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Research Article

Existence of Pseudo-Almost Automorphic Mild Solutions to Some Nonautonomous Partial Evolution Equations

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We use the Krasnoselskii fixed point principle to obtain the existence of pseudo almost automorphic mild solutions to some classes of nonautonomous partial evolutions equations in a Banach space.

1. Introduction

Let $\mathbb{X}$ be a Banach space. In the recent paper by Diagana [1], the existence of *almost automorphic* mild solutions to the nonautonomous abstract differential equations

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of closed linear operators with domains $D(A(t))$ satisfying Acquistapace-Terreni conditions, and the function $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is almost automorphic in $t \in \mathbb{R}$ uniformly in the second variable, was studied. For that, the author made extensive use of techniques utilized in [2], exponential dichotomy tools, and the Schauder fixed point theorem.

In this paper we study the existence of pseudo-almost automorphic mild solutions to the nonautonomous partial evolution equations

$$\frac{d}{dt}[u(t) + G(t, u(t))] = A(t)u(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of linear operators satisfying Acquistpace-Terreni conditions and F, G are pseudo-almost automorphic functions. For that, we make use of exponential dichotomy tools as well as the well-known *Krasnoselskii* fixed point principle to obtain some reasonable sufficient conditions, which do guarantee the existence of pseudo-almost automorphic mild solutions to (1.2).

The concept of pseudo-almost automorphy is a powerful generalization of both the notion of almost automorphy due to Bochner [3] and that of pseudo-almost periodicity due to Zhang (see [4]), which has recently been introduced in the literature by Liang et al. [5–7]. Such a concept, since its introduction in the literature, has recently generated several developments; see, for example, [8–12]. The question which consists of the existence of pseudo-almost automorphic solutions to abstract partial evolution equations has been made; see for instance [10, 11, 13]. However, the use of Krasnoselskii fixed point principle to establish the existence of pseudo-almost automorphic solutions to nonautonomous partial evolution equations in the form (1.2) is an original untreated problem, which is the main motivation of the paper.

The paper is organized as follows: Section 2 is devoted to preliminaries facts related to the existence of an evolution family. Some preliminary results on intermediate spaces are also stated there. Moreover, basic definitions and results on the concept of pseudo-almost automorphy are also given. Section 3 is devoted to the proof of the main result of the paper.

2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. If L is a linear operator on the Banach space $\mathbb{X}$, then, $D(L)$, $\rho(L)$, $\sigma(L)$, $N(L)$, and $R(L)$ stand, respectively, for its domain, resolvent, spectrum, null-space or kernel, and range. If $L : D = D(L) \subset \mathbb{X} \mapsto \mathbb{X}$ is a linear operator, one sets $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(A)$.

If $\mathbb{Y}, \mathbb{Z}$ are Banach spaces, then the space $B(\mathbb{Y}, \mathbb{Z})$ denotes the collection of all bounded linear operators from $\mathbb{Y}$ into $\mathbb{Z}$ equipped with its natural topology. This is simply denoted by $B(\mathbb{Y})$ when $\mathbb{Y} = \mathbb{Z}$. If P is a projection, we set $Q = I - P$.

2.1. Evolution Families

This section is devoted to the basic material on evolution equations as well the dichotomy tools. We follow the same setting as in the studies of Diagana [1].

Assumption (H.1) given below will be crucial throughout the paper.

(H.1) The family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace-Terreni conditions, that is, there exist constants $\omega \geq 0$, $\theta \in (\pi/2, \pi)$, $K, L \geq 0$, and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$S_\theta \bigcup \{0\} \subset \rho(A(t) - \omega) \ni \lambda, \qquad \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1 + |\lambda|},$$
$$\|(A(t) - \omega) R(\lambda, A(t) - \omega)[R(\omega, A(t)) - R(\omega, A(s))]\| \leq L|t - s|^{\mu} |\lambda|^{-\nu}, \tag{2.1}$$

for $t, s \in \mathbb{R}$, $\lambda \in S_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

It should mentioned that (H.1) was introduced in the literature by Acquistapace et al. in [14, 15] for $\omega = 0$. Among other things, it ensures that there exists a unique evolution family

$$\mathcal{U} = \{U(t,s) : t, s \in \mathbb{R} \text{ such that } t \geq s\}, \tag{2.2}$$

on $\mathbb{X}$ associated with $A(t)$ such that $U(t,s)\mathbb{X} \subset D(A(t))$ for all $t, s \in \mathbb{R}$ with $t \geq s$, and

(a) $U(t,s)U(s,r) = U(t,r)$ for $t, s, r \in \mathbb{R}$ such that $t \geq s \geq r$;

(b) $U(t,t) = I$ for $t \in \mathbb{R}$ where I is the identity operator of $\mathbb{X}$;

(c) $(t,s) \mapsto U(t,s) \in B(\mathbb{X})$ is continuous for $t > s$;

(d) $U(\cdot,s) \in C^1((s,\infty), B(\mathbb{X}))$, $(\partial U/\partial t)(t,s) = A(t)U(t,s)$ and

$$\left\| A(t)^k U(t,s) \right\| \leq K(t-s)^{-k}, \tag{2.3}$$

for $0 < t - s \leq 1$, $k = 0, 1$;

(e) $(\partial^+ U(t,s)/\partial s)(x) = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

It should also be mentioned that the above-mentioned proprieties were mainly established in [16, Theorem 2.3] and [17, Theorem 2.1]; see also [15, 18]. In that case we say that $A(\cdot)$ generates the evolution family $U(\cdot,\cdot)$. For some nice works on evolution equations, which make use of evolution families, we refer the reader to, for example, [19–29].

Definition 2.1. One says that an evolution family $\mathcal{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ $(t \in \mathbb{R})$ that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(f) $U(t,s)P(s) = P(t)U(t,s)$;

(g) the restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible (we then set $\widetilde{U}_Q(s,t) := U_Q(t,s)^{-1}$);

(h) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|\widetilde{U}_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

Under Acquistpace-Terreni conditions, the family of operators defined by

$$\Gamma(t,s) = \begin{cases} U(t,s)P(s), & \text{if } t \geq s,\ t, s \in \mathbb{R}, \\ -\widetilde{U}_Q(t,s)Q(s) & \text{if } t < s,\ t, \in \mathbb{R} \end{cases} \tag{2.4}$$

are called Green function corresponding to U and $P(\cdot)$.

