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Research Article

Exponential Stability of Two Coupled Second-Order Evolution Equations

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By using the multiplier technique, we prove that the energy of a system of two coupled second order evolution equations (one is an integro-differential equation) decays exponentially if the convolution kernel k decays exponentially. An example is give to illustrate that the result obtained can be applied to concrete partial differential equations.

1. Introduction

Of concern is the exponential stability of two coupled second-order evolution equations (one is an integro-differential equation) in Hilbert space H

$$u''(t) + Au(t) + \alpha u(t) - \int_0^t k(t-s)Au(s)\mathrm{d}s + \beta\sqrt{A}v(t) = 0, \tag{1.1}$$

$$v''(t) + Av(t) + \beta\sqrt{A}u(t) = 0, \tag{1.2}$$

with initial data

$$u(0) = u_0, \quad v(0) = v_0, \quad u'(0) = u_1, \quad v'(0) = v_1. \tag{1.3}$$

Here $A : D(A) \subset H \to H$ is a positive self-adjoint linear operator, $\alpha \geqslant 0$, $\beta > 0$, $k(t)$ is a nonnegative function on $[0, \infty)$. Moreover, the fractional power $A^{1/2}$ is defined as in the well known operator theory (cf, e.g., [1, 2]).

An interesting and difficult point for it is to stabilize the whole system via the damping effect given by only one equation (1.1). We remark that there is very few work concerning the situation when the damping mechanism is given by memory terms; see [3], where a coupled Timoshenko beam system is investigated.

On the other hand, the stability of the single integro-differential equation has been studied extensively; see, for instance, [4, 5].

In this paper, through suitably choosing multipliers for the energy together with other techniques, we obtain the desired exponential decay result for the system (1.1)–(1.3). Nonlinear coupled systems with general decay rates will be discussed in a forthcoming paper.

In Section 2, we present our exponential decay theorem and its proof. An application is given in Section 3.

2. Exponential Decay Result

We start with stating our assumptions:

(1) A is a self-adjoint linear operator in H, satisfying

$$\langle Au, u\rangle \geqslant a\|u\|^2, \quad u \in D(A), \tag{2.1}$$

where $a > 0$.

(2) $\alpha > 0$, $\beta > 0$ are constants. $k(t) : [0,\infty) \to [0,\infty)$ is locally absolutely continuous, satisfying

$$k(0) > 0, \quad 1 - \int_0^\infty k(t)\mathrm{d}t - \frac{1}{a}\left|\alpha - \beta^2\right| > 0, \tag{2.2}$$

and there exists a positive constant λ, such that

$$k'(t) \leqslant -\lambda k(t), \quad \text{for a.e. } t \geqslant 0. \tag{2.3}$$

We define the energy of a mild solution u of (1.1)–(1.3) as

$$\begin{aligned} E(t) = E_u(t) :={}& \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\left\|v'(t)\right\|^2 + \frac{1 - \int_0^t k(s)\mathrm{d}s}{2}\left\|\sqrt{A}u(t)\right\|^2 \\ &+ \frac{1}{2}\int_0^t k(t-s)\left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|^2 \mathrm{d}s \\ &+ \frac{1}{2}\left\|\sqrt{A}v(t) + \beta u(t)\right\|^2 + \frac{1}{2}\left(\alpha - \beta^2\right)\|u(t)\|^2. \end{aligned} \tag{2.4}$$

The following is our exponential decay theorem.

Theorem 2.1. *Let the assumptions be satisfied. Then,*

(i) *for any $u_0, v_0 \in D(\sqrt{A})$ and $u_1, v_1 \in H$, problem* (1.1)–(1.3) *admits a unique mild solution on* $[0,\infty)$.
The solution is a classical one, if $u_0 \in D(A), v_0 \in D(A)$ and $u_1, v_1 \in D(\sqrt{A})$,

(ii) *there exists a constant $C > 0$ such that the energy*

$$E(t) \leqslant E(0)e^{1-Ct}, \quad \forall t \geqslant 0, \tag{2.5}$$

for any mild solution of (1.1)–(1.3).

