f_Q\right| > t\right\}\right| \le F(t)|Q|, \tag{2.2}$$

for any cube $Q \subset Q_0$. It is easy to see that $F(t) \le 1 (t > 0)$ and $F(t)$ is decreasing.

Fix a cube $Q \subset Q_0$. Applying the Calderón-Zygmund decomposition (cf., e.g., [2, 9]) to $|f(x) - f_Q|$ on Q, with 2^n as the separating number, we get a sequence of disjoint cubes $\{Q_j\}$ and E such that

$$Q = \left(\bigcup_j Q_j\right) \cup E, \tag{2.3}$$

$$\left|f(x) - f_Q\right| \le 2^n, \quad \text{for a.e. } x \in E, \tag{2.4}$$

$$2^n < \frac{1}{\left|Q_j\right|} \int_{Q_j} \left|f - f_Q\right| dx \le 4^n. \tag{2.5}$$

Using (2.5), we have

$$\sum_j \left|Q_j\right| < \frac{1}{2^n}|Q|. \tag{2.6}$$

From (2.3), (2.4), and (2.6), we deduce that for $t > 4^n$,

$$\begin{aligned}
\left|\left\{x \in Q : \left|f(x) - f_Q\right| > t\right\}\right| &= \left|\bigcup_j \left\{x \in Q_j : \left|f(x) - f_Q\right| > t\right\}\right| \\
&\le \sum_j \left|\left\{x \in Q_j : \left|f(x) - f_Q\right| > t - 4^n\right\}\right| \\
&= \sum_j \left|Q_j\right| \frac{1}{\left|Q_j\right|} \left|\left\{x \in Q_j : \left|f(x) - f_Q\right| > t - 4^n\right\}\right| \\
&\le \frac{1}{2^n} F(t - 4^n)|Q|.
\end{aligned} \tag{2.7}$$

This yields that

$$F(t) \leq \frac{1}{2^n} F(t - 4^n), \quad t > 4^n. \tag{2.8}$$

Let $\gamma = [(t-1)4^{-n}]$ $(t > 4^n)$, $\mu = 1 + \gamma 4^n$. Then $0 < \mu \leq t$. By iterating, we get

$$\begin{aligned} F(t) &\leq F(\mu) = F(1 + \gamma 4^n) \leq 2^{-n\gamma} \leq 2^{-n((t-1)4^{-n}-1)} \\ &= 2^{n(1+4^{-n})} \exp(-(\log 2) n 4^{-n} t), \quad t > 4^n. \end{aligned} \tag{2.9}$$

Thus, letting

$$c_1 = 2^{n(1+4^{-n})}, \qquad c_2 = (\log 2) n 4^{-n} \tag{2.10}$$

gives that

$$F(t) \leq c_1 e^{-c_2 t}, \quad t > 0, \tag{2.11}$$

since

$$F(t) \leq 1 \leq c_1 e^{-c_2 t}, \quad 0 < t \leq 4^n. \tag{2.12}$$

This completes the proof. □

Remark 2.2. (1) As we have seen, the recursive estimation (2.8) justifies the desired exponential decay of $F(t)$.

(2) There exists a gap in the proof of the John-Nirenberg inequality given in [2]. Actually, for a decreasing function $G(t) : (0, \infty) \to [0, 1]$, the following estimate:

$$G(2 \cdot 2^n \alpha) \leq \frac{1}{\alpha} G(2^n \alpha), \quad \alpha > 1 \tag{2.13}$$

does not generally imply such a property, that is, the existence of constants $c_1, c_2 > 0$ such that

$$G(t) \leq c_1 e^{-c_2 t}, \quad t > 0. \tag{2.14}$$

We present the following function as a counter example:

$$G(t) = \exp\left\{ -\left(\log \frac{5}{3} \right)^{-1} \log^2(t+1) \right\}, \quad t > 0. \tag{2.15}$$

In fact, it is easy to see that there are no constants $c_1, c_2 > 0$ such that (2.14) holds. On the other hand, we have

$$\frac{G'(t)}{G(t)} = \left\{-\left(\log\frac{5}{3}\right)^{-1} 2\frac{\log(t+1)}{t+1}\right\}, \quad t > 0. \tag{2.16}$$

Integrating both sides of the above equation from $2^n\alpha$ to $2 \cdot 2^n\alpha$, we obtain

$$\begin{aligned}
G(2\cdot 2^n\alpha) &= \exp\left\{-2\left(\log\frac{5}{3}\right)^{-1}\int_{2^n\alpha}^{2\cdot 2^n\alpha}\frac{\log(t+1)}{t+1}dt\right\}G(2^n\alpha) \\
&= \exp\left\{-\left(\log\frac{5}{3}\right)^{-1}\left(\log^2(2\cdot 2^n\alpha+1)-\log^2(2^n\alpha+1)\right)\right\}G(2^n\alpha) \\
&= \exp\left\{-\left(\log\frac{5}{3}\right)^{-1}\log((2\cdot 2^n\alpha+1)(2^n\alpha+1))\cdot\log\left(\frac{2\cdot 2^n\alpha+1}{2^n\alpha+1}\right)\right\}G(2^n\alpha) \\
&\leq \exp\{-\log((2\cdot 2^n\alpha+1)(2^n\alpha+1))\}G(2^n\alpha) \\
&= \frac{1}{(2\cdot 2^n\alpha+1)(2^n\alpha+1)}G(2^n\alpha) \\
&\leq \frac{1}{\alpha}G(2^n\alpha),
\end{aligned} \tag{2.17}$$

where the fact that

$$\frac{2\cdot 2^n\alpha+1}{2^n\alpha+1} > \frac{5}{3} \quad (\alpha > 1) \tag{2.18}$$

is used to get the first inequality above. This means that

$$G(2\cdot 2^n\alpha) \leq \frac{1}{\alpha}G(2^n\alpha), \quad \alpha > 1. \tag{2.19}$$

Next, we make use of the John-Nirenberg inequality to obtain an interpolation inequality for L^p and BMO norms.

