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Research Article

Solutions to a Three-Point Boundary Value Problem

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By using the fixed-point index theory and Leggett-Williams fixed-point theorem, we study the existence of multiple solutions to the three-point boundary value problem $u'''(t) + a(t)f(t, u(t), u'(t)) = 0$, $0 < t < 1$; $u(0) = u'(0) = 0$; $u'(1) - \alpha u'(\eta) = \lambda$, where $\eta \in (0, 1/2]$, $\alpha \in [1/2\eta, 1/\eta)$ are constants, $\lambda \in (0, \infty)$ is a parameter, and a, f are given functions. New existence theorems are obtained, which extend and complement some existing results. Examples are also given to illustrate our results.

1. Introduction

It is known that when differential equations are required to satisfy boundary conditions at more than one value of the independent variable, the resulting problem is called a multipoint boundary value problem, and a typical distinction between initial value problems and multipoint boundary value problems is that in the former case one is able to obtain the solutions depend only on the initial values, while in the latter case, the boundary conditions at the starting point do not determine a unique solution to start with, and some random choices among the solutions that satisfy these starting boundary conditions are normally not to satisfy the boundary conditions at the other specified point(s). As it is noticed elsewhere (see, e.g., Agarwal [1], Bisplinghoff and Ashley [2], and Henderson [3]), multi point boundary value problem has deep physical and engineering background as well as realistic mathematical model. For the development of the research of multi point boundary value problems for differential equations in last decade, we refer the readers to, for example, [1, 4–9] and references therein.

In this paper, we study the existence of multiple solutions to the following three-point boundary value problem for a class of third-order differential equations with inhomogeneous three-point boundary values,

$$\begin{gathered} u'''(t) + a(t)f\left(t, u(t), u'(t)\right) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u'(1) - \alpha u'\left(\eta\right) = \lambda, \end{gathered} \tag{1.1}$$

where $\eta \in (0, 1/2]$, $\alpha \in [1/2\eta, 1/\eta)$, $\lambda \in (0, \infty)$, and a, f are given functions. To the authors' knowledge, few results on third-order differential equations with inhomogeneous three-point boundary values can be found in the literature. Our purpose is to establish new existence theorems for (1.1) which extend and complement some existing results.

Let X be an Banach space, and let Υ be a cone in X. A mapping β is said to be a nonnegative continuous concave functional on Υ if $\beta : \Upsilon \to [0, +\infty)$ is continuous and

$$\beta\left(tx + (1-t)y\right) \geq t\beta(x) + (1-t)\beta\left(y\right), \quad x, y \in \Upsilon, \ t \in [0, 1]. \tag{1.2}$$

Assume that

(H)

$$\begin{gathered} a \in C((0, 1), [0, \infty)), \quad 0 < \int_0^1 (1-s)sa(s)ds < \infty, \\ f \in C([0, 1] \times [0, \infty) \times [0, \infty), [0, \infty)). \end{gathered} \tag{1.3}$$

Define

$$\begin{gathered} \max f_0 = \lim_{v \to 0+} \max_{t \in [0,1]} \sup_{u \in [0,+\infty)} \frac{f(t, u, v)}{v}, \\ \min f_0 = \lim_{v \to 0+} \min_{t \in [0,1]} \inf_{u \in [0,+\infty)} \frac{f(t, u, v)}{v}, \\ \max f_\infty = \lim_{v \to +\infty} \max_{t \in [0,1]} \sup_{u \in [0,+\infty)} \frac{f(t, u, v)}{v}, \\ \min f_\infty = \lim_{v \to +\infty} \min_{t \in [0,1]} \inf_{u \in [0,+\infty)} \frac{f(t, u, v)}{v}. \end{gathered} \tag{1.4}$$

This paper is organized in the following way. In Section 2, we present some lemmas, which will be used in Section 3. The main results and proofs are given in Section 3. Finally, in Section 4, we give some examples to illustrate our results.

2. Lemmas

Let $E = C^1[0,1]$ be a Banach Space with norm

$$\|u\|_1 = \max\{\|u\|, \|u'\|\}, \tag{2.1}$$

where

$$\|u\| = \max_{t\in[0,1]}|u(t)|, \qquad \|u'\| = \max_{t\in[0,1]}|u'(t)|. \tag{2.2}$$

It is not hard to see Lemmas 2.1 and 2.2.

Lemma 2.1. *Let $u \in C^1[0,1]$ be the unique solution of* (1.1). *Then*

$$\begin{aligned} u(t) = &\int_0^1 G(t,s)a(s)f(s,u(s),u'(s))ds \\ &+ \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda t^2}{2(1-\alpha\eta)}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} G(t,s) &= \frac{1}{2}\begin{cases} (2t - t^2 - s)s, & s \le t, \\ (1-s)t^2, & t \le s, \end{cases} \\ G_1(t,s) &= \frac{\partial G(t,s)}{\partial t} = \begin{cases} (1-t)s, & s \le t, \\ (1-s)t, & t \le s. \end{cases} \end{aligned} \tag{2.4}$$

Lemma 2.2. *One has the following.*

(i) $0 \le G_1(t,s) \le (1-s)s$, $(1/2)t^2(1-s)s \le G(t,s) \le G(1,s) = (1/2)(1-s)s$.

(ii) $G_1(t,s) \ge (1/4)G_1(s,s) = (1/4)(1-s)s$, for $t \in [1/4, 3/4]$, $s \in [0,1]$.

(iii) $G_1(1/2,s) \ge (1/2)(1-s)s$, for $s \in [0,1]$.