This setting requires some estimates related to $U(t,s)$. For that, we introduce the interpolation spaces for $A(t)$. We refer the reader to the following excellent books [30–32] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{X}$ (for that, in assumption (H.1), replace $A(t)$ with (A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{X}_\alpha^A := \left\{ x \in \mathbb{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha (A-\omega)R(r, A-\omega)x\| < \infty \right\}, \tag{2.5}$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{X}_0^A := \mathbb{X}, \qquad \|x\|_0^A := \|x\|, \qquad \mathbb{X}_1^A := D(A),$$
$$\|x\|_1^A := \|(\omega - A)x\|. \tag{2.6}$$

Moreover, let $\widehat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we have the following continuous embedding:

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D\big((\omega - A)^\alpha\big) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \widehat{\mathbb{X}}^A \hookrightarrow \mathbb{X}, \tag{2.7}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_\alpha^A$ and $\mathbb{X}$. However, we have the following continuous injection:

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.8}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (H.1), we set

$$\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}, \qquad \widehat{\mathbb{X}}^t := \widehat{\mathbb{X}}^{A(t)} \tag{2.9}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms.

Now the embedding in (2.7) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_\alpha$ [32, Definition 1.1.1], and hence there is a constant $c(\alpha)$ such that

$$\|y\|_\alpha^t \leq c(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^\alpha, \quad y \in D(A(t)). \tag{2.10}$$

We have the following fundamental estimates for the evolution family $U(t,s)$.

Proposition 2.2 (see [33]). *Suppose that the evolution family $U = U(t,s)$ has exponential dichotomy. For $x \in \mathbb{X}$, $0 \leq \alpha \leq 1$, and $t > s$, the following hold.*

(i) *There is a constant $c(\alpha)$, such that*

$$\|U(t,s)P(s)x\|_\alpha^t \leq c(\alpha)e^{-(\delta/2)(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.11}$$

(ii) *There is a constant $m(\alpha)$, such that*

$$\left\|\widetilde{U}_Q(s,t)Q(t)x\right\|_\alpha^s \leq m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{2.12}$$

In addition to above, we also assume that the next assumption holds.

(H.2) The domain $D(A(t)) = D$ is constant in $t \in \mathbb{R}$. Moreover, the evolution family $U = (U(t,s))_{t\geq s}$ generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$.

2.2. *Pseudo-Almost Automorphic Functions*

Let $BC(\mathbb{R}, \mathbb{X})$ denote the collection of all $\mathbb{X}$-valued bounded continuous functions. The space $BC(\mathbb{R}, \mathbb{X})$ equipped with its natural norm, that is, the sup norm is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$.

Definition 2.3. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n) \tag{2.13}$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t) \tag{2.14}$$

for each $t \in \mathbb{R}$.

If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic in the classical Bochner's sense. Denote by $AA(\mathbb{X})$ the collection of all almost automorphic functions $\mathbb{R} \mapsto \mathbb{X}$. Note that $AA(\mathbb{X})$ equipped with the sup-norm $\|\cdot\|_\infty$ turns out to be a Banach space.

Among other things, almost automorphic functions satisfy the following properties.

Theorem 2.4 (see [34]). *If $f, f_1, f_2 \in AA(\mathbb{X})$, then*

(i) $f_1 + f_2 \in AA(\mathbb{X})$,

(ii) $\lambda f \in AA(\mathbb{X})$ *for any scalar* λ,

(iii) $f_\alpha \in AA(\mathbb{X})$, *where* $f_\alpha : \mathbb{R} \to \mathbb{X}$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$,

(iv) *the range* $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* $\mathbb{X}$, *thus* f *is bounded in norm,*

(v) *if* $f_n \to f$ *uniformly on* $\mathbb{R}$, *where each* $f_n \in AA(\mathbb{X})$, *then* $f \in AA(\mathbb{X})$ *too.*

Let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be another Banach space.

Definition 2.5. A jointly continuous function $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ is said to be almost automorphic in $t \in \mathbb{R}$ if $t \mapsto F(t, x)$ is almost automorphic for all $x \in K$ ($K \subset \mathbb{Y}$ being any bounded subset). Equivalently, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$G(t, x) := \lim_{n\to\infty} F(t + s_n, x) \tag{2.15}$$

is well defined in $t \in \mathbb{R}$ and for each $x \in K$, and

$$\lim_{n \to \infty} G(t - s_n, x) = F(t, x) \tag{2.16}$$

for all $t \in \mathbb{R}$ and $x \in K$.

The collection of such functions will be denoted by $AA(\mathbb{Y}, \mathbb{X})$.

For more on almost automorphic functions and related issues, we refer the reader to, for example, [1, 4, 9, 13, 34–39].

Define

$$PAP_0(\mathbb{R}, \mathbb{X}) := \left\{ f \in \mathrm{BC}(\mathbb{R}, \mathbb{X}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \| f(s) \| ds = 0 \right\}. \tag{2.17}$$

Similarly, $PAP_0(\mathbb{Y}, \mathbb{X})$ will denote the collection of all bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ such that

$$\lim_{T \to \infty} \frac{1}{2r} \int_{-r}^{r} \| F(s, x) \| ds = 0 \tag{2.18}$$

uniformly in $x \in K$, where $K \subset \mathbb{Y}$ is any bounded subset.

Definition 2.6 (see Liang et al. [5, 6]). A function $f \in \mathrm{BC}(\mathbb{R}, \mathbb{X})$ is called pseudo-almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{X})$.

The functions g and ϕ appearing in Definition 2.6 are, respectively, called the *almost automorphic* and the *ergodic perturbation* components of f.

Definition 2.7. A bounded continuous function $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ belongs to $AA(Y, \mathbb{X})$ whenever it can be expressed as $F = G + \Phi$, where $G \in AA(\mathbb{Y}, \mathbb{X})$ and $\Phi \in PAP_0(\mathbb{Y}, \mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{Y}, \mathbb{X})$.

An important result is the next theorem, which is due to Xiao et al. [6].

Theorem 2.8 (see [6]). *The space $PAA(\mathbb{X})$ equipped with the sup norm $\| \cdot \|_\infty$ is a Banach space.*

The next composition result, that is Theorem 2.9, is a consequence of [12, Theorem 2.4].

Theorem 2.9. *Suppose that $f : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ belongs to $PAA(\mathbb{Y}, \mathbb{X})$; $f = g + h$, with $x \mapsto g(t, x)$ being uniformly continuous on any bounded subset K of $\mathbb{Y}$ uniformly in $t \in \mathbb{R}$. Furthermore, one supposes that there exists $L > 0$ such that*

$$\| f(t, x) - f(t, y) \| \leq L \| x - y \|_{\mathbb{Y}} \tag{2.19}$$

for all $x, y \in \mathbb{Y}$ and $t \in \mathbb{R}$.