Theorem 2.3. *Suppose that $1 \leq p < r < \infty$ and $f \in L^p(Q_0) \cap BMO(Q_0)$. Then we have*

$$\|f\|_{L^r} \leq (\text{const})\|f\|_{L^p}^{p/r}\|f\|_{BMO}^{1-p/r}. \tag{2.20}$$

Proof. If $\|f\|_{\mathrm{BMO}} = 0$, the proof is trivial; so we assume that $\|f\|_{\mathrm{BMO}} \neq 0$. In view of the Calderón-Zygmund decomposition theorem, there exists a sequence of disjoint cubes $\{Q_j\}$ and E such that

$$Q_0 = \left(\bigcup_j Q_j\right) \cup E, \tag{2.21}$$

$$|f(x)|^p \leq \|f\|_{\mathrm{BMO}}^p \quad \text{for a.e. } x \in E, \tag{2.22}$$

$$\|f\|_{\mathrm{BMO}}^p < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|^p dx \leq 2^n \|f\|_{\mathrm{BMO}}^p. \tag{2.23}$$

From (2.23), we get

$$\begin{gathered}
\sum_j |Q_j| < \frac{1}{\|f\|_{\mathrm{BMO}}^p}\int_{Q_0} |f(x)|^p dx = \frac{\|f\|_{L_p}^p}{\|f\|_{\mathrm{BMO}}^p}, \\
|f|_{Q_j} = \frac{1}{|Q_j|}\int_{Q_j} |f(x)| dx \leq \left(\frac{1}{|Q_j|}\int_{Q_j} |f(x)|^p dx\right)^{1/p} \leq 2^{n/p}\|f\|_{\mathrm{BMO}}.
\end{gathered} \tag{2.24}$$

Using (2.21)–(2.24), together with Lemma 2.1, yields that, for $\lambda > 2^{n/p}\|f\|_{\mathrm{BMO}}$,

$$\begin{aligned}
|\{x \in Q_0 : |f(x)| > \lambda\}| &= \left|\bigcup_j \{x \in Q_j : |f(x)| > \lambda\}\right| \\
&\leq \sum_j \left|\left\{x \in Q_j : \ \left|f(x) - f_{Q_j}\right| > \lambda - \left|f_{Q_j}\right|\right\}\right| \\
&\leq \sum_j |Q_j| \frac{1}{|Q_j|}\left|\left\{x \in Q_j : \left|f(x) - f_{Q_j}\right| > \lambda - 2^{n/p}\|f\|_{\mathrm{BMO}}\right\}\right| \\
&\leq \sum_j c_1 \exp\left\{-\frac{c_2}{\|f\|_{\mathrm{BMO}}}\left(\lambda - 2^{n/p}\|f\|_{\mathrm{BMO}}\right)\right\}|Q_j| \\
&\leq c_1 \exp\left\{-\frac{c_2}{\|f\|_{\mathrm{BMO}}}\left(\lambda - 2^{n/p}\|f\|_{\mathrm{BMO}}\right)\right\}\frac{\|f\|_{L^p}^p}{\|f\|_{\mathrm{BMO}}^p}.
\end{aligned} \tag{2.25}$$

From (2.25), we obtain

$$\begin{aligned}
\|f\|_{L^r}^r &= r\int_0^\infty \lambda^{r-1}\left|\{x\in Q_0 : |f(x)|>\lambda\}\right| d\lambda \\
&= r\int_0^{2^{n/p}\|f\|_{\mathrm{BMO}}} \lambda^{r-1}\left|\{x\in Q_0 : |f(x)|>\lambda\}\right| d\lambda \\
&\quad + r\int_{2^{n/p}\|f\|_{\mathrm{BMO}}}^\infty \lambda^{r-1}\left|\{x\in Q_0 : |f(x)|>\lambda\}\right| d\lambda \\
&\le r\int_0^{2^{n/p}\|f\|_{\mathrm{BMO}}} \lambda^{r-1}\frac{\|f\|_{L^p}^p}{\lambda^p} d\lambda \\
&\quad + r\int_{2^{n/p}\|f\|_{\mathrm{BMO}}}^\infty \lambda^{r-1} c_1 \exp\left\{-\frac{c_2}{\|f\|_{\mathrm{BMO}}}\left(\lambda - 2^{n/p}\|f\|_{\mathrm{BMO}}\right)\right\}\frac{f\|f\|_{L^p}^p}{\|f\|_{\mathrm{BMO}}^p} d\lambda \\
&= \frac{r}{r-p}2^{(n/p)(r-p)}\|f\|_{\mathrm{BMO}}^{r-p}\|f\|_{L^p}^p + \frac{rc_1}{c_2}2^{(n/p)(r-1)}\|f\|_{\mathrm{BMO}}^{r-p}\|f\|_{L^p}^p \\
&\le (\mathrm{const})\|f\|_{\mathrm{BMO}}^{r-p}\|f\|_{L^p}^p.
\end{aligned} \tag{2.26}$$

Hence, the proof is complete. □

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