Lemma 2.3. *Let $u \in C^1[0,1]$ be the unique solution of* (1.1). *Then $u(t)$ is nonnegative and satisfies $\|u\|_1 = \|u'\|$.*

Proof. Let $u \in C^1[0,1]$ be the unique solution of (1.1). Then it is obvious that $u(t)$ is nonnegative. By Lemmas 2.1 and 2.2, we have the following.

(i) For $t \le \eta$,

$$u(t) = \frac{1}{2(1-\alpha\eta)}\left\{\int_0^t\left[\left(2ts-s^2\right)(1-\alpha\eta)+t^2s(\alpha-1)\right]a(s)f\left(s,u(s),u'(s)\right)ds\right.$$
$$+\int_t^\eta\left[t^2(1-\alpha\eta)+t^2s(\alpha-1)\right]a(s)f\left(s,u(s),u'(s)\right)ds$$
$$\left.+\int_\eta^1 t^2(1-s)a(s)f\left(s,u(s),u'(s)\right)ds+\lambda t^2\right\},$$
$$u'(t) = \frac{1}{2(1-\alpha\eta)}\left\{\int_0^t\left[2s(1-\alpha\eta)+2ts(\alpha-1)\right]a(s)f\left(s,u(s),u'(s)\right)ds\right.$$
$$+\int_t^\eta\left[2t(1-\alpha\eta)+2ts(\alpha-1)\right]a(s)f\left(s,u(s),u'(s)\right)ds$$
$$\left.+\int_\eta^1 2t(1-s)a(s)f\left(s,u(s),u'(s)\right)ds+2\lambda t\right\}, \tag{2.5}$$

that is, $u(t) \le u'(t)$.

(ii) For $t \ge \eta$,

$$u(t) = \frac{1}{2(1-\alpha\eta)}\left\{\int_0^\eta\left[\left(2ts-s^2\right)(1-\alpha\eta)+t^2s(\alpha-1)\right]a(s)f\left(s,u(s),u'(s)\right)ds\right.$$
$$+\int_\eta^t\left[\left(2ts-s^2\right)(1-\alpha\eta)+t^2(\alpha\eta-s)\right]a(s)f\left(s,u(s),u'(s)\right)ds$$
$$\left.+\int_t^1 t^2(1-s)a(s)f\left(s,u(s),u'(s)\right)ds+\lambda t^2\right\},$$
$$u'(t) = \frac{1}{2(1-\alpha\eta)}\left\{\int_0^\eta\left[2s(1-\alpha\eta)+2ts(\alpha-1)\right]a(s)f\left(s,u(s),u'(s)\right)ds\right.$$
$$+\int_\eta^t\left[2s(1-\alpha\eta)+2t(\alpha\eta-s)\right]a(s)f\left(s,u(s),u'(s)\right)ds$$
$$\left.+\int_t^1 2t(1-s)a(s)f\left(s,u(s),u'(s)\right)ds+2\lambda t\right\}. \tag{2.6}$$

On the other hand, for $\eta \le s \le t$, we have

$$\begin{aligned} &2s(1-\alpha\eta) + 2t(\alpha\eta - s) - \left[\left(2ts - s^2\right)(1-\alpha\eta) + t^2(\alpha\eta - s)\right] \\ &\quad = \alpha\eta(t-s)(2+s-t) + s(1-t)(2-t) + s(s-t) \\ &\quad = \alpha\eta(t-s)\left(2+s-t-\frac{s}{\alpha\eta}\right) + s(1-t)(2-t). \end{aligned} \tag{2.7}$$

Since $\alpha \in [1/2\eta, 1/\eta)$,

$$2s(1-\alpha\eta) + 2t(\alpha\eta - s) \ge \left(2ts - s^2\right)(1-\alpha\eta) + t^2(\alpha\eta - s). \tag{2.8}$$

So, $u(t) \le u'(t)$. Therefore, $u(t) \le u'(t)$, which means

$$\|u\| \le \|u'\|, \qquad \|u\|_1 = \|u'\|. \tag{2.9}$$

The proof is completed. □

Lemma 2.4. *Let* $u \in C^1[0,1]$ *be the unique solution of* (1.1)*. Then*

$$\min_{t\in[1/4,3/4]} u'(t) \ge \frac{1}{4}\|u\|_1. \tag{2.10}$$

Proof. From (2.3), it follows that

$$\begin{aligned} u'(t) &= \int_0^1 G_1(t,s)a(s)f(s,u(s),u'(s))ds \\ &\quad + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &\le \int_0^1 (1-s)sa(s)f(s,u(s),u'(s))ds \\ &\quad + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda}{1-\alpha\eta}. \end{aligned} \tag{2.11}$$

Hence,

$$\begin{aligned} \|u\|_1 = \|u'\| &\le \int_0^1 (1-s)sa(s)f(s,u(s),u'(s))ds \\ &\quad + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda}{1-\alpha\eta}. \end{aligned} \tag{2.12}$$

By Lemmas 2.2 and 2.3, we get, for any $t \in [1/4, 3/4]$,

$$\begin{aligned}
\min_{t\in[1/4,3/4]} u'(t) &= \min_{t\in[1/4,3/4]} \left(\int_0^1 G_1(t,s)a(s)f(s,u(s),u'(s))ds \right. \\
&\qquad \left. + \frac{\alpha t}{1-\alpha\eta} \int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda t}{1-\alpha\eta} \right) \\
&\geq \frac{1}{4} \left(\int_0^1 (1-s)sa(s)f(s,u(s),u'(s))ds \right. \\
&\qquad \left. + \frac{\alpha}{1-\alpha\eta} \int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda}{1-\alpha\eta} \right).
\end{aligned} \tag{2.13}$$