Then the function defined by $h(t) = f(t, \varphi(t))$ belongs to $PAA(\mathbb{X})$ provided $\varphi \in PAA(\mathbb{Y})$.

We also have the following.

Theorem 2.10 (see [6]). *If $f : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ belongs to $PAA(\mathbb{Y}, \mathbb{X})$ and if $x \mapsto f(t, x)$ is uniformly continuous on any bounded subset K of $\mathbb{Y}$ for each $t \in \mathbb{R}$, then the function defined by $h(t) = f(t, \varphi(t))$ belongs to $PAA(\mathbb{X})$ provided that $\varphi \in PAA(\mathbb{Y})$.*

3. Main Results

Throughout the rest of the paper we fix α, β, real numbers, satisfying $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$.

To study the existence of pseudo-almost automorphic solutions to (1.2), in addition to the previous assumptions, we suppose that the injection

$$\mathbb{X}_\alpha \hookrightarrow \mathbb{X} \tag{3.1}$$

is compact, and that the following additional assumptions hold:

(H.3) $R(\omega, A(\cdot)) \in AA(B(\mathbb{X}, \mathbb{X}_\alpha))$. Moreover, for any sequence of real numbers $(\tau'_n)_{n\in\mathbb{N}}$ there exist a subsequence $(\tau_n)_{n\in\mathbb{N}}$ and a well-defined function $R(t, s)$ such that for each $\varepsilon > 0$, one can find $N_0, N_1 \in \mathbb{N}$ such that

$$\|R(t, s) - \Gamma(t + \tau_n, s + \tau_n)\|_{B(\mathbb{X},\mathbb{X}_\alpha)} \leq \varepsilon H_0(t - s) \tag{3.2}$$

whenever $n > N_0$ for $t, s \in \mathbb{R}$, and

$$\|\Gamma(t, s) - R(t - \tau_n, s - \tau_n)\|_{B(\mathbb{X},\mathbb{X}_\alpha)} \leq \varepsilon H_1(t - s) \tag{3.3}$$

whenever $n > N_1$ for all $t, s \in \mathbb{R}$, where $H_0, H_1 : [0, \infty) \mapsto [0, \infty)$ with $H_0, H_1 \in L^1[0, \infty)$.

(H.4) (a) The function $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is pseudo-almost automorphic in the first variable uniformly in the second one. The function $u \mapsto F(t, u)$ is uniformly continuous on any bounded subset K of $\mathbb{X}_\alpha$ for each $t \in \mathbb{R}$. Finally,

$$\|F(t, u)\|_\infty \leq \mathcal{M}(\|u\|_{\alpha,\infty}), \tag{3.4}$$

where $\|u\|_{\alpha,\infty} = \sup_{t\in\mathbb{R}} \|u(t)\|_\alpha$ and $\mathcal{M} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{\mathcal{M}(r)}{r} = 0. \tag{3.5}$$

(b) The function $G : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{X}_\beta$ is pseudo-almost automorphic in the first variable uniformly in the second one. Moreover, G is globally Lipschitz in the following sense: there exists $L > 0$ for which

$$\|G(t,u) - G(t,v)\|_\beta \leq L\|u - v\| \tag{3.6}$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

(H.5) The operator $A(t)$ is invertible for each $t \in \mathbb{R}$, that is, $0 \in \rho(A(t))$ for each $t \in \mathbb{R}$. Moreover, there exists $c_0 > 0$ such that

$$\sup_{t,s\in\mathbb{R}} \left\| A(s)A(t)^{-1} \right\|_{B(\mathbb{X},\mathbb{X}_\beta)} < c_0. \tag{3.7}$$

To study the existence and uniqueness of pseudo-almost automorphic solutions to (1.2) we first introduce the notion of a mild solution, which has been adapted to the one given in the studies of Diagana et al. [35, Definition 3.1].

Definition 3.1. A continuous function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ is said to be a mild solution to (1.2) provided that the function $s \to A(s)U(t,s)P(s)G(s,u(s))$ is integrable on (s,t), the function $s \to A(s)U_Q(t,s)Q(s)G(s,u(s))$ is integrable on (t,s) and

$$\begin{aligned} u(t) = {} & -G(t,u(t)) + U(t,s)(u(s) + G(s,u(s))) \\ & - \int_s^t A(s)U(t,s)P(s)G(s,u(s))ds + \int_t^s A(s)U_Q(t,s)Q(s)G(s,u(s))ds \\ & + \int_s^t U(t,s)P(s)F(s,u(s))ds - \int_t^s U_Q(t,s)Q(s)F(s,u(s))ds, \end{aligned} \tag{3.8}$$

for $t \geq s$ and for all $t, s \in \mathbb{R}$.

Under assumptions (H.1), (H.2), and (H.5), it can be readily shown that (1.2) has a mild solution given by

$$\begin{aligned} u(t) = {} & -G(t,u(t)) - \int_{-\infty}^t A(s)U(t,s)P(s)G(s,u(s))ds \\ & + \int_t^\infty A(s)U_Q(t,s)Q(s)G(s,u(s))ds + \int_{-\infty}^t U(t,s)P(s)F(s,u(s))ds \\ & - \int_t^\infty U_Q(t,s)Q(s)F(s,u(s))ds \end{aligned} \tag{3.9}$$

for each $t \in \mathbb{R}$.

We denote by S and T the nonlinear integral operators defined by

$$(Su)(t) = \int_{-\infty}^{t} U(t,s)P(s)F(s,u(s))ds - \int_{t}^{\infty} U_Q(t,s)Q(s)F(s,u(s))ds,$$
$$(Tu)(t) = -G(t,u(t)) - \int_{-\infty}^{t} A(s)U(t,s)P(s)G(s,u(s))ds \quad (3.10)$$
$$+ \int_{t}^{\infty} A(s)U_Q(t,s)Q(s)G(s,u(s))ds.$$

The main result of the present paper will be based upon the use of the well-known fixed point theorem of Krasnoselskii given as follows.

Theorem 3.2. *Let C be a closed bounded convex subset of a Banach space $\mathbb{X}$. Suppose the (possibly nonlinear) operators T and S map C into $\mathbb{X}$ satisfying*

(1) *for all $u, v \in C$, then $Su + Tv \in C$;*

(2) *the operator T is a contraction;*

(3) *the operator S is continuous and $S(C)$ is contained in a compact set.*

Then there exists $u \in C$ such that $u = Tu + Su$.

We need the following new technical lemma.