Proof. We denote

$$\begin{gathered} w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \qquad w_0(t) = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix}, \qquad w_1(t) = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix}, \\ \mathscr{A} = \begin{pmatrix} A + \alpha & \beta\sqrt{A} \\ \beta\sqrt{A} & A \end{pmatrix}, \qquad \mathcal{B}(t) = \begin{pmatrix} k(t)A & 0 \\ 0 & 0 \end{pmatrix}. \end{gathered} \tag{2.6}$$

Then, (1.1)–(1.3) becomes

$$\begin{gathered} w''(t) + \mathscr{A}w(t) - \int_0^t \mathcal{B}(t-s)w(s)\mathrm{d}s = 0 \quad t \in [0,\infty), \\ w(0) = w_0, \qquad w'(0) = w_1, \end{gathered} \tag{2.7}$$

in $\mathscr{H} := H \times H$. From the assumptions, one sees that—$\mathscr{A}$ is the generator of a strongly continuous cosine function on $\mathscr{H}$, and $\mathcal{B}(\cdot)$ is bounded from $D(\mathscr{A})$ into $W^{1,1}_{\mathrm{loc}}(0,\infty;\mathscr{H})$. Therefore, we justify the assertion (i) (cf., e.g. [6]).

Suppose now that u is a classical solution of (1.1)–(1.3). We observe

$$\begin{aligned} E'(t) &= -\frac{1}{2}k(t)\left\|\sqrt{A}u(t)\right\|^2 + \frac{1}{2}\int_0^t k'(t-s)\left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|^2 \mathrm{d}s \\ &\leqslant 0, \end{aligned} \tag{2.8}$$

by Assumption (2) and so

$$E(t) \leqslant E(s), \quad 0 \leqslant s \leqslant t \leqslant T. \tag{2.9}$$

Let

$$\mu := 1 - \int_0^\infty k(t)\mathrm{d}t - \frac{|\alpha - \beta^2|}{a}, \tag{2.10}$$

and take $1 < \delta < 1 + a\mu/2\beta^2$. We have

$$\begin{aligned} E(t) \geqslant{} & \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|v'(t)\|^2 + \frac{\mu}{4}\left\|\sqrt{A}u(t)\right\|^2 \\ & + \left(\frac{1}{2} - \frac{1}{2\delta}\right)\left\|\sqrt{A}v(t)\right\|^2 + \left(\frac{(1-\delta)\beta^2}{2a}\right)\left\|\sqrt{A}u(t)\right\|^2, \quad t \in [0,\infty). \end{aligned} \tag{2.11}$$

□

Furthermore, we need the following lemmas.

Lemma 2.2. *For any $T \geqslant S \geqslant 0$ and for any $\varepsilon_1 > 0$, there exist positive numbers $D_1(\varepsilon_1)$, $D_2(\varepsilon_1)$ such that*

$$\begin{aligned} & \int_S^T \left\|\sqrt{A}v(t) + \beta u(t)\right\|^2 \mathrm{d}t \\ & \quad \leqslant D_1 E(S) + D_2 \int_S^T \int_0^t k(t-s)\left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|^2 \mathrm{d}s\,\mathrm{d}t \\ & \quad + G_1 \int_S^T \|u'(t)\|^2 \mathrm{d}t + G_2 \frac{1}{\varepsilon_1}\int_S^T \|v'(t)\|^2 \mathrm{d}t + (G_3\varepsilon_1 + G_4)\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t, \end{aligned} \tag{2.12}$$

for some positive constants G_i $(i = 1,2,3,4)$ which only depend on α, β, a, and k.

Proof. At first, let us take the inner product of both sides of (1.1) with $\sqrt{A}v(t)$ and integrate over $[S,T]$. Then, noticing (1.2), we obtain

$$\begin{aligned} & \int_S^T \left\langle u''(t), \sqrt{A}v(t)\right\rangle \mathrm{d}t - \int_S^T \left\langle \sqrt{A}u(t), v''(t)\right\rangle \mathrm{d}t - \beta\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t \\ & \quad + \alpha\int_S^T \left\langle u(t), \sqrt{A}v(t)\right\rangle \mathrm{d}t + \int_S^T \left\langle \int_0^t k(t-s)\sqrt{A}u(s)\mathrm{d}s, v''(t)\right\rangle \mathrm{d}t \\ & \quad + \beta\int_S^T \left\langle \int_0^t k(t-s)\sqrt{A}u(s)\mathrm{d}s, \sqrt{A}u(t)\right\rangle \mathrm{d}t + \beta\int_S^T \left\|\sqrt{A}v(t)\right\|^2 \mathrm{d}t = 0. \end{aligned} \tag{2.13}$$

For the first item, integrating by parts, we have

$$\int_S^T \left\langle u''(t), \sqrt{A}v(t) \right\rangle \mathrm{d}t = \int_S^T \left(\left\langle u'(t), \sqrt{A}v(t) \right\rangle\right)' \mathrm{d}t - \int_S^T \left\langle u'(t), \sqrt{A}v'(t) \right\rangle \mathrm{d}t. \tag{2.14}$$