Thus,

$$\min_{t\in[1/4,3/4]} u'(t) \geq \frac{1}{4}\|u\|_1. \tag{2.14}$$

Define a cone by

$$K = \left\{ u \in E : u \geq 0, \min_{t\in[1/4,3/4]} u'(t) \geq \frac{1}{4}\|u\|_1 \right\}. \tag{2.15}$$

Set

$$\begin{aligned}
K_r &= \{u \in K : \|u\|_1 < r\}, \quad \partial K_r = \{u \in K : \|u\|_1 = r\}, \quad r > 0, \\
\overline{K_r} &= \{u \in K : \|u\|_1 \leq r\}, \quad K(\beta, r, s) = \{u \in K : r \leq \beta(u),\ \|u\|_1 \leq s\}, \quad s > r > 0.
\end{aligned} \tag{2.16}$$

Define an operator T by

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),u'(s))ds \\
&\quad + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda t^2}{2(1-\alpha\eta)}.
\end{aligned} \tag{2.17}$$

Lemma 2.1 implies that (1.1) has a solution $u = u(t)$ if and only if u is a fixed point of T. □

From Lemmas 2.1 and 2.2 and the Ascoli-Arzela theorem, the following follow.

Lemma 2.5. *The operator defined in* (2.17) *is completely continuous and satisfies* $T(K) \subset K$.

Theorem 2.6 (see [10]). *Let E be a real Banach Space, let $K \subset E$ be a cone, and $\Omega_r = \{u \in K : \|u\| \leq r\}$. Let operator $T : K \cap \Omega_r \to K$ be completely continuous and satisfy $Tx \neq x$, for all $x \in \partial\Omega_r$. Then*

(i) *if $\|Tx\| \leq \|x\|$, for all $x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 1$,*

(ii) *if $\|Tx\| \geq \|x\|$, for all $x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 0$.*

Theorem 2.7 (see [8]). *Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and β a nonnegative continuous concave functional on P such that $\beta(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose that there exist $0 < d_0 < a_0 < b_0 \leq c$ such that*

(a) *$\{x \in P(\beta, a_0, b_0) : \beta(x) > a_0\} \neq \emptyset$ and $\beta(Tx) > a_0$ for $x \in P(\beta, a_0, b_0)$,*

(b) *$\|Tx\| < d_0$ for $\|x\| \leq d_0$,*

(c) *$\beta(Tx) > a_0$ for $x \in P(\beta, a_0, c)$ with $\|Tx\| > b_0$.*

Then, T has at least three fixed points $x_1, x_2,$ and x_3 in $\overline{P_c}$ satisfying

$$\|x_1\| < d_0, \quad a_0 < \beta(x_2), \quad \|x_3\| > d_0, \quad \beta(x_3) < a_0. \tag{2.18}$$

3. Main Results

In this section, we give new existence theorem about two positive solutions or three positive solutions for (1.1).

Write

$$\Lambda_1 = \left(\int_0^1 (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta, s)a(s)ds\right)^{-1},$$
$$\Lambda_2 = \left(\int_{1/4}^{3/4} (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_{1/4}^{3/4} G_1(\eta, s)a(s)ds\right)^{-1}. \tag{3.1}$$

Theorem 3.1. *Assume that*

(H_1) $\min f_0 = \min f_\infty = +\infty$;

(H_2) *there exists a constant $\rho_1 > 0$ such that $f(t, u, v) \leq (1/2)\Lambda_1\rho_1$, for $t \in [0,1]$, $u \in [0, \rho_1]$ and $v \in [0, \rho_1]$.*

Then, the problem (1.1) *has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\|_1 < \rho_1 < \|u_2\|_1, \tag{3.2}$$

for λ small enough.

Proof. Since

$$\min f_0 = \lim_{v \to 0+} \min_{t \in [0,1]} \inf_{u \in [0,+\infty)} \frac{f(t, u, v)}{v} = +\infty, \tag{3.3}$$

there is $\rho_0 \in (0, \rho_1)$ such that

$$f(t, u, v) \geq 8\Lambda_2 v, \quad \text{for } t \in [0,1],\ u \in [0, +\infty),\ v \in \left[0, \rho_0\right],\ \Lambda_2 > 0. \tag{3.4}$$

Let

$$\Omega_{\rho_0} = \left\{u \in K : \|u\|_1 < \rho_0\right\}. \tag{3.5}$$

Then, for any $u \in \partial\Omega_{\rho_0}$, it follows from Lemmas 2.2 and 2.3 and (3.4) that

$$\begin{aligned}
Tu'\left(\frac{1}{2}\right) &= \int_0^1 G_1\left(\frac{1}{2}, s\right) a(s) f(s, u(s), u'(s)) ds \\
&\quad + \frac{\alpha}{2(1-\alpha\eta)} \int_0^1 G_1(\eta, s) a(s) f(s, u(s), u'(s)) ds + \frac{\lambda}{2(1-\alpha\eta)} \\
&\geq \int_0^1 G_1\left(\frac{1}{2}, s\right) a(s) f(s, u(s), u'(s)) ds \\
&\quad + \frac{\alpha}{2(1-\alpha\eta)} \int_0^1 G_1(\eta, s) a(s) f(s, u(s), u'(s)) ds \\
&\geq \frac{1}{2} \int_{1/4}^{3/4} (1-s) s a(s) 8\Lambda_2 u'(s) ds + \frac{\alpha}{2(1-\alpha\eta)} \int_{1/4}^{3/4} G_1(\eta, s) a(s) 8\Lambda_2 u'(s) ds \\
&\geq 4\Lambda_2 \left(\int_{1/4}^{3/4} (1-s) s a(s) ds + \frac{\alpha}{2(1-\alpha\eta)} \int_{1/4}^{3/4} G_1(\eta, s) a(s) ds \right) \frac{1}{4} \|u_1\|_1 \\
&= \|u\|_1.
\end{aligned} \tag{3.6}$$