Lemma 3.3. *For each $x \in \mathbb{X}$, suppose that assumptions (H.1), (H.2) hold, and let α, β be real numbers such that $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$. Then there are two constants $r'(\alpha,\beta), d'(\beta) > 0$ such that*

$$\|A(t)U(t,s)P(s)x\|_\beta \le r'(\alpha,\beta)e^{-(\delta/4)(t-s)}(t-s)^{-\beta}\|x\|, \quad t > s, \quad (3.11)$$

$$\left\|A(t)\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta \le d'(\beta)e^{-\delta(s-t)}\|x\|, \quad t \le s. \quad (3.12)$$

Proof. Let $x \in \mathbb{X}$. First of all, note that $\|A(t)U(t,s)\|_{B(\mathbb{X},\mathbb{X}_\beta)} \le K(t-s)^{-(1-\beta)}$ for all t, s such that $0 < t - s \le 1$ and $\beta \in [0,1]$.

Letting $t - s \ge 1$ and using (H.2) and the above-mentioned approximate, we obtain

$$\begin{aligned}
\|A(t)U(t,s)x\|_\beta &= \|A(t)U(t,t-1)U(t-1,s)x\|_\beta \\
&\le \|A(t)U(t,t-1)\|_{B(\mathbb{X},\mathbb{X}_\beta)}\|U(t-1,s)x\| \\
&\le MKe^{\delta}e^{-\delta(t-s)}\|x\| \\
&= K_1e^{-\delta(t-s)}\|x\| \\
&= K_1e^{-(3\delta/4)(t-s)}(t-s)^{\beta}(t-s)^{-\beta}e^{-(\delta/4)(t-s)}\|x\|.
\end{aligned} \quad (3.13)$$

Now since $e^{-(3\delta/4)(t-s)}(t-s)^{\beta} \to 0$ as $t \to \infty$, it follows that there exists $c_4(\beta) > 0$ such that

$$\|A(t)U(t,s)x\|_{\beta} \leq c_4(\beta)(t-s)^{-\beta}e^{-(\delta/4)(t-s)}\|x\|. \tag{3.14}$$

Now, let $0 < t-s \leq 1$. Using (2.11) and the fact $2\beta > \alpha + 1$, we obtain

$$\begin{aligned}
\|A(t)U(t,s)x\|_{\beta} &= \left\|A(t)U\left(t,\frac{t+s}{2}\right)U\left(\frac{t+s}{2},s\right)x\right\|_{\beta} \\
&\leq \left\|A(t)U\left(t,\frac{t+s}{2}\right)\right\|_{B(\mathbb{X},\mathbb{X}_{\beta})}\left\|U\left(\frac{t+s}{2},s\right)x\right\| \\
&\leq k_1\left\|A(t)U\left(t,\frac{t+s}{2}\right)\right\|_{B(\mathbb{X},\mathbb{X}_{\beta})}\left\|U\left(\frac{t+s}{2},s\right)x\right\|_{\alpha} \\
&\leq k_1K\left(\frac{t-s}{2}\right)^{\beta-1}c(\alpha)\left(\frac{t-s}{2}\right)^{-\alpha}e^{-(\delta/4)(t-s)}\|x\| \\
&= c_5(\alpha,\beta)(t-s)^{\beta-1-\alpha}e^{-(\delta/4)(t-s)}\|x\| \\
&\leq c_5(\alpha,\beta)(t-s)^{-\beta}e^{-(\delta/4)(t-s)}\|x\|.
\end{aligned} \tag{3.15}$$

In summary, there exists $r'(\beta,\alpha) > 0$ such that

$$\|A(t)U(t,s)x\|_{\beta} \leq r'(\alpha,\beta)(t-s)^{-\beta}e^{-(\delta/4)(t-s)}\|x\|, \tag{3.16}$$

for all $t, s \in \mathbb{R}$ with $t > s$.

Let $x \in \mathbb{X}$. Since the restriction of $A(s)$ to $R(Q(s))$ is a bounded linear operator it follows that

$$\begin{aligned}
\left\|A(t)\widetilde{U}_Q(t,s)Q(s)x\right\|_{\beta} &= \left\|A(t)A(s)^{-1}A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_{\beta} \\
&\leq \left\|A(t)A(s)^{-1}\right\|_{B(\mathbb{X},\mathbb{X}_{\beta})}\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\| \\
&\leq c_1\left\|A(t)A(s)^{-1}\right\|_{B(\mathbb{X},\mathbb{X}_{\beta})}\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_{\beta} \\
&\leq c_1c_0\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_{\beta} \\
&\leq \widetilde{c}\left\|\widetilde{U}_Q(t,s)Q(s)x\right\|_{\beta} \\
&\leq \widetilde{c}m(\beta)e^{-\delta(s-t)}\|x\| \\
&= d'(\beta)e^{-\delta(s-t)}\|x\|
\end{aligned} \tag{3.17}$$

for $t \leq s$ by using (2.12). □

A straightforward consequence of Lemma 3.3 is the following.

Corollary 3.4. *For each $x \in \mathbb{X}$, suppose that assumptions (H.1), (H.2), and (H.5) hold, and let α, β be real numbers such that $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$. Then there are two constants $r(\alpha, \beta), d(\beta) > 0$ such that*

$$\|A(s)U(t,s)P(s)x\|_\beta \le r(\alpha,\beta)e^{-(\delta/4)(t-s)}(t-s)^{-\beta}\|x\|, \quad t > s, \tag{3.18}$$

$$\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta \le d(\beta)e^{-\delta(s-t)}\|x\|, \quad t \le s. \tag{3.19}$$

Proof. We make use of (H.5) and Lemma 3.3. Indeed, for each $x \in \mathbb{X}$,

$$\begin{aligned}
\|A(s)U(t,s)P(s)x\|_\beta &= \left\|A(s)A^{-1}(t)A(t)U(t,s)P(s)x\right\|_\beta \\
&\le \left\|A(s)A^{-1}(t)\right\|_{B(\mathbb{X},\mathbb{X}_\beta)}\|A(t)U(t,s)P(s)x\| \\
&\le c_0k'\|A(t)U(t,s)P(s)x\|_\beta \\
&\le c_0k'r'(\alpha,\beta)e^{-(\delta/4)(t-s)}(t-s)^{-\beta}\|x\| \\
&= r(\alpha,\beta)e^{-(\delta/4)(t-s)}(t-s)^{-\beta}\|x\|, \quad t > s.
\end{aligned} \tag{3.20}$$

Equation (3.19) has already been proved (see the proof of (3.12)). □

Lemma 3.5. *Under assumptions (H.1), (H.2), (H.3), and (H.4), the mapping $S : BC(\mathbb{R}, \mathbb{X}_\alpha) \mapsto BC(\mathbb{R}, \mathbb{X}_\alpha)$ is well defined and continuous.*

