Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2011, Article ID 867136, 9 pages
doi:10.1155/2011/867136

Research Article

Asymptotic Behavior of a Discrete Nonlinear Oscillator with Damping Dynamical System

Hadi Khatibzadeh[1, 2]

[1] *Department of Mathematics, Zanjan University, P.O. Box 45195-313, Zanjan, Iran*
[2] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*

Correspondence should be addressed to Hadi Khatibzadeh, hkhatibzadeh@znu.ac.ir

Received 24 December 2010; Accepted 10 February 2011

Academic Editor: Istvan Gyori

We propose a new discrete version of nonlinear oscillator with damping dynamical system governed by a general maximal monotone operator. We show the weak convergence of solutions and their weighted averages to a zero of a maximal monotone operator A. We also prove some strong convergence theorems with additional assumptions on A. This iterative scheme gives also an extension of the proximal point algorithm for the approximation of a zero of a maximal monotone operator. These results extend previous results by Brézis and Lions (1978), Lions (1978) as well as Djafari Rouhani and H. Khatibzadeh (2008).

1. Introduction

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. We denote weak convergence in H by $\rightharpoonup$ and strong convergence by $\to$. Let A be a nonempty subset of $H\times H$ which we will refer to as a (nonlinear) possibly multivalued operator in H. A is called monotone (resp. strongly monotone) if $(y_2-y_1, x_2-x_1)\geq 0$ (resp. $(y_2-y_1, x_2-x_1)\geq \alpha|x_1-x_2|^2$ for some $\alpha>0$) for all $[x_i, y_i]\in A$, $i=1,2$. A is maximal monotone if A is monotone and $I+A$ is surjective, where I is the identity operator on H.

Nonlinear oscillator with damping dynamical system,

$$\begin{gathered} u''(t)+\gamma u'(t)+Au(t)\ni 0, \\ u(0)=u_0, \quad u'(0)=u_1, \end{gathered} \tag{1.1}$$

where A is a maximal monotone operator and $\gamma>0$, has been investigated by many authors specially for asymptotic behavior. We refer the reader to [1–6] and references in there.

Following discrete version of (1.1),

$$u_{n+1} = (I + \lambda_n A)^{-1}(u_n + \alpha_n(u_n - u_{n-1})) \tag{1.2}$$

is called inertial proximal method and has been studied in [3]. This iterative algorithm gives a method for approximation of a zero of a maximal monotone operator. In this paper, we propose another discrete version of (1.1) and study asymptotic behavior of its solutions. By using approximations

$$\begin{aligned} u'(t) &= \frac{u(t+h) - u(t-h)}{2h} + o(h), \\ u''(t) &= \frac{u(t+h) - 2u(t) + u(t-h)}{h^2} + o(h), \end{aligned} \tag{1.3}$$

for (1.1), we get

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h_n^2} + \gamma\frac{u_{n+1} - u_{n-1}}{2h_n} + Au_{n+1} \ni 0. \tag{1.4}$$

By letting $\beta = \gamma/2$, $\lambda_{n+1} = h_n^2/(1+\beta h_n)$ and $\alpha_n = (\beta h_n - 1)/(\beta h_n + 1)$, we get

$$\begin{aligned} u_{n+1} &= J_{\lambda_{n+1}}((1-\alpha_n)u_n + \alpha_n u_{n-1}), \quad n \geq 0, \\ u_{-1} &= 0, \quad u_0 = x \in H, \end{aligned} \tag{1.5}$$

where α_n (resp. λ_n) is nonnegative (resp. positive) sequence and $J_\lambda = (I + \lambda A)^{-1}$. This discrete version gives also an algorithm for approximation of a zero of maximal monotone operator A. This algorithm extends proximal point algorithm which was introduced by Martinet in [7] with $\lambda_n = \lambda$ and $\alpha_n = 0$ and then generalized by Rockafellar [8]. We investigate asymptotic behavior of solutions of (1.5) as discrete version of (1.1) which also extend previous results of [9–11] on proximal point algorithm.