Hence,

$$\left\|Tu'\right\| \geq \|u\|_1. \tag{3.7}$$

So

$$\|Tu\|_1 \geq \|u\|_1, \quad \forall u \in \partial\Omega_{\rho_0}. \tag{3.8}$$

By Theorem 2.6, we have

$$i\left(T, \Omega_{\rho_0}, K\right) = 0. \tag{3.9}$$

On the other hand, since

$$\min f_\infty = \lim_{v \to +\infty} \min_{t \in [0,1]} \inf_{u \in [0, +\infty)} \frac{f(t, u, v)}{v} = +\infty, \tag{3.10}$$

there exist ρ_0^*, $\rho_0^* > \rho_1$ such that

$$f(t,u,v) \geq 8\Lambda_2 v, \quad \text{for } t \in [0,1],\ u \in [0,+\infty),\ v \geq \frac{1}{4}\rho_0^*. \tag{3.11}$$

Let $\Omega_{\rho_0^*} = \{u \in K : \|u\|_1 < \rho_0^*\}$. Then, by a argument similar to that above, we obtain

$$\|Tu\|_1 \geq \|u\|_1, \quad \forall u \in \partial\Omega_{\rho_0^*}. \tag{3.12}$$

By Theorem 2.6,

$$i\left(T, \Omega_{\rho_0^*}, K\right) = 0. \tag{3.13}$$

Finally, let $\Omega_{\rho_1} = \{u \in K : \|u\|_1 < \rho_1\}$, and let λ satisfy $0 < \lambda \leq (1/2)(1-\alpha\eta)\rho_1$ for any $u \in \partial\Omega_{\rho_1}$. Then, (H_2) implies

$$\begin{aligned}
Tu'(t) &= \int_0^1 G_1(t,s)a(s)f(s,u(s),u'(s))ds \\
&\quad + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda t}{1-\alpha\eta} \\
&\leq \int_0^1 (1-s)sa(s)\frac{1}{2}\Lambda_1\rho_1 ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)\frac{1}{2}\Lambda_1\rho_1 ds + \frac{\lambda}{1-\alpha\eta} \\
&\leq \frac{1}{2}\Lambda_1\left(\int_0^1 (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)ds\right)\rho_1 + \frac{(1/2)(1-\alpha\eta)\rho_1}{1-\alpha\eta} \\
&= \frac{1}{2}\rho_1 + \frac{1}{2}\rho_1 \\
&= \|u\|_1,
\end{aligned} \tag{3.14}$$

which means that $\|Tu'\| \leq \|u\|_1$. Thus, $\|Tu\|_1 \leq \|u\|_1$, for all $u \in \partial\Omega_{\rho_1}$.

Using Theorem 2.6, we get

$$i\left(T, \Omega_{\rho_1}, K\right) = 1. \tag{3.15}$$

From (3.9)–(3.15) and $\rho_0 < \rho_1 < \rho_0^*$, it follows that

$$i\left(T, \Omega_{\rho_0^*} \setminus \overline{\Omega}_{\rho_1}, K\right) = -1, \qquad i\left(T, \Omega_{\rho_1} \setminus \overline{\Omega}_{\rho_0}, K\right) = 1. \tag{3.16}$$

Therefore, T has fixed point $u_1 \in \Omega_{\rho_1} \setminus \overline{\Omega}_{\rho_0}$ and fixed point $u_2 \in \Omega_{\rho_0^*} \setminus \overline{\Omega}_{\rho_1}$. Clearly, u_1, u_2 are both positive solutions of the problem (1.1) and

$$0 < \|u_1\|_1 < \rho_1 < \|u_2\|_1. \tag{3.17}$$

The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Assume that*

(H_3) $\max f_0 = \max f_\infty = 0$;

(H_4) *there exists a constant* $\rho_2 > 0$ *such that* $f(t,u,v) \geq 2\Lambda_2\rho_2$, *for* $t \in [0,1], u \in [0,\rho_2]$ *and* $v \in [(1/4)\rho_2, \rho_2]$.

Then, the problem (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\|_1 < \rho_2 < \|u_2\|_1 \tag{3.18}$$

for λ *small enough.*

Proof. By

$$\max f_0 = \lim_{v\to 0+} \max_{t\in[0,1]} \sup_{u\in[0,+\infty)} \frac{f(t,u,v)}{v} = 0, \tag{3.19}$$

we see that there exists $\rho_* \in (0, \rho_2)$ such that

$$f(t,u,v) \leq \frac{1}{2}\Lambda_1 v, \quad \text{for } t \in [0,1],\ u \in [0,+\infty),\ v \in [0,\rho_*]. \tag{3.20}$$