Proof. We first show that $S(\mathrm{BC}(\mathbb{R}, \mathbb{X}_\alpha)) \subset \mathrm{BC}(\mathbb{R}, \mathbb{X}_\alpha)$. For that, let S_1 and S_2 be the integral operators defined, respectively, by

$$\begin{aligned}
(S_1u)(t) &= \int_{-\infty}^{t} U(t,s)P(s)F(s,u(s))ds, \\
(S_2u)(t) &= \int_{t}^{\infty} U_Q(t,s)Q(s)F(s,u(s))ds.
\end{aligned} \tag{3.21}$$

Now, using (2.11) it follows that for all $v \in \mathrm{BC}(\mathbb{R}, \mathbb{X}_\alpha)$,

$$\begin{aligned}
\|(S_1 v)(t)\|_\infty &= \left\| \int_{-\infty}^{t} U(t,s)P(s)F(s,v(s))ds \right\|_\alpha \\
&\leq \int_{-\infty}^{t} c(\alpha)(t-s)^{-\alpha} e^{-(\delta/2)(t-s)} \|F(s,v(s))\| ds \\
&\leq \int_{-\infty}^{t} c(\alpha)(t-s)^{-\alpha} e^{-(\delta/2)(t-s)} \mathcal{M}\left(\|v\|_{\alpha,\infty}\right) ds \\
&= \mathcal{M}\left(\|v\|_{\alpha,\infty}\right) c(\alpha) \left(2\delta^{-1}\right)^{1-\alpha} \Gamma(1-\alpha),
\end{aligned} \tag{3.22}$$

and hence

$$\|S_1 u\|_{\alpha,\infty} \leq s(\alpha) \mathcal{M}\left(\|v\|_{\alpha,\infty}\right), \tag{3.23}$$

where $s(\alpha) = c(\alpha)(2\delta^{-1})^{1-\alpha}\Gamma(1-\alpha)$.

It remains to prove that S_1 is continuous. For that consider an arbitrary sequence of functions $u_n \in \mathrm{BC}(\mathbb{R}, \mathbb{X}_\alpha)$ which converges uniformly to some $u \in \mathrm{BC}(\mathbb{R}, \mathbb{X}_\alpha)$, that is, $\|u_n - u\|_{\alpha,\infty} \to 0$ as $n \to \infty$.

Now

$$\begin{aligned}
&\left\| \int_{-\infty}^{t} U(t,s)P(s)[F(s,u_n(s)) - F(s,u(s))]ds \right\|_\alpha \\
&\quad \leq c(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-(\delta/2)(t-s)} \|F(s,u_n(s)) - F(s,u(s))\| ds.
\end{aligned} \tag{3.24}$$

Now, using the continuity of F and the Lebesgue Dominated Convergence Theorem we conclude that

$$\left\| \int_{-\infty}^{t} U(t,s)P(s)[F(s,u_n(s)) - F(s,u(s))]ds \right\|_\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.25}$$

and hence $\|S_1 u_n - S_1 u\|_{\alpha,\infty} \to 0$ as $n \to \infty$.

The proof for S_2 is similar to that of S_1 and hence omitted. For S_2, one makes use of (2.12) rather than (2.11). □

Lemma 3.6. *Under assumptions (H.1), (H.2), (H.3), and (H.4), the integral operator S defined above maps $PAA(\mathbb{X}_\alpha)$ into itself.*

Proof. Let $u \in PAA(\mathbb{X}_\alpha)$. Setting $\phi(t) = F(t, u(t))$ and using Theorem 2.10 it follows that $\phi \in PAA(\mathbb{X})$. Let $\phi = u_1 + u_2 \in PAA(\mathbb{X})$, where $u_1 \in AA(\mathbb{X})$ and $u_2 \in PAP_0(\mathbb{X})$. Let us show that $S_1u_1 \in AA(\mathbb{X}_\alpha)$. Indeed, since $u_1 \in AA(\mathbb{X})$, for every sequence of real numbers $(\tau_n')_{n\in\mathbb{N}}$ there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$v_1(t) := \lim_{n\to\infty} u_1(t+\tau_n) \tag{3.26}$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} v_1(t-\tau_n) = u_1(t) \tag{3.27}$$

for each $t \in \mathbb{R}$.

Set $M(t) = \int_{-\infty}^t U(t,s)P(s)u_1(s)ds$ and $N(t) = \int_{-\infty}^t U(t,s)P(s)v_1(s)ds$ for all $t \in \mathbb{R}$. Now

$$\begin{aligned}
M(t+\tau_n) - N(t) &= \int_{-\infty}^{t+\tau_n} U(t+\tau_n,s)P(s)u_1(s)ds - \int_{-\infty}^{t} U(t,s)P(s)v_1(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau,s+\tau_n)P(s+\tau_n)u_1(s+\tau_n)ds - \int_{-\infty}^{t} U(t,s)P(s)v_1(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)P(s+\tau_n)(u_1(s+\tau_n)-v_1(s))ds \\
&\quad + \int_{-\infty}^{t} (U(t+\tau_n,s+\tau_n)P(s+\tau_n) - U(t,s)P(s))v_1(s)ds.
\end{aligned} \tag{3.28}$$

Using (2.11) and the Lebesgue Dominated Convergence Theorem, one can easily see that

$$\left\| \int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)P(s+\tau_n)(u_1(s+\tau_n)-v_1(s))ds \right\|_\alpha \longrightarrow 0 \quad \text{as } n \to \infty,\ t \in \mathbb{R}. \tag{3.29}$$

Similarly, using (H.3) and [40] it follows that

$$\left\| \int_{-\infty}^{t} (U(t+\tau_n,s+\tau_n)P(s+\tau_n) - U(t,s)P(s))v_1(s)ds \right\|_\alpha \longrightarrow 0 \quad \text{as } n \to \infty,\ t \in \mathbb{R}. \tag{3.30}$$

Therefore,

$$N(t) = \lim_{n\to\infty} M(t+\tau_n), \quad t \in \mathbb{R}. \tag{3.31}$$

Using similar ideas as the previous ones, one can easily see that

$$M(t) = \lim_{n\to\infty} N(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.32}$$

Again using (2.11) it follows that

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|(S_1u_2)(t)\|_\alpha dt &\le \lim_{r\to\infty}\frac{c(\alpha)}{2r}\int_{-r}^{r}\int_{0}^{+\infty}s^{-\alpha}e^{-(\delta/2)s}\|u_2(t-s)\|ds\,dt\\ &\le \lim_{r\to\infty}c(\alpha)\int_{0}^{+\infty}s^{-\alpha}e^{-(\delta/2)s}\frac{1}{2r}\int_{-r}^{r}\|u_2(t-s)\|dt\,ds.\end{aligned} \tag{3.33}$$

Set

$$\Gamma_s(r)=\frac{1}{2r}\int_{-r}^{r}\|u_2(t-s)\|dt. \tag{3.34}$$

Since $PAP_0(\mathbb{X})$ is translation invariant it follows that $t\mapsto u_2(t-s)$ belongs to $PAP_0(\mathbb{X})$ for each $s\in\mathbb{R}$, and hence

$$\lim_{r\mapsto\infty}\frac{1}{2r}\int_{-r}^{r}\|u_2(t-s)\|dt=0 \tag{3.35}$$

for each $s\in\mathbb{R}$.