The second and the fifth items can be treated similarly. Therefore,

$$\begin{aligned}
&\beta \int_S^T \left\| \sqrt{A}v(t) \right\|^2 \mathrm{d}t + \alpha \int_S^T \left\langle u(t), \sqrt{A}v(t) \right\rangle \mathrm{d}t \\
&\quad = -\int_S^T \left(\left\langle u'(t), \sqrt{A}v(t) \right\rangle\right)' \mathrm{d}t + \int_S^T \left(\left\langle \sqrt{A}u(t), v'(t) \right\rangle\right)' \mathrm{d}t \\
&\qquad - \int_S^T \left(\left\langle \int_0^t k(t-s)\sqrt{A}u(s)\mathrm{d}s, v'(t) \right\rangle\right)' \mathrm{d}t \\
&\qquad + \beta \int_S^T \left(1 - \int_0^t k(s)\mathrm{d}s\right) \left\| \sqrt{A}u(t) \right\|^2 \mathrm{d}t \\
&\qquad + \int_S^T \left\langle \int_0^t k'(t-s)\left(\sqrt{A}u(s) - \sqrt{A}u(t)\right)\mathrm{d}s, v'(t) \right\rangle \mathrm{d}t \\
&\qquad - \beta \int_S^T \left\langle \int_0^t k(t-s)\left(\sqrt{A}u(s) - \sqrt{A}u(t)\right)\mathrm{d}s, \sqrt{A}u(t) \right\rangle \mathrm{d}t \\
&\qquad + \int_S^T k(t) \left\langle \sqrt{A}u(t), v'(t) \right\rangle \mathrm{d}t.
\end{aligned} \tag{2.15}$$

Then, taking the inner product of both sides of (1.1) with $u(t)$ and integrating over $[S, T]$, we obtain

$$\begin{aligned}
&\beta \int_S^T \left\langle \sqrt{A}v(t), u(t) \right\rangle \mathrm{d}t + \alpha \int_S^T \|u(t)\|^2 \mathrm{d}t \\
&\quad = -\int_S^T (\langle u'(t), u(t) \rangle)' \mathrm{d}t + \int_S^T \left\| u'(t) \right\|^2 \mathrm{d}t \\
&\qquad + \int_S^T \left\langle \int_0^t k(t-s)\left(\sqrt{A}u(s) - \sqrt{A}u(t)\right)\mathrm{d}s, \sqrt{A}u(t) \right\rangle \mathrm{d}t \\
&\qquad - \int_S^T \left(1 - \int_0^t k(s)\mathrm{d}s\right) \left\| \sqrt{A}u(t) \right\|^2 \mathrm{d}t.
\end{aligned} \tag{2.16}$$

Equation (2.15) $\times\ \beta/\alpha$ + (2.16) yields that

$$
\begin{aligned}
&\int_S^T \left\| \sqrt{A}v(t) + \beta u(t) \right\|^2 \mathrm{d}t \\
&\quad = -\int_S^T (\langle u'(t), u(t)\rangle)' \mathrm{d}t + \int_S^T \|u'(t)\|^2 \mathrm{d}t \\
&\quad - \frac{\beta}{\alpha}\int_S^T \left(\left\langle u'(t), \sqrt{A}v(t)\right\rangle\right)' \mathrm{d}t + \frac{\beta}{\alpha}\int_S^T \left(\left\langle \sqrt{A}u(t), v'(t)\right\rangle\right)' \mathrm{d}t \\
&\quad - \frac{\beta}{\alpha}\int_S^T \left(\left\langle \int_0^t k(t-s)\sqrt{A}u(s)\mathrm{d}s, v'(t)\right\rangle\right)' \mathrm{d}t \\
&\quad + \frac{\beta^2-\alpha}{\alpha}\int_S^T \left(1 - \int_0^t k(s)\mathrm{d}s\right)\left\| \sqrt{A}u(t)\right\|^2 \mathrm{d}t \\
&\quad + \frac{\beta}{\alpha}\int_S^T \left\langle \int_0^t k'(t-s)\left(\sqrt{A}u(s) - \sqrt{A}u(t)\right)\mathrm{d}s, v'(t)\right\rangle \mathrm{d}t \\
&\quad + \frac{\beta}{\alpha}\int_S^T k(t)\left\langle \sqrt{A}u(t), v'(t)\right\rangle \mathrm{d}t \\
&\quad + \frac{\alpha-\beta^2}{\alpha}\int_S^T \left\langle \int_0^t k(t-s)\left(\sqrt{A}u(s) - \sqrt{A}u(t)\right)\mathrm{d}s, \sqrt{A}u(t)\right\rangle \mathrm{d}t \\
&\quad - \frac{\beta^2-\alpha}{\alpha}\int_S^T \left\|\sqrt{A}v(t)\right\|^2 \mathrm{d}t - \left(\alpha-\beta^2\right)\int_S^T \|u(t)\|^2 \mathrm{d}t.
\end{aligned}
\tag{2.17}
$$