Put

$$\Omega_{\rho_*} = \{u \in K : \|u\|_1 < \rho_*\}, \tag{3.21}$$

and let λ satisfy

$$0 < \lambda \leq \frac{1}{2}(1-\alpha\eta)\rho_*. \tag{3.22}$$

Then Lemmas 2.2 and 2.3 and (3.20) implies that for any $u \in \partial\Omega_{\rho_*}$,

$$\begin{aligned} Tu'(t) &= \int_0^1 G_1(t,s)a(s)f(s,u(s),u'(s))ds \\ &\quad + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda t}{1-\alpha\eta} \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 (1-s)sa(s)\frac{1}{2}\Lambda_1 u'(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)\frac{1}{2}\Lambda_1 u'(s)ds + \frac{\lambda}{1-\alpha\eta} \\
&\leq \frac{1}{2}\Lambda_1\left(\int_0^1 (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)ds\right)\|u\|_1 + \frac{(1-\alpha\eta)\rho_*}{2(1-\alpha\eta)} \\
&\leq \frac{1}{2}\|u\|_1 + \frac{1}{2}\rho_* \\
&= \|u\|_1.
\end{aligned} \tag{3.23}$$

So $\|Tu'\| \leq \|u\|_1$. Hence, $\|Tu\|_1 \leq \|u\|_1$, for all $u \in \partial\Omega_{\rho_*}$.

Applying Theorem 2.6, we have

$$i(T, \Omega_{\rho_*}, K) = 1. \tag{3.24}$$

Next, by

$$\max f_\infty = \lim_{v\to+\infty}\max_{t\in[0,1]}\sup_{u\in[0,+\infty)}\frac{f(t,u,v)}{v} = 0, \tag{3.25}$$

we know that there exists $r_0 > \rho_2$ such that

$$f(t,u,v) \leq \frac{1}{2}\Lambda_1 v, \quad \text{for } t \in [0,1],\ u \in [0,+\infty),\ v \geq r_0. \tag{3.26}$$

Case 1. $\max_{t\in[0,1]} f(t,u,v)$ is unbounded.

Define a function $f^* : [0,+\infty) \to [0,+\infty)$ by

$$f^*(\rho) = \max\{f(t,u,v) : t \in [0,1],\ u,v \in [0,\rho]\}. \tag{3.27}$$

Clearly, f^* is nondecreasing and $\lim_{\rho\to+\infty} f^*(\rho)/\rho = 0$, and

$$f^*(\rho) \leq \frac{1}{2}\Lambda_1\rho, \quad \text{for } \rho > r_0. \tag{3.28}$$

Taking $\rho^* \geq \max\{2r_0, 2\lambda/(1-\alpha\eta), 2\rho_2\}$, it follows from (3.26)–(3.28) that

$$f(t,u,v) \leq f^*(\rho^*) \leq \frac{1}{2}\Lambda_1\rho^*, \quad \text{for } t \in [0,1],\ u,v \in [0,\rho^*]. \tag{3.29}$$

By Lemmas 2.2 and 2.3 and (3.28), we have

$$
\begin{aligned}
Tu'(t) &= \int_0^1 G_1(t,s)a(s)f\big(s,u(s),u'(s)\big)ds \\
&\quad + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f\big(s,u(s),u'(s)\big)ds + \frac{\lambda t}{1-\alpha\eta} \\
&\le \int_0^1 (1-s)sa(s)\frac{1}{2}\Lambda_1\rho^* ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)\frac{1}{2}\Lambda_1\rho^* ds + \frac{\lambda}{1-\alpha\eta} \\
&\le \frac{1}{2}\Lambda_1\left(\int_0^1 (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)ds\right)\rho^* + \frac{\rho^*}{2} \\
&= \rho^*.
\end{aligned}
\tag{3.30}
$$

So $\|Tu'\| \le \rho^*$, and then $\|Tu\|_1 \le \rho^*$.

Case 2. $\max_{t\in[0,1]} f(t,u,v)$ is bounded.

In this case, there exists an $M > 0$ such that

$$
\max_{t\in[0,1]} f(t,u,v) \le M, \quad \text{for } t \in [0,1],\ u,\ v \in [0,+\infty).
\tag{3.31}
$$

Choosing $\rho^* \ge \max\{2\rho_2, 2M/\Lambda_1, 2\lambda/(1-\alpha\eta)\}$, we see by Lemmas 2.2 and 2.3 and (3.31) that

$$
\begin{aligned}
Tu'(t) &= \int_0^1 G_1(t,s)a(s)f\big(s,u(s),u'(s)\big)ds \\
&\quad + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f\big(s,u(s),u'(s)\big)ds + \frac{\lambda t}{1-\alpha\eta} \\
&\le M\left(\int_0^1 (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)ds\right) + \frac{\rho^*}{2} \\
&\le \frac{\rho^*}{2} + \frac{\rho^*}{2} \\
&= \rho^*,
\end{aligned}
\tag{3.32}
$$

which implies $\|Tu'\| \le \rho^*$, and then $\|Tu\|_1 \le \rho^*$.