One completes the proof by using the well-known Lebesgue dominated convergence theorem and the fact $\Gamma_s(r)\mapsto 0$ as $r\to\infty$ for each $s\in\mathbb{R}$.

The proof for S_2 is similar to that of S_1 and hence omitted. For S_2, one makes use of (2.12) rather than (2.11). □

Let $\gamma\in(0,1]$, and let $\mathrm{BC}^\gamma(\mathbb{R},\mathbb{X}_\alpha)=\{u\in\mathrm{BC}(\mathbb{R},\mathbb{X}_\alpha):\|u\|_{\alpha,\gamma}<\infty\}$, where

$$\|u\|_{\alpha,\gamma}=\sup\|u(t)\|_\alpha+\gamma\sup_{t,s\in\mathbb{R},\,t\neq s}\frac{\|u(t)-u(s)\|_\alpha}{|t-s|^\gamma}. \tag{3.36}$$

Clearly, the space $\mathrm{BC}^\gamma(\mathbb{R},\mathbb{X}_\alpha)$ equipped with the norm $\|\cdot\|_{\alpha,\gamma}$ is a Banach space, which is the Banach space of all bounded continuous Hölder functions from $\mathbb{R}$ to $\mathbb{X}_\alpha$ whose Hölder exponent is γ.

Lemma 3.7. *Under assumptions (H.1), (H.2), (H.3), (H.4), and (H.5), $V=S_1-S_2$ maps bounded sets of $BC(\mathbb{R},\mathbb{X}_\alpha)$ into bounded sets of $BC^\gamma(\mathbb{R},\mathbb{X}_\alpha)$ for some $0<\gamma<1$, where S_1,S_2 are the integral operators introduced previously.*

Proof. Let $u \in BC(\mathbb{R}, \mathbb{X}_\alpha)$, and let $g(t) = F(t, u(t))$ for each $t \in \mathbb{R}$. Then we have

$$\begin{aligned}
\|S_1 u(t)\|_\alpha &\leq k(\alpha)\|S_1 u(t)\|_\beta \\
&\leq k(\alpha)\int_{-\infty}^{t} \|U(t,s)P(s)g(s)\|_\beta ds \\
&\leq k(\alpha)c(\beta)\int_{-\infty}^{t} e^{-(\delta/2)(t-s)}(t-s)^{-\beta}\|g(s)\| ds \\
&\leq \mathcal{M}(\|u\|_{\alpha,\infty})\left[k(\alpha)c(\beta)\int_{0}^{+\infty} e^{-\sigma}\left(\frac{2\sigma}{\delta}\right)^{-\beta}\frac{2d\sigma}{\delta}\right] \\
&\leq \mathcal{M}(\|u\|_{\alpha,\infty})\left[k(\alpha)c(\beta)\left(2^{-1}\delta\right)^{1-\beta}\Gamma(1-\beta)\right],
\end{aligned} \tag{3.37}$$

and hence

$$\|S_1 u\|_{\alpha,\infty} \leq \left[k(\alpha)c(\beta)\left(2^{-1}\delta\right)^{1-\beta}\Gamma(1-\beta)\right]\mathcal{M}(\|u\|_{\alpha,\infty}). \tag{3.38}$$

Similarly,

$$\begin{aligned}
\|S_2 u(t)\|_\alpha &\leq k(\alpha)\|S_2 u(t)\|_\beta \\
&\leq k(\alpha)\int_{t}^{\infty} \|U_Q(t,s)Q(s)g(s)\|_\beta ds \\
&\leq k(\alpha)m(\beta)\int_{t}^{\infty} e^{-\delta(s-t)}\|g(s)\| ds \\
&\leq \mathcal{M}(\|u\|_{\alpha,\infty})k(\alpha)m(\beta)\delta^{-1},
\end{aligned} \tag{3.39}$$

and hence

$$\|Vu\|_{\alpha,\infty} \leq p(\alpha,\beta,\delta)\mathcal{M}(\|u\|_{\alpha,\infty}). \tag{3.40}$$

Let $t_1 < t_2$. Clearly,

$$\begin{aligned}
&\|S_1u(t_2) - S_1u(t_1)\|_\alpha \\
&\quad\le \left\| \int_{t_1}^{t_2} U(t_2,s)P(s)g(s)ds + \int_{-\infty}^{t_1} [U(t_2,s) - U(t_1,s)]P(s)g(s)ds \right\|_\alpha \\
&\quad= \left\| \int_{t_1}^{t_2} U(t_2,s)P(s)g(s)ds + \int_{-\infty}^{t_1} \left(\int_{t_1}^{t_2} \frac{\partial U(\tau,s)}{\partial \tau} d\tau \right) P(s)g(s)ds \right\|_\alpha \\
&\quad\le \left\| \int_{t_1}^{t_2} U(t_2,s)P(s)g(s)ds \right\|_\alpha + \left\| \int_{-\infty}^{t_1} \left(\int_{t_1}^{t_2} A(\tau)U(\tau,s)P(s)g(s)d\tau \right) ds \right\|_\alpha \\
&\quad= N_1 + N_2.
\end{aligned} \tag{3.41}$$

Clearly,

$$\begin{aligned}
N_1 &\le \int_{t_1}^{t_2} \|U(t_2,s)P(s)g(s)\|_\alpha ds \\
&\le c(\alpha) \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} e^{-(\delta/2)(t_2-s)} \|g(s)\| ds \\
&\le c(\alpha)\mathcal{M}(\|u\|_{\alpha,\infty}) \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} e^{-(\delta/2)(t_2-s)} ds \\
&\le c(\alpha)\mathcal{M}(\|u\|_{\alpha,\infty}) \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} ds \\
&\le (1-\alpha)^{-1} c(\alpha)\mathcal{M}(\|u\|_{\alpha,\infty})(t_2 - t_1)^{1-\alpha}.
\end{aligned} \tag{3.42}$$