Next, we will estimate all the terms on the right side of (2.17). From (2.11), we have the following estimate:

$$
\left| \int_S^T \left(\left\langle u'(t), \sqrt{A}v(t)\right\rangle\right)' \mathrm{d}t \right| = \left| \left[\left\langle u'(t), \sqrt{A}v(t)\right\rangle\right]_S^T \right| \leqslant 2ME(S), \tag{2.18}
$$

where M is a positive constant. Those terms of the form $\int_S^T (\langle\cdot,\cdot\rangle)'\mathrm{d}t$ can be similarly treated. Denote by J the sum of the other terms on the right of (2.17).

Using Young's inequality and noting (2.8), we get, for $\varepsilon_1 > 0$,

$$\begin{aligned}
&\int_S^T \left\langle \int_0^t k'(t-s)\left(\sqrt{A}u(s) - \sqrt{A}u(t)\right)\mathrm{d}s, v'(t) \right\rangle \mathrm{d}t \\
&\quad \leqslant \frac{\varepsilon_1}{2}\int_S^T \left(\int_0^t |k'(t-s)|\left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|\mathrm{d}s\right)^2 \mathrm{d}t + \frac{1}{2\varepsilon_1}\int_S^T \|v'(t)\|^2 \mathrm{d}t. \\
&\quad \leqslant -\frac{\varepsilon_1}{2}\int_S^T \int_0^t k'(s)\mathrm{d}s \int_0^t |k'(t-s)|\left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|^2 \mathrm{d}s\,\mathrm{d}t + \frac{1}{2\varepsilon_1}\int_S^T \|v'(t)\|^2 \mathrm{d}t. \\
&\quad \leqslant \varepsilon_1 k(0)E(S) + \frac{1}{2\varepsilon_1}\int_S^T \|v'(t)\|^2 \mathrm{d}t.
\end{aligned} \tag{2.19}$$

The treatment of the other terms of J is similar, giving

$$\begin{aligned}
&\frac{\beta}{\alpha}\int_S^T \left\langle k(t)\sqrt{A}u(t), v'(t) \right\rangle \mathrm{d}t \leqslant \frac{\varepsilon_1}{2}\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t + \frac{k^2(0)\beta^2}{2\varepsilon_1\alpha^2}\int_S^T \|v'(t)\|^2 \mathrm{d}t, \\
&\frac{\alpha-\beta^2}{\alpha}\int_S^T \left\langle \int_0^t k(t-s)\left(\sqrt{A}u(s) - \sqrt{A}u(t)\right)\mathrm{d}s, \sqrt{A}u(t) \right\rangle \mathrm{d}t \\
&\quad \leqslant \frac{(\alpha-\beta^2)^2}{2\alpha^2\varepsilon_1}\int_S^T \int_0^t k(s)\mathrm{d}s \int_0^t k(t-s)\left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|^2 \mathrm{d}s\,\mathrm{d}t + \frac{\varepsilon_1}{2}\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t.
\end{aligned} \tag{2.20}$$

Thus, we obtain

$$\begin{aligned}
J \leqslant{}& \frac{\varepsilon_1 k(0)\beta}{\alpha}E(S) \\
&+ \frac{(\alpha-\beta^2)^2}{2\alpha^2\varepsilon_1}\int_S^T \int_0^t k(s)\mathrm{d}s \int_0^t k(t-s)\left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|^2 \mathrm{d}s\,\mathrm{d}t \\
&+ \left[\left(\frac{1}{\alpha} + \frac{1}{a}\right)\left|\beta^2 - \alpha\right| + \varepsilon_1\right]\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t \\
&+ \left(\frac{k^2(0)\beta^2}{2\varepsilon_1\alpha^2} + \frac{\beta}{2\varepsilon_1\alpha}\right)\int_S^T \|v'(t)\|^2 \mathrm{d}t + \int_S^T \|u'(t)\|^2 \mathrm{d}t + \theta\int_S^T \left\|\sqrt{A}v(t)\right\|^2 \mathrm{d}t,
\end{aligned} \tag{2.21}$$

where $\theta = \max\{(\alpha-\beta^2)/\alpha, 0\}$. Make use of the estimate

$$\left\|\sqrt{A}v\right\|^2 \leqslant (1+\zeta_1)\left\|\sqrt{A}v + \beta u\right\|^2 + \left(1 + \frac{1}{\zeta_1}\right)\frac{\beta^2}{a}\left\|\sqrt{A}u\right\|^2, \tag{2.22}$$

where ζ_1 is a positive constant, small enough to satisfy $(1+\zeta_1)\theta < 1$. We thus verify our conclusion. □

Lemma 2.3. *For any $T \geqslant S \geqslant 0$ and for any $\varepsilon_2 > 0$, there exist positive numbers $D_3(\varepsilon_2)$, $D_4(\varepsilon_2)$, such that*

$$\frac{\mu}{2}\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t \leqslant D_3 E(S) + D_4 \int_S^T \int_0^t k(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 \mathrm{d}s\,\mathrm{d}t + \left(1+\frac{\beta^2}{2\varepsilon_2 a}\right)\int_S^T \left\|u'(t)\right\|^2 \mathrm{d}t + \frac{\varepsilon_2}{2}\int_S^T \left\|v'(t)\right\|^2 \mathrm{d}t. \tag{2.23}$$

Proof. We denote $w = A^{-1/2}u$ and take the inner product of both sides of (1.2) with $w(t)$, and integrate over $[S,T]$. It follows that

$$-\int_S^T \left\langle \sqrt{A}v(t), u(t)\right\rangle \mathrm{d}t = \int_S^T \left(\langle v'(t), w(t)\rangle\right)' \mathrm{d}t - \int_S^T \langle v'(t), w'(t)\rangle \mathrm{d}t + \beta\int_S^T \|u(t)\|^2 \mathrm{d}t. \tag{2.24}$$

Plugging this equation into (2.16), we find

$$\begin{aligned}
\int_S^T &\left(1-\int_0^t k(s)\mathrm{d}s\right)\left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t \\
&= -\int_S^T \left(\langle u'(t), u(t)\rangle\right)' \mathrm{d}t + \beta\int_S^T \left(\langle v'(t), w(t)\rangle\right)' \mathrm{d}t \\
&\quad + \left(\beta^2-\alpha\right)\int_S^T \|u(t)\|^2 \mathrm{d}t + \int_S^T \left\|u'(t)\right\|^2 \mathrm{d}t \\
&\quad + \int_S^T \left\langle \int_0^t k(t-s)\left(\sqrt{A}u(s)-\sqrt{A}u(t)\right)\mathrm{d}s, \sqrt{A}u(t)\right\rangle \mathrm{d}t \\
&\quad - \beta\int_S^T \langle v'(t), w'(t)\rangle \mathrm{d}t.
\end{aligned} \tag{2.25}$$

Observe

$$\begin{aligned}
\int_S^T &\left\langle \int_0^t k(t-s)\left(\sqrt{A}u(s)-\sqrt{A}u(t)\right)\mathrm{d}s, \sqrt{A}u(t)\right\rangle \mathrm{d}t \\
&\leqslant \frac{1}{2\delta_3}\int_S^T \int_0^t k(s)\mathrm{d}s \int_0^t k(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 \mathrm{d}s\,\mathrm{d}t + \frac{\delta_3}{2}\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t,
\end{aligned} \tag{2.26}$$

where $\delta_3 = \mu$, and for $\varepsilon_2 > 0$

$$\beta \int_S^T \langle v'(t), w'(t) \rangle \mathrm{d}t \leqslant \frac{\varepsilon_2}{2} \int_S^T \|v'(t)\|^2 \mathrm{d}t + \frac{\beta^2}{2a\varepsilon_2} \int_S^T \|u'(t)\|^2 \mathrm{d}t. \tag{2.27}$$

The other items on the right of (2.25) can be dealt with as in the proof of Lemma 2.2. Hence, we get the conclusion. □