Therefore, in both cases, taking

$$
\Omega_{\rho^*} = \{u \in K : \|u\|_1 < \rho^*\},
\tag{3.33}
$$

we get

$$\|Tu\|_1 \leq \|u\|_1, \quad \forall u \in \partial\Omega_{\rho^*}. \tag{3.34}$$

By Theorem 2.6, we have

$$i(T, \Omega_{\rho^*}, K) = 1. \tag{3.35}$$

Finally, put $\Omega_{\rho_2} = \{u \in K : \|u\|_1 < \rho_2\}$. Then (H_4) implies that

$$\begin{aligned}
Tu'\left(\frac{1}{2}\right) &= \int_0^1 G_1\left(\frac{1}{2}, s\right)a(s)f(s, u(s), u'(s))ds \\
&\quad + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 G_1(\eta, s)a(s)f(s, u(s), u'(s))ds + \frac{\lambda}{2(1-\alpha\eta)} \\
&\geq \int_{1/4}^{3/4} \frac{1}{2}(1-s)sa(s)2\Lambda_2\rho_2 ds + \frac{\alpha}{2(1-\alpha\eta)}\int_{1/4}^{3/4} G_1(\eta, s)a(s)2\Lambda_2\rho_2 ds \\
&\geq \Lambda_2\rho_2\left(\int_{1/4}^{3/4}(1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_{1/4}^{3/4} G_1(\eta, s)a(s)ds\right) \\
&= \rho_2,
\end{aligned} \tag{3.36}$$

that is, $\|Tu'\| \geq \rho_2$, and then $\|Tu\|_1 \geq \|u\|_1$, for all $u \in \partial\Omega_{\rho_2}$. By virtue of Theorem 2.6, we have

$$i(T, \Omega_{\rho_2}, K) = 0. \tag{3.37}$$

From (3.24), (3.35), (3.37), and $\rho_* < \rho_2 < \rho^*$, it follows that

$$i\left(T, \Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_2}, K\right) = 1, \qquad i\left(T, \Omega_{\rho_2} \setminus \overline{\Omega}_{\rho_*}, K\right) = -1. \tag{3.38}$$

Hence, T has fixed point $u_1 \in \Omega_{\rho_2} \setminus \overline{\Omega}_{\rho_*}$ and fixed point $u_2 \in \Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_2}$. Obviously, u_1, u_2 are both positive solutions of the problem (1.1) and

$$0 < \|u_1\|_1 < \rho_2 < \|u_2\|_1. \tag{3.39}$$

The proof of Theorem 3.2 is completed. □

Theorem 3.3. *Let there exist d_0, a_0, b_0, and c with*

$$0 < d_0 < a_0 < 32a_0 < b_0 \leq c, \tag{3.40}$$

such that

$$f(t,u,v) \leq \frac{1}{4}\Lambda_1 d_0, \quad t \in [0,1],\ u \in [0,d_0],\ v \in [0,d_0], \tag{3.41}$$

$$f(t,u,v) \geq 35\Lambda_2 a_0, \quad t \in [0,1],\ u \in [a_0,b_0],\ v \in [a_0,b_0], \tag{3.42}$$

$$f(t,u,v) \leq \frac{1}{2}\Lambda_1 c, \quad t \in [0,1],\ u \in [0,c],\ v \in [0,c]. \tag{3.43}$$

Then problem (1.1) *has at least three positive solutions* u_1, u_2, u_3 *satisfying*

$$\|u_1\|_1 < d_0, \quad a_0 < \beta(u_2), \quad \|u_3\|_1 > d_0, \quad \beta(u_3) < a_0, \tag{3.44}$$

for $\lambda \leq (1/2)(1-\alpha\eta)d_0$.

Proof. Let

$$\beta(u) = \min_{t\in[1/4,3/4]} |u(t)|, \quad u \in K. \tag{3.45}$$

Then, β is a nonnegative continuous concave functional on K and $\beta(u) \leq \|u\|_1$ for each $u \in K$. Let u be in $\overline{K_c}$. Equation (3.43) implies that

$$\begin{aligned} Tu'(t) &= \int_0^1 G_1(t,s)a(s)f(s,u(s),u'(s))ds \\ &\quad + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &\leq \frac{1}{2}\Lambda_1\left(\int_0^1 (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)ds\right)c + \frac{(1/2)(1-\alpha\eta)c}{(1-\alpha\eta)} \\ &= c. \end{aligned} \tag{3.46}$$

Hence, $\|Tu\|_1 \leq c$. This means that $T : \overline{K_c} \to \overline{K_c}$.

Take

$$u_0(t) = \frac{1}{2}(a_0+b_0), \quad t \in [0,1]. \tag{3.47}$$

Then,

$$u_0 \in \{u \in K(\beta,a_0,b_0) : \beta(u) > a_0\} \neq \emptyset. \tag{3.48}$$

By (3.42), we have, for any $u \in K(\beta, a_0, b_0)$,

$$
\begin{aligned}
\beta(Tu) &= \min_{t\in[1/4,3/4]}\left[\int_0^1 G(t,s)a(s)f(s,u(s),u'(s))ds\right.\\
&\qquad\left.+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(\mathrm{s}))ds+\frac{\lambda t^2}{2(1-\alpha\eta)}\right]\\
&\ge \min_{t\in[1/4,3/4]}\left[\int_0^1 \frac{1}{2}t^2 s(1-s)a(s)f(s,u(s),u'(s))ds\right.\\
&\qquad\left.+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds+\frac{\lambda t^2}{2(1-\alpha\eta)}\right]\\
&\ge \int_{1/4}^{3/4}\frac{1}{2}\left(\frac{1}{4}\right)^2 s(1-s)a(s)35\Lambda_2 a_0 ds+\frac{\alpha}{2(1-\alpha\eta)}\left(\frac{1}{4}\right)^2\int_{1/4}^{3/4} G_1(\eta,s)a(s)35\Lambda_2 a_0 ds\\
&= \frac{35}{32}a_0\\
&> a_0.
\end{aligned}
\tag{3.49}
$$

Therefore, (a) in Theorem 2.7 holds.