Similarly,

$$\begin{aligned}
N_2 &\le k(\alpha) \int_{-\infty}^{t_1} \left(\int_{t_1}^{t_2} \|A(\tau)U(\tau,s)P(s)g(s)\|_\beta d\tau \right) ds \\
&\le k(\alpha) r(\alpha,\beta) \int_{-\infty}^{t_1} \left(\int_{t_1}^{t_2} (\tau - s)^{-\beta} e^{-(\delta/4)(\tau-s)} \|g(s)\| d\tau \right) ds \\
&\le k(\alpha) r(\alpha,\beta) \mathcal{M}(\|u\|_{\alpha,\infty}) \int_{t_1}^{t_2} \left(\int_{-\infty}^{t_1} (\tau - s)^{-\beta} e^{-(\delta/4)(\tau-s)} ds \right) d\tau \\
&\le k(\alpha) r(\alpha,\beta) \mathcal{M}(\|u\|_{\alpha,\infty}) \int_{t_1}^{t_2} (\tau - t_1)^{-\beta} \left(\int_{\tau-t_1}^{\infty} e^{-(\delta/4)r} dr \right) d\tau \\
&\le 4\delta^{-1} k(\alpha) r(\alpha,\beta) \mathcal{M}(\|u\|_{\alpha,\infty})(t_2 - t_1)^{1-\beta}.
\end{aligned} \tag{3.43}$$

Now

$$\begin{aligned}\|S_2u(t_2) - S_2u(t_1)\|_\alpha &\le m(\alpha)\int_{t_1}^{t_2} e^{-\delta(s-t_1)}\|g(s)\|ds \\ &\quad + m(\alpha)\int_{t_2}^{\infty}\left(\int_{t_1}^{t_2} e^{-\delta(s-\tau)}\|g(s)\|\tau\right)ds \\ &\le N(\alpha,\delta)(t_2 - t_1)\mathcal{M}\left(\|u\|_{\alpha,\infty}\right),\end{aligned} \tag{3.44}$$

where $N(\alpha,\delta)$ is a positive constant.

Consequently, letting $\gamma = 1-\beta$ it follows that

$$\|Vu(t_2) - Vu(t_1)\|_\alpha \le s(\alpha,\beta,\delta)\mathcal{M}\left(\|u\|_{\alpha,\infty}\right)|t_2 - t_1|^\gamma, \tag{3.45}$$

where $s(\alpha,\beta,\delta)$ is a positive constant.

Therefore, for each $u \in \mathrm{BC}(\mathbb{R},\mathbb{X}_\alpha)$ such that

$$\|u(t)\|_\alpha \le R \tag{3.46}$$

for all $t \in \mathbb{R}$, then Vu belongs to $\mathrm{BC}^\gamma(\mathbb{R},\mathbb{X}_\alpha)$ with

$$\|Vu(t)\|_\alpha \le R' \tag{3.47}$$

for all $t \in \mathbb{R}$, where R' depends on R. □

The proof of the next lemma follows along the same lines as that of Lemma 3.6 and hence omitted.

Lemma 3.8. *The integral operator $V = S_1 - S_2$ maps bounded sets of $AA(\mathbb{X}_\alpha)$ into bounded sets of $BC^{1-\beta}(\mathbb{R},\mathbb{X}_\alpha) \cap AA(\mathbb{X}_\alpha)$.*

Similarly, the next lemma is a consequence of [2, Proposition 3.3].

Lemma 3.9. *The set $BC^{1-\beta}(\mathbb{R},\mathbb{X}_\alpha)$ is compactly contained in $BC(\mathbb{R},\mathbb{X})$, that is, the canonical injection $id : BC^{1-\beta}(\mathbb{R},\mathbb{X}_\alpha) \mapsto BC(\mathbb{R},\mathbb{X})$ is compact, which yields*

$$id : BC^{1-\beta}(\mathbb{R},\mathbb{X}_\alpha)\bigcap AA(\mathbb{X}_\alpha) \longmapsto AA(\mathbb{X}_\alpha) \tag{3.48}$$

is compact, too.

Theorem 3.10. *Suppose that assumptions (H.1), (H.2), (H.3), (H.4), and (H.5) hold, then the operator V defined by $V = S_1 - S_2$ is compact.*

Proof. The proof follows along the same lines as that of [2, Proposition 3.4]. Recalling that in view of Lemma 3.7, we have

$$\|Vu\|_{\alpha,\infty} \le p(\alpha,\beta,\delta)\mathcal{M}(\|u\|_{\alpha,\infty}),$$
$$\|Vu(t_2) - V(t_1)\|_\alpha \le s(\alpha,\beta,\delta)\mathcal{M}(\|u\|_{\alpha,\infty})|t_2 - t_1|, \tag{3.49}$$

for all $u \in BC(\mathbb{R},\mathbb{X}_\alpha)$, $t_1, t_2 \in \mathbb{R}$ with $t_1 \ne t_2$, where $p(\alpha,\beta,\delta)$, $s(\alpha,\beta,\delta)$ are positive constants. Consequently, $u \in BC(\mathbb{R},\mathbb{X}_\alpha)$ and $\|u\|_{\alpha,\infty} < R$ yield $Vu \in BC^{1-\beta}(\mathbb{R},\mathbb{X}_\alpha)$ and

$$\|Vu\|_\alpha < R_1, \tag{3.50}$$

where $R_1 = c(\alpha,\beta,\delta)\mathcal{M}(R)$.

Therefore, there exists $r > 0$ such that for all $R \ge r$, the following hold:

$$V\left(B_{AA(\mathbb{X}_\alpha)}(0,R)\right) \subset B_{BC^{1-\beta}(\mathbb{R},\mathbb{X}_\alpha)}(0,R) \bigcap B_{AA(\mathbb{X}_\alpha)}(0,R). \tag{3.51}$$

In view of the above, it follows that $V : D \mapsto D$ is continuous and compact, where D is the ball in $AA(\mathbb{X}_\alpha)$ of radius R with $R \ge r$. □

Define

$$(W_1u)(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)G(s,u(s))ds,$$
$$(W_2u)(t) = \int_{t}^{s} A(s)U_Q(t,s)Q(s)G(s,u(s))ds \tag{3.52}$$

for all $t \in \mathbb{R}$.