Let $w_n := (\sum_{k=1}^n \lambda_k)^{-1}(\sum_{k=1}^n \lambda_k u_k)$. Under suitable assumptions, we investigate weak and strong convergence of w_n and u_n to an element of $A^{-1}(0)$ if and only if $\{u_n\}$ is bounded. Therefore, $A^{-1}(0) \neq \phi$ if and only if $\{u_n\}$ is bounded provided $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. Our results extend previous results in [2, 3, 5].

Throughout the paper, we denote $Au_{n+1} = ((1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1})/\lambda_{n+1}$, and we assume the following assumptions on the sequence $\{\alpha_n\}$:

$$0 \leq \alpha_n \leq 1, \quad \{\alpha_n\} \text{ is nonincreasing and } \alpha_n \longrightarrow 0 \text{ as } n \longrightarrow +\infty. \tag{1.6}$$

2. Main Results

In this section, we establish convergence of the sequence $\{u_n\}$ or its weighted average to an element of $A^{-1}(0)$. First we recall the following elementary lemma without proof.

Lemma 2.1. *Suppose that $\{\alpha_n\}$ is a nonnegative sequence and $\{\lambda_n\}$ is a positive sequence such that $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. If $\alpha_n/\lambda_n \to 0$ as $n \to +\infty$, then $\sum_{k=1}^{n} \alpha_k / \sum_{k=1}^{n} \lambda_k \to 0$ as $n \to +\infty$.*

We start with a weak ergodic theorem which extends a theorem of Lions [11] (see also [12] page 139 Theorem 3.1 as well as [10] Theorem 2.1).

Theorem 2.2. *Assume that u_n is a solution to* (1.5) *and $\{\alpha_n\}$ satisfies* (1.6). *If $\sum_{k=1}^{+\infty} \lambda_k = +\infty$ and $\alpha_n/\lambda_n \to 0$, then $w_n \rightharpoonup p \in A^{-1}(0)$ as $n \to \infty$ if and only if u_n is bounded.*

Proof. Suppose that $w_n \rightharpoonup p \in A^{-1}(0)$ by (1.5); we get

$$\left|u_{n+1} - p\right| \leq \left|J_{\lambda_{n+1}}((1-\alpha_n)u_n + \alpha_n u_{n-1}) - p\right| \leq (1-\alpha_n)\left|u_n - p\right| + \alpha_n\left|u_{n-1} - p\right|. \tag{2.1}$$

This implies that

$$\left|u_{n+1} - p\right| \leq \max\{\left|u_1 - p\right|, \left|u_0 - p\right|\}. \tag{2.2}$$

Then $\{u_n\}$ is bounded and this proves necessity. Now, we prove sufficiency. By monotonicity of A, we have

$$(Au_{n+1}, u_{m+1}) + (Au_{m+1}, u_{n+1}) \leq (Au_{m+1}, u_{m+1}) + (Au_{n+1}, u_{n+1}) \tag{2.3}$$

for all $m, n \geq 0$. Multiplying both sides of the above inequality by $\lambda_{m+1}\lambda_{n+1}$ and using (1.5), we deduce

$$\begin{aligned}
&(1-\alpha_n)(u_n - u_{n+1}, \lambda_{m+1}u_{m+1}) + \alpha_n(u_{n-1} - u_{n+1}, \lambda_{m+1}u_{m+1}) \\
&\quad + (1-\alpha_m)(u_m - u_{m+1}, \lambda_{n+1}u_{n+1}) + \alpha_m(u_{m-1} - u_{m+1}, \lambda_{n+1}u_{n+1}) \\
&\leq \lambda_{m+1}(1-\alpha_n)(u_n - u_{n+1}, u_{n+1}) + \lambda_{m+1}\alpha_n(u_{n-1} - u_{n+1}, u_{n+1}) \\
&\quad + \lambda_{n+1}(1-\alpha_m)(u_m - u_{m+1}, u_{m+1}) + \lambda_{n+1}\alpha_m(u_{m-1} - u_{m+1}, u_{m+1}).
\end{aligned} \tag{2.4}$$

Summing both sides of this inequality from $m = 0$ to $m = k - 1$, we get

$$
\begin{aligned}
&(1-\alpha_n)\left(u_n - u_{n+1}, \sum_{m=0}^{k-1}\lambda_{m+1}u_{m+1}\right) + \alpha_n\left(u_{n-1} - u_{n+1}, \sum_{m=0}^{k-1}\lambda_{m+1}u_{m+1}\right)\\
&\quad\le \lambda_{n+1}|u_{n+1}|\sum_{m=0}^{k-1}\alpha_m|u_{m-1}-u_m| + \sum_{m=0}^{k-1}(u_{m+1}-u_m, \lambda_{n+1}u_{n+1})\\
&\qquad + \left(\sum_{m=0}^{k-1}\lambda_{m+1}\right)(1-\alpha_n)(u_n-u_{n+1},u_{n+1}) + \left(\sum_{m=0}^{k-1}\lambda_{m+1}\right)\alpha_n(u_{n-1}-u_{n+1},u_{n+1})\\
&\qquad + \lambda_{n+1}\sum_{m=0}^{k-1}\left(\frac{(1-\alpha_m)}{2}|u_m|^2 - \frac{(1-\alpha_m)}{2}|u_{m+1}|^2\right) + \lambda_{n+1}\sum_{m=0}^{k-1}\left(\frac{\alpha_m}{2}|u_{m-1}|^2 - \frac{\alpha_m}{2}|u_{m+1}|^2\right)\\
&\quad= \lambda_{n+1}|u_{n+1}|\sum_{m=0}^{k-1}\alpha_m|u_{m-1}-u_m| + (u_k-u_0, \lambda_{n+1}u_{n+1})\\
&\qquad + \left(\sum_{m=0}^{k-1}\lambda_{m+1}\right)(1-\alpha_n)(u_n-u_{n+1},u_{n+1}) + \left(\sum_{m=0}^{k-1}\lambda_{m+1}\right)\alpha_n(u_{n-1}-u_{n+1},u_{n+1})\\
&\qquad + \lambda_{n+1}\sum_{m=0}^{k-1}\left(\frac{1}{2}|u_m|^2 - \frac{1}{2}|u_{m+1}|^2\right) + \lambda_{n+1}\sum_{m=0}^{k-1}\left(\frac{\alpha_m}{2}|u_{m-1}|^2 - \frac{\alpha_m}{2}|u_m|^2\right).
\end{aligned}
\tag{2.5}
$$

Divide both sides of the above inequality by $\sum_{m=0}^{k-1}\lambda_{m+1}$ and suppose that $k = n_j$ and $w_{n_j} \rightharpoonup p$ as $j \to +\infty$. By assumptions on $\{\alpha_n\}$, $\{\lambda_n\}$ and Lemma 2.1, we have

$$(1-\alpha_n)(u_n-u_{n+1},p) + \alpha_n(u_{n-1}-u_{n+1},p) \le (1-\alpha_n)(u_n-u_{n+1},u_{n+1}) + \alpha_n(u_{n-1}-u_{n+1},u_{n+1}). \tag{2.6}$$

This implies that

$$\left((1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1}, u_{n+1} - p\right) \ge 0. \tag{2.7}$$

From (1.6), we get

$$\left|u_{n+1}-p\right| + \alpha_n\left|u_n-p\right| \le \left|u_n-p\right| + \alpha_{n-1}\left|u_{n-1}-p\right|. \tag{2.8}$$

By (1.6) and boundedness of $\{u_n\}$, we get $\lim_{n\to+\infty}|u_n-p|$ exists. If $w_{n_k} \rightharpoonup q$, we obtain again $\lim_{n\to+\infty}|u_n-q|$ exists. Therefore, $\lim_{n\to+\infty}(1/2)(|u_n-p|^2 - |u_n-q|^2)$, and hence $\lim_{n\to+\infty}(u_n, p-q)$ exists. This follows that $\lim_{n\to+\infty}(w_n, p-q)$ exists. It implies that

$(q, p-q) = (p, p-q)$ and hence $p = q$ and $w_n \rightharpoonup p \in H$ as $n \to +\infty$. Now we prove $p \in A^{-1}(0)$. Suppose that $[x, y] \in A$. By monotonicity of A and Assumption (1.6), we get

$$\begin{aligned}
&\left(x - \left(\sum_{i=0}^{n-1}\lambda_{i+1}\right)^{-1}\sum_{i=0}^{n-1}\lambda_{i+1}u_{i+1}, y\right) \\
&\quad = \left(\sum_{i=0}^{n-1}\lambda_{i+1}\right)^{-1}\sum_{i=0}^{n-1}\lambda_{i+1}(x - u_{i+1}, y) \\
&\quad \geq \left(\sum_{i=0}^{n-1}\lambda_{i+1}\right)^{-1}\sum_{i=0}^{n-1}\lambda_{i+1}(x - u_{i+1}, Au_{i+1}) \\
&\quad = \left(\sum_{i=0}^{n-1}\lambda_{i+1}\right)^{-1}\sum_{i=0}^{n-1}(x - u_{i+1}, (1-\alpha_i)u_i + \alpha_i u_{i-1} - u_{i+1}) \\
&\quad = \left(\sum_{i=0}^{n-1}\lambda_{i+1}\right)^{-1}\sum_{i=0}^{n-1}\left(-(1-\alpha_i)(u_{i+1} - x, u_i - x) - \alpha_i(u_{i+1} - x, u_{i-1} - x) + |u_{i+1} - x|^2\right) \\
&\quad \geq \left(\sum_{i=0}^{n-1}\lambda_{i+1}\right)^{-1}\sum_{i=0}^{n-1}\left(\frac{1}{2}\left(|u_{i+1} - x|^2 - |u_i - x|^2\right) + \frac{1}{2}\left(\alpha_i|u_i - x|^2 - \alpha_{i-1}|u_{i-1} - x|^2\right)\right).
\end{aligned} \tag{2.9}$$

Letting $n \to +\infty$, we get: $(x - p, y) \geq 0$. By maximality of A, we get $p \in A^{-1}(0)$. □

Remark 2.3. Since range of J_{λ_n} is $D(A)$ (the domain of A), as a trivial consequence of Theorem 2.2, we have that If $D(A)$ is bounded then $A^{-1}(0) \neq \phi$.

In the following, we prove a weak convergence theorem. Since the necessity is obvious, we omit the proof of necessity in the next theorems.

Theorem 2.4. *Let u_n be a solution to* (1.5) *and $\lambda_n \geq \lambda_0 > 0$. If $\{\alpha_n\}$ satisfies* (1.6)*, then $u_n \rightharpoonup p \in A^{-1}(0)$ as $n \to +\infty$ if and only if $\{u_n\}$ is bounded.*

Proof. Since assumption on $\{\lambda_n\}$ implies that $\sum_{n=1}^{+\infty}\lambda_n = +\infty$, from (1.5) and (2.7), we get

$$\begin{aligned}
\lambda_{n+1}^2|Au_{n+1}|^2 &= \left|u_{n+1} - p + \lambda_{n+1}Au_{n+1}\right|^2 - \left|u_{n+1} - p\right|^2 - 2\lambda_{n+1}(Au_{n+1}, u_{n+1} - p) \\
&\leq \left|(1-\alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p)\right|^2 - \left|u_{n+1} - p\right|^2 \\
&\leq (1-\alpha_n)\left|u_n - p\right|^2 + \alpha_n\left|u_{n-1} - p\right|^2 - \left|u_{n+1} - p\right|^2 \\
&\leq \alpha_{n-1}\left|u_{n-1} - p\right|^2 - \alpha_n\left|u_n - p\right|^2 + \left|u_n - p\right|^2 - \left|u_{n+1} - p\right|^2.
\end{aligned} \tag{2.10}$$

(The last inequality follows from Assumption (1.6)). Summing both sides of this inequality from $n = 1$ to m and letting $m \to +\infty$, since $\{\alpha_n\}$ satisfies (1.6), we have

$$\sum_{n=1}^{+\infty}\lambda_{n+1}^2|Au_{n+1}|^2 < +\infty. \tag{2.11}$$

By assumption on $\{\lambda_n\}$, we have $|Au_n| \to 0$ as $n \to +\infty$. Assume $u_{n_j} \rightharpoonup q$ as $j \to +\infty$, by the monotonicity of A, we have $(Au_m - Au_{n_j}, u_m - u_{n_j}) \geq 0$. Letting $j \to +\infty$, we get $(Au_m, u_m - q) \geq 0$. Similar to the proof of Theorem 2.2, $\lim_{m\to+\infty}|u_m - q|$ exists. This implies that $u_n \rightharpoonup q = p \in A^{-1}(0)$ as $n \to +\infty$. □

In two following, theorems we show strong convergence of $\{u_n\}$ under suitable assumptions on operator A and the sequence $\{\lambda_n\}$.

Theorem 2.5. *Assume that* $(I + A)^{-1}$ *is compact and* $\sum_{n=1}^{+\infty} \lambda_n^2 = +\infty$*. If* α_n *satisfies* (1.6)*, then* $u_n \to p \in A^{-1}(0)$ *as* $n \to +\infty$ *if and only if* $\{u_n\}$ *is bounded.*

Proof. By (2.11) and assumption on $\{\lambda_n\}$, we get $\liminf_{n\to+\infty}|Au_n| = 0$ and $u_n \rightharpoonup p$ as $n \to +\infty$. Therefore, there exists a subsequence $\{Au_{n_j}\}$ of $\{Au_n\}$ such that $|Au_{n_j}| \to 0$ as $j \to +\infty$ and $\{u_{n_j} + Au_{n_j}\}$ is bounded. The compacity of $(I + A)^{-1}$ implies that $\{u_{n_j}\}$ has a strongly convergent subsequence (we denote again by $\{u_{n_j}\}$) to p. By the monotonicity of A, we have $(Au_n - Au_{n_j}, u_n - u_{n_j}) \geq 0$. Letting $j \to +\infty$, we obtain $(Au_n, u_n - p) \geq 0$. Now, the proof of Theorem 2.2 shows that $\lim_{n\to+\infty}|u_n - p|^2$ exists. This implies that $u_n \to p$ as $n \to +\infty$. □

Theorem 2.6. *Assume that* A *is strongly monotone operator and* $\sum_{n=1}^{+\infty} \lambda_n = +\infty$*. If* $\{\alpha_n\}$ *satisfies* (1.6)*, then* $u_n \to p \in A^{-1}(0)$ *as* $n \to +\infty$ *if and only if* $\{u_n\}$ *is bounded.*

Proof. By the proof of Theorem 2.2, $w_n \rightharpoonup p \in A^{-1}(0)$ as $n \to +\infty$, and $\lim_{n\to+\infty}|u_n - p|^2$ exists. Since A is strongly monotone, we have

$$(Au_{n+1}, u_{n+1} - p) \geq \alpha|u_{n+1} - p|^2. \tag{2.12}$$

Multiplying both sides of (2.12) by λ_{n+1} and summing from $n = 1$ to m, we have

$$\begin{aligned}
\alpha\sum_{n=1}^{m}\lambda_{n+1}|u_{n+1} - p|^2 &\leq \sum_{n=1}^{m}((1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1}, u_{n+1} - p) \\
&= \sum_{n=1}^{m}\left[(1-\alpha_n)(u_n - p, u_{n+1} - p) + \alpha_n(u_{n-1} - p, u_{n+1} - p) - |u_{n+1} - p|^2\right] \\
&\leq \frac{1}{2}\sum_{n=1}^{m}\left[(1-\alpha_n)|u_n - p|^2 + \alpha_n|