By (3.41), we see that for any $\|u\|_1 \le d_0$

$$
\begin{aligned}
Tu'(t) &= \int_0^1 G_1(t,s)a(s)f(s,u(s),u'(s))ds\\
&\quad+\frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds+\frac{\lambda t}{1-\alpha\eta}\\
&\le \int_0^1 (1-s)sa(s)\frac{1}{4}\Lambda_1 d_0 ds+\frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)\frac{1}{4}\Lambda_1 d_0 ds+\frac{(1/2)(1-\alpha\eta)d_0}{(1-\alpha\eta)}\\
&= \frac{3}{4}d_0.
\end{aligned}
\tag{3.50}
$$

So, $\|Tu\|_1 \le (3/4)d_0 < d_0$. This means that (b) of Theorem 2.7 holds.

Moreover, for any $u \in K(\beta, a_0, c)$ with $\|Tu\|_1 > b_0$, we have

$$
\begin{aligned}
\beta(Tu) &= \min_{t\in[1/4,3/4]}\left[\int_0^1 G(t,s)a(s)f(s,u(s),u'(s))ds\right.\\
&\qquad\left.+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)a(s)f(s,u(s),u'(s))ds+\frac{\lambda t^2}{2(1-\alpha\eta)}\right]
\end{aligned}
$$

$$\begin{aligned}
&\geq \min_{t\in[1/4,3/4]}\left[\int_0^1 \frac{1}{2}t^2(1-s)sa(s)f\left(s,u(s),u'(s)\right)ds\right.\\
&\qquad\left.+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)a(s)f\left(s,u(s),u'(s)\right)ds+\frac{\lambda t^2}{2(1-\alpha\eta)}\right]\\
&\geq \frac{1}{2}\times\frac{1}{16}\left[\int_0^1 (1-s)sa(s)f\left(s,u(s),u'(s)\right)ds\right.\\
&\qquad\left.+\frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f\left(s,u(s),u'(s)\right)ds+\frac{\lambda}{1-\alpha\eta}\right]\\
&\geq \frac{1}{32}\left[\int_0^1 G_1(t,s)a(s)f\left(s,u(s),u'(s)\right)ds\right.\\
&\qquad\left.+\frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f\left(s,u(s),u'(s)\right)ds+\frac{\lambda}{1-\alpha\eta}\right]\\
&\geq \frac{1}{32}\left[\int_0^1 G_1(t,s)a(s)f\left(s,u(s),u'(s)\right)ds\right.\\
&\qquad\left.+\frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)a(s)f\left(s,u(s),u'(s)\right)ds+\frac{\lambda t}{1-\alpha\eta}\right]\\
&= \frac{1}{32}Tu'(t),
\end{aligned} \tag{3.51}$$

which implies

$$\beta(Tu)\geq\frac{1}{32}\|Tu\|_1>\frac{1}{32}b_0>a_0. \tag{3.52}$$

So, (c) in Theorem 2.7 holds. Thus, by Theorem 2.7, we know that the operator T has at least three positive fixed points $u_1,u_2,u_3\in\overline{K_c}$ satisfying

$$\|u_1\|_1<d_0,\quad a_0<\beta(u_2),\quad \|u_3\|_1>d_0,\quad \beta(u_3)<a_0. \tag{3.53}$$

□

4. Examples

In this section, we give three examples to illustrate our results.

Example 4.1. Consider the problem

$$\begin{gathered}
u'''(t)+\frac{1}{10}2^{t-1}\left(1+2^{-u(t)}\right)\left[\left(u'(t)\right)^{1/2}+\left(u'(t)\right)^2\right]=0,\quad 0<t<1,\\
u(0)=u'(0)=0,\quad u'(1)-u'\left(\frac{1}{2}\right)=\lambda,
\end{gathered} \tag{4.1}$$

where $\eta = 1/2$, $\alpha = 1$. Set

$$a(t) = \frac{1}{10}, \quad f\left(t, u(t), u'(t)\right) = 2^{t-1}\left(1 + 2^{-u(t)}\right)\left[\left(u'(t)\right)^{1/2} + \left(u'(t)\right)^{2}\right]. \tag{4.2}$$

Then,

$$\min f_0 = \min f_\infty = +\infty. \tag{4.3}$$

So, the condition (H_1) is satisfied. Observe

$$\begin{aligned}
\Lambda_1 &= \left(\int_0^1 (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G_1(\eta, s)a(s)ds\right)^{-1} \\
&= \left[\int_0^1 (1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\left(\int_0^\eta (1-\eta)sa(s)ds + \int_\eta^1 \eta(1-s)a(s)ds\right)\right]^{-1} \\
&= \left[\int_0^1 (1-s)s\frac{1}{10}ds + \frac{1}{1-(1/2)}\left(\int_0^{1/2}\left(1-\frac{1}{2}\right)s\frac{1}{10}ds + \int_{1/2}^1 \frac{1}{2}(1-s)\frac{1}{10}ds\right)\right]^{-1} \\
&= 24.
\end{aligned} \tag{4.4}$$

Taking

$$\rho_1 = 4, \quad \text{for } t \in [0,1],\ u \in \left[0, \rho_1\right],\ v \in \left[0, \rho_1\right], \tag{4.5}$$

we have

$$f(t, u, v) \le (1+1)(2+16) = 36 = 9\rho_1 < \frac{1}{2}\Lambda_1\rho_1 = 12\rho_1. \tag{4.6}$$

Thus, condition (H_2) is satisfied.