Lemma 3.11. *Under assumptions (H.1), (H.2), (H.3), (H.4), and (H.5), the integral operators W_1 and W_2 defined above map $PAA(\mathbb{X}_\alpha)$ into itself.*

Proof. Let $u \in PAA(\mathbb{X}_\alpha)$. Again, using the composition of pseudo-almost automorphic functions (Theorem 2.10) it follows that $\varphi(\cdot) = G(\cdot,u(\cdot))$ is in $PAA(\mathbb{X}_\beta)$ whenever $u \in PAA(\mathbb{X}_\alpha)$. In particular,

$$\|\varphi\|_{\beta,\infty} = \sup_{t\in\mathbb{R}}\|G(t,u(t))\|_\beta < \infty. \tag{3.53}$$

Now write $\varphi = \phi + z$, where $\phi \in AA(\mathbb{X}_\beta)$ and $z \in PAP_0(\mathbb{X}_\beta)$, that is, $W_1\varphi = \Xi(\phi) + \Xi(z)$ where

$$\Xi\phi(t) := \int_{-\infty}^{t} A(s)U(t,s)P(s)\phi(s)ds,$$
$$\Xi z(t) := \int_{-\infty}^{t} A(s)U(t,s)P(s)z(s)ds. \tag{3.54}$$

Clearly, $\Xi(\phi) \in AA(\mathbb{X}_\alpha)$. Indeed, since $\phi \in AA(\mathbb{X}_\beta)$, for every sequence of real numbers $(\tau_n')_{n\in\mathbb{N}}$ there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\psi(t) := \lim_{n\to\infty} \phi(t+\tau_n) \tag{3.55}$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \psi(t-\tau_n) = \phi(t) \tag{3.56}$$

for each $t \in \mathbb{R}$.

Set $J(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\phi(s)ds$ and $K(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\psi(s)ds$ for all $t \in \mathbb{R}$.

Now

$$\begin{aligned} J(t+\tau_n) - K(t) &= \int_{-\infty}^{t+\tau_n} A(s)U(t+\tau_n,s)P(s)\phi(s)ds - \int_{-\infty}^{t} A(s)U(t,s)P(s)\psi(s)ds \\ &= \int_{-\infty}^{t} A(s+\tau_n)U(t+\tau,s+\tau_n)P(s+\tau_n)\phi(s+\tau_n)ds \\ &\quad - \int_{-\infty}^{t} A(s)U(t,s)P(s)\psi(s)ds \\ &= \int_{-\infty}^{t} A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n)\big(\phi(s+\tau_n)-\psi(s)\big)ds \\ &\quad + \int_{-\infty}^{t} (A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n) - A(s)U(t,s)P(s))\psi(s)ds. \end{aligned} \tag{3.57}$$

Using (3.18) and the Lebesgue Dominated Convergence Theorem, one can easily see that

$$\left\| \int_{-\infty}^{t} A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n)\big(\phi(s+\tau_n)-\psi(s)\big)ds \right\|_\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\ t \in \mathbb{R}. \tag{3.58}$$

Similarly, using (H.3) it follows that

$$\left\| \int_{-\infty}^{t} (A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n) - A(s)U(t,s)P(s))\psi(s)ds \right\|_\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\ t \in \mathbb{R}. \tag{3.59}$$

Therefore,

$$K(t) = \lim_{n\to\infty} J(t+\tau_n), \quad t \in \mathbb{R}. \tag{3.60}$$

Using similar ideas as the previous ones, one can easily see that

$$J(t) = \lim_{n\to\infty} K(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.61}$$

Now, let $r > 0$. Again from (3.18), we have

$$\begin{aligned} \frac{1}{2r}\int_{-r}^{r} \|(\Xi z)(t)\|_\alpha dt &\le \frac{k(\alpha)}{2r}\int_{-r}^{r}\int_{-\infty}^{t} \|A(s)U(t,s)P(s)z(t-s)\|_\beta \, ds\, dt \\ &\le \frac{k(\alpha)r(\alpha,\beta)}{2r}\int_{-r}^{r}\int_{-\infty}^{t} e^{-(\delta/4)(t-s)}(t-s)^{-\beta}\|z(t-s)\|ds\,dt \\ &\le l(\alpha,\beta)\cdot\int_0^{+\infty} e^{-(\delta/4)s}s^{-\beta}\left(\frac{1}{2r}\int_{-r}^{r}\|z(t-s)\|_\beta dt\right)ds. \end{aligned} \tag{3.62}$$

Now

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|z(t-s)\|_\beta dt = 0, \tag{3.63}$$

as $t \mapsto z(t-s) \in PAP_0(\mathbb{X}_\beta)$ for every $s \in \mathbb{R}$. One completes the proof by using the Lebesgue's dominated convergence theorem.

The proof for $W_2u(\cdot)$ is similar to that of $W_1u(\cdot)$ except that one makes use of (3.19) instead of (3.18). □

Theorem 3.12. *Under assumptions (H.1), (H.2), (H.3), (H.4), and (H.5) and if L is small enough, then* (1.2) *has at least one pseudo-almost automorphic solution.*

Proof. We have seen in the proof of Theorem 3.10 that $S : D \mapsto D$ is continuous and compact, where D is the ball in $PAA(\mathbb{X}_\alpha)$ of radius R with $R \ge r$.

Now, if we set $a_G := \sup_{t\in\mathbb{R}}\|G(t,0)\|_\beta$ it follows that

$$\|Tu\|_\alpha \le k(\alpha)(kLR + a_G)\left[1 + r(\alpha,\beta)\left(\frac{4}{\delta}\right)^{1-\beta}\Gamma(1-\beta) + \frac{d(\beta)}{\delta}\right] \tag{3.64}$$

for all $u \in D$.

Choose R' such that

$$k(\alpha)(kLR + a_G)\left[1 + r(\alpha,\beta)\left(\frac{4}{\delta}\right)^{1-\beta}\Gamma(1-\beta) + \frac{d(\beta)}{\delta}\right] \le R' \tag{3.65}$$

and let D' be the closed ball in $PAA(\mathbb{X}_\alpha)$ of radius R'. It is then clear that

$$\|Tu + Su\|_\alpha \le R' \tag{3.66}$$

for all $u \in D'$ and hence $(S+T)(D') \subset D'$.

To complete the proof we have to show that T is a strict contraction. Indeed, for all $u, v \in \mathbb{X}_\alpha$

$$\|Tu - Tv\|_{\alpha,\infty} \leq Lk(\alpha)\left[1 + r(\alpha,\beta)\left(\frac{4}{\delta}\right)^{1-\beta}\Gamma(1-\beta) + \frac{d(\beta)}{\delta}\right]\|u - v\|_{\alpha,\infty} \quad (3.67)$$

and hence T is a strict contraction whenever L is small enough.

Using the Krasnoselskii fixed point theorem (Theorem 3.2) it follows that there exists at least one pseudo-almost automorphic mild solution to (1.2). □

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