Lemma 2.4. *For any $T \geqslant S \geqslant 0$, there exist positive numbers D_5, D_6 such that*

$$\begin{aligned}
&\int_S^T \|u'(t)\|^2 \mathrm{d}t + \int_S^T \|v'(t)\|^2 \mathrm{d}t \\
&\quad \leqslant D_5 E(S) + D_6 \int_S^T \int_0^t k(t-s) \left\| \sqrt{A}u(s) - \sqrt{A}u(t) \right\|^2 \mathrm{d}s\, \mathrm{d}t \\
&\qquad + \left(\frac{3}{2} + \frac{|\alpha - \beta^2|}{a} \right) \int_S^T \left\| \sqrt{A}u(t) \right\|^2 \mathrm{d}t + \int_S^T \left\| \sqrt{A}v(t) + \beta u(t) \right\|^2 \mathrm{d}t.
\end{aligned} \tag{2.28}$$

Proof. Taking the inner product of both sides of (1.2) with $v(t)$ and integrating over $[S, T]$, we see

$$\int_S^T \|v'(t)\|^2 \mathrm{d}t = \int_S^T (\langle v'(t), v(t) \rangle)' \mathrm{d}t + \int_S^T \left\| \sqrt{A}v(t) \right\|^2 \mathrm{d}t + \beta \int_S^T \left\langle \sqrt{A}u(t), v(t) \right\rangle \mathrm{d}t. \tag{2.29}$$

Combining this equation and (2.16) gives

$$\begin{aligned}
&\int_S^T \|v'(t)\|^2 \mathrm{d}t + \int_S^T \|u'(t)\|^2 \mathrm{d}t \\
&\quad = \int_S^T (\langle u'(t), u(t) \rangle)' \mathrm{d}t + \int_S^T (\langle v'(t), v(t) \rangle)' \mathrm{d}t + \int_S^T \left(1 - \int_0^t k(s) \mathrm{d}s \right) \left\| \sqrt{A}u(t) \right\|^2 \mathrm{d}t \\
&\qquad - \int_S^T \left\langle \int_0^t k(t-s) \left(\sqrt{A}u(s) - \sqrt{A}u(t) \right) \mathrm{d}s, \sqrt{A}u(t) \right\rangle \mathrm{d}t \\
&\qquad + \int_S^T \left\| \sqrt{A}v(t) + \beta u(t) \right\|^2 \mathrm{d}t + \left(\alpha - \beta^2 \right) \int_S^T \|u(t)\|^2 \mathrm{d}t.
\end{aligned} \tag{2.30}$$

This yields the estimate as desired. □

Lemma 2.5. *Let $S_0 > 0$ be fixed. For any $T \geqslant S \geqslant S_0$ and for any $\varepsilon_3 > 0$, there exist positive numbers $D_7(\varepsilon_3)$, $D_8(\varepsilon_3)$ such that*

$$\begin{aligned}\int_S^T \|u'(t)\|^2 \mathrm{d}t \leqslant D_7 E(S) + D_8 \int_S^T \int_0^t k(t-s) \left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|^2 \mathrm{d}s\, \mathrm{d}t \\ + \varepsilon_3 \int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t + \varepsilon_3 \int_S^T \left\|\sqrt{A}v(t) + \beta u(t)\right\|^2 \mathrm{d}t.\end{aligned} \tag{2.31}$$

Proof. Take the inner product of both sides of (1.1) with $\int_0^t k(t-s)(u(s)-u(t))\mathrm{d}s$ and integrate over $[S,T]$. This leads to

$$\begin{aligned}&\int_S^T \int_0^t k(s)\mathrm{d}s \|u'(t)\|^2 \mathrm{d}t \\ &\quad= -\int_S^T \left(\left\langle u'(t), \int_0^t k(t-s)(u(s)-u(t))\mathrm{d}s \right\rangle\right)' \mathrm{d}t \\ &\quad+ \int_S^T \left\langle u'(t), \int_0^t k'(t-s)(u(s)-u(t))\mathrm{d}s \right\rangle \mathrm{d}t \\ &\quad- \int_S^T \left(1 - \int_0^t k(s)\mathrm{d}s\right) \left\langle \sqrt{A}u(t), \int_0^t k(t-s)\left(\sqrt{A}u(s) - \sqrt{A}u(t)\right)\mathrm{d}s \right\rangle \mathrm{d}t \\ &\quad- \alpha \int_S^T \left\langle u(t), \int_0^t k(t-s)(u(s)-u(t))\mathrm{d}s \right\rangle \mathrm{d}t \\ &\quad+ \int_S^T \left\| \int_0^t k(t-s)(\sqrt{A}u(s) - \sqrt{A}u(t))\mathrm{d}s \right\|^2 \mathrm{d}t \\ &\quad+ \beta \int_S^T \left\langle \sqrt{A}v(t), \int_0^t k(t-s)(u(s)-u(t))\mathrm{d}s \right\rangle \mathrm{d}t.\end{aligned} \tag{2.32}$$

Just as in the proofs of the above lemmas, using Young's inequality and noting that

$$\int_S^T \int_0^t k(s)\mathrm{d}s \|u'(t)\|^2 \mathrm{d}t \geqslant \int_0^{S_0} k(s)\mathrm{d}s \int_S^T \|u'(t)\|^2 \mathrm{d}t, \tag{2.33}$$

we prove the conclusion. □

Proof of Theorem 2.1 (continued). From Assumption (2) and (2.8), we have

$$\begin{aligned}
&\int_S^T \int_0^t k(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 \mathrm{d}s\,\mathrm{d}t \\
&-\frac{1}{\lambda}\int_S^T \int_0^t k'(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 \mathrm{d}s\,\mathrm{d}t \\
&\leqslant -\frac{1}{\lambda}\int_S^T E'(t)\mathrm{d}t \\
&\leqslant \frac{2}{\lambda}E(S).
\end{aligned} \tag{2.34}$$

Now, fix $S_0 > 0$. Thanks to Lemmas 2.2 and 2.3, we know that for any $T \geqslant S \geqslant S_0$ and for $\eta > 0$,

$$\begin{aligned}
&\frac{\mu}{2}\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t + \eta\int_S^T \left\|\sqrt{A}v(t)+\beta u(t)\right\|^2 \mathrm{d}t \\
&\quad\leqslant \left[(D_3+\eta D_1)+(D_4+\eta D_2)\frac{2}{\lambda}\right]E(S)+\eta(G_3\varepsilon_1+G_4)\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t. \\
&\qquad+\left(\frac{\varepsilon_2}{2}+\eta G_2\frac{1}{\varepsilon_1}\right)\int_S^T \left\|v'(t)\right\|^2 \mathrm{d}t+\left(1+\frac{\beta^2}{2\varepsilon_2 a}+\eta G_1\right)\int_S^T \left\|u'(t)\right\|^2 \mathrm{d}t.
\end{aligned} \tag{2.35}$$

Moreover, by the use of Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
&\frac{\mu}{2}\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t + \eta\int_S^T \left\|\sqrt{A}v(t)+\beta u(t)\right\|^2 \mathrm{d}t \\
&\quad\leqslant p_0E(S)+p_1\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t+p_2\int_S^T \left\|\sqrt{A}v(t)+\beta u(t)\right\|^2 \mathrm{d}t,
\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}
p_0 &= \left[D_3+\eta D_1+D_5\left(\frac{\varepsilon_2}{2}+\eta G_2\frac{1}{\varepsilon_1}\right)+D_7\left(1+\frac{\beta^2}{2\varepsilon_2 a}+\eta G_1\right)\right] \\
&\quad+\left[D_4+\eta D_2+D_6\left(\frac{\varepsilon_2}{2}+\eta G_2\frac{1}{\varepsilon_1}\right)+D_8\left(1+\frac{\beta^2}{2\varepsilon_2 a}+\eta G_1\right)\right]\frac{2}{\lambda}, \\
p_1 &= \eta(G_3\varepsilon_1+G_4)+\left(\frac{3}{2}+\frac{\left|\alpha-\beta^2\right|}{a}\right)\left(\frac{\varepsilon_2}{2}+\eta G_2\frac{1}{\varepsilon_1}\right)+\varepsilon_3\left(1+\frac{\beta^2}{2\varepsilon_2 a}+\eta G_1\right), \\
p_2 &= \frac{\varepsilon_2}{2}+\eta G_2\frac{1}{\varepsilon_1}+\varepsilon_3\left(1+\frac{\beta^2}{2\varepsilon_2 a}+\eta G_1\right).
\end{aligned} \tag{2.37}$$

Let

$$\varepsilon_1 = \varepsilon^{-1}, \qquad \eta = \varepsilon_2 = \varepsilon^2, \qquad \varepsilon_3 = \varepsilon^5. \tag{2.38}$$

Taking ε small enough gives

$$p_1 < \frac{\mu}{2}, \qquad p_2 < \eta. \tag{2.39}$$

Therefore, there is a constant $N_1 > 0$ such that

$$\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t + \int_S^T \left\|\sqrt{A}v(t) + \beta u(t)\right\|^2 \mathrm{d}t \leqslant N_1 E(S) \tag{2.40}$$

by (2.36). Using Lemma 2.4 and (2.34), we deduce that for some $N_2 > 0$,

$$\begin{aligned}
&\int_S^T \|u'(t)\|^2 \mathrm{d}t + \int_S^T \|v'(t)\|^2 \mathrm{d}t \\
&\quad\leqslant \left(D_5 + 2D_6\frac{1}{\lambda}\right)E(S) \\
&\qquad + \left(\frac{3}{2} + \frac{\left|\alpha - \beta^2\right|}{a}\right)\int_S^T \left\|\sqrt{A}u(t)\right\|^2 \mathrm{d}t + \int_S^T \left\|\sqrt{A}v(t) + \beta u(t)\right\|^2 \mathrm{d}t \\
&\quad\leqslant N_2 E(S).
\end{aligned} \tag{2.41}$$

Next, define

$$\begin{aligned}
H(t) := \|u'(t)\|^2 + \|v'(t)\|^2 + \left\|\sqrt{A}u(t)\right\|^2 + \left\|\sqrt{A}v(t) + \beta u(t)\right\|^2 \\
+ \int_0^t k(t-s)\left\|\sqrt{A}u(s) - \sqrt{A}u(t)\right\|^2 \mathrm{d}s.
\end{aligned} \tag{2.42}$$

It is easy to see that there exist $M_1, M_2 > 0$ such that $M_1 E(t) \leqslant H(t) \leqslant M_2 E(t)$. Therefore,

$$\begin{aligned}
\int_S^T H(t)\mathrm{d}t &\leqslant \left(N_1 + N_2 + \frac{2}{\lambda}\right)E(S), \\
\int_S^T E(t)\mathrm{d}t \leqslant \frac{1}{M_1}\int_S^T H(t)\mathrm{d}t &\leqslant \frac{N_1 + N_2 + 2/\lambda}{M_1}E(S).
\end{aligned} \tag{2.43}$$

On the other hand, when $0 \leqslant S \leqslant S_0$,

$$\begin{aligned}\int_S^T E(t)\mathrm{d}t &= \int_S^{S_0} E(t)\mathrm{d}t + \int_{S_0}^T E(t)\mathrm{d}t \\ &\leqslant (S_0 - S)E(S) + \frac{N_1 + N_2 + 2/\lambda}{M_1} E(S_0) \\ &\leqslant \left(S_0 + \frac{N_1 + N_2 + 2/\lambda}{M_1}\right) E(S),\end{aligned} \tag{2.44}$$

that is,

$$\int_S^T E(t)\mathrm{d}t \leqslant NE(S), \quad \forall S \geqslant 0. \tag{2.45}$$

By a standard approximation argument, we see that (2.45) is also true for mild solutions. From this integral inequality, we complete the proof (cf., e.g., [7, Theorem 8.1]). □

3. An Example

Example 3.1. Consider a coupled system of Petrovsky equations with a memory term

$$\begin{gathered}\partial_t^2 u(t,\xi) + \Delta^2 u(t,\xi) + \alpha u(t,\xi) - \int_0^t k(t-s)\Delta^2 u(t,\xi)\mathrm{d}s + \beta\Delta v(t,\xi) = 0, \quad t \geqslant 0,\ \xi \in \Omega, \\ \partial_t^2 v(t,\xi) + \Delta^2 v(t,\xi) + \beta\Delta u(t,\xi) = 0, \quad t \geqslant 0,\ \xi \in \Omega, \\ u(t,\xi) = v(t,\xi) = \Delta u(t,\xi) = \Delta v(t,\xi) = 0, \quad t \geqslant 0,\ \xi \in \partial\Omega, \\ u(0,\xi) = u_0(\xi), \quad v(0,\xi) = v_0(\xi), \quad \partial_t u(0,\xi) = u_1(\xi), \quad \partial_t v(0,\xi) = v_1(\xi), \quad \xi \in \Omega,\end{gathered} \tag{3.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$, with sufficiently smooth boundary $\partial\Omega$ and α, β, k as in Assumption (2). Let $H = L^2(\Omega)$ with the usual inner product and norm. Here, we denote by $\partial_t u$ the time derivative of u and by Δu the Laplacian of u with respect to space variable ξ. Define $A : D(A) \subset H \to H$ by

$$A = \Delta^2, \quad \text{with } D(A) = H^4(\Omega) \cap H_0^2(\Omega). \tag{3.2}$$

Then, Assumption (1) is satisfied. Therefore, we claim in view of Theorem 2.1 that the energy of the system decays exponentially at infinity.

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