Therefore, by Theorem 3.1, the problem (4.1) has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\|_1 < 4 < \|u_2\|_1, \tag{4.7}$$

for

$$0 < \lambda \le \frac{1}{2}(1-\alpha\eta)\rho_1 = \frac{1}{4}\rho_1. \tag{4.8}$$

Example 4.2. Consider the problem

$$u'''(t) + 2 \times 5^8 (1+t)(2+\sin u(t))\left(u'(t)\right)^2 5^{-u'(t)} = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \quad u'(1) - u'\left(\frac{1}{2}\right) = \lambda, \tag{4.9}$$

where $\eta = 1/2$, $\alpha = 1$. Set

$$a(t) = 2, \quad f(t,u,v) = 5^8(1+t)(2+\sin u(t))\left(u'(t)\right)^2 5^{-u'(t)}. \tag{4.10}$$

Then,

$$\max f_0 = \max f_\infty = 0, \tag{4.11}$$

that is, the condition (H_3) is satisfied. Moreover,

$$\begin{aligned}\Lambda_2 &= \left(\int_{1/4}^{3/4}(1-s)sa(s)ds + \frac{\alpha}{1-\alpha\eta}\int_{1/4}^{3/4} G_1(\eta,s)a(s)ds\right)^{-1} \\ &= \left[\int_{1/4}^{3/4}(1-s)s2ds + \frac{1}{1-(1/2)}\left(\int_{1/4}^{1/2}\left(1-\frac{1}{2}\right)s2ds + \int_{1/2}^{3/4}\frac{1}{2}(1-s)2ds\right)\right]^{-1} \\ &= \frac{48}{29}.\end{aligned} \tag{4.12}$$

Taking

$$\rho_2 = 8, \quad \text{for } t \in [0,1],\ u \in [0,\rho_2],\ v \in \left[\frac{1}{4}\rho_2, \rho_2\right], \tag{4.13}$$

we get

$$f(t,u,v) \geq 5^8 8^2 5^{-8} = 8^2 = 8\rho_2 > 2\Lambda_2\rho_2 = \frac{96}{29}\rho_2. \tag{4.14}$$

Thus, condition (H_4) is satisfied.

Consequently, by Theorem 3.2, we see that for

$$0 < \lambda \leq \frac{1}{2}(1-\alpha\eta)\rho_* \leq \frac{1}{4}\rho_2, \tag{4.15}$$

the problem (4.9) has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\|_1 < 8 < \|u_2\|_1. \tag{4.16}$$

Example 4.3. For the problem (1.1), take $\eta = 1/2$, $\alpha = 1$, and $a(t) = 2$. Then,

$$\Lambda_1 = \frac{6}{5}, \qquad \Lambda_2 = \frac{48}{29}. \tag{4.17}$$

Let

$$d_0 = 1, \quad a_0 = 2, \quad b_0 = 99, \quad c = \frac{1425}{6},$$
$$f(t,u,v) = \begin{cases} \dfrac{1-t}{10} + \dfrac{t^2 u}{10} + \dfrac{tv}{10}, & t \in [0,1],\ u \in [0,1],\ v \in [0,1]. \\ \dfrac{1-t}{10} + \dfrac{t^2}{10} + \dfrac{t}{10} \\ \quad +70(u-1) + 70(v-1), & t \in [0,1],\ u \in [1,2],\ v \in [1,2], \\ \dfrac{1-t}{10} + \dfrac{t^2}{10} + \dfrac{t}{10} + 140, & t \in [0,1],\ u \in [2,99],\ v \in [2,99], \\ \dfrac{1-t}{10} + \dfrac{t^2}{10} + \dfrac{t}{10} + 140 \\ \quad +2\left|\sin \dfrac{2v(u-99)(v-99)}{1+t^2u^2}\right|, & t \in [0,1],\ u \in [99,+\infty),\ v \in [99,+\infty). \end{cases} \tag{4.18}$$

Then,

$$\begin{aligned} f(t,u,v) &= \frac{1-t}{10} + \frac{t^2 u}{10} + \frac{tv}{10} \le \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10} \\ &= \frac{1}{4}\Lambda_1 d_0, \quad t \in [0,1],\ u \in [0,1],\ v \in [0,1], \\ f(t,u,v) &= \frac{1-t}{10} + \frac{t^2}{10} + \frac{t}{10} + 140 \ge 140 > 35\Lambda_2 a_0 \\ &= 35 \times \frac{48}{29} \times 2, \quad t \in [0,1],\ u \in [2,99],\ v \in [2,99], \\ f(t,u,v) &= \frac{1-t}{10} + \frac{t^2}{10} + \frac{t}{10} + 140 + 2\left|\sin \frac{2v(u-99)(v-99)}{1+t^2u^2}\right| \\ &\le \frac{3}{10} + 140 + 2 = \frac{1423}{10} < \frac{1}{2}\Lambda_1 c \\ &= \frac{1}{2} \times \frac{6}{5} \times \frac{1425}{6}, \quad t \in [0,1],\ u \in [99,+\infty),\ v \in [99,+\infty), \end{aligned} \tag{4.19}$$

which implies

$$f(t,u,v) < \frac{1}{2}\Lambda_1 c, \quad t \in [0,1],\ u \in [0,c],\ v \in [0,c]. \tag{4.20}$$

That is, the conditions of Theorem 3.3 are satisfied. Consequently, the problem (1.1) has at least three positive solutions $u_1, u_2, u_3 \in \overline{K_c}$ for

$$\lambda \leq \frac{1}{2}(1 - \alpha\eta)d_0 = \frac{1}{4} \tag{4.21}$$

satisfying

$$\|u_1\|_1 < 1, \quad 2 < \beta(u_2), \quad \|u_3\|_1 > 1, \quad \beta(u_3) < 2. \tag{4.22}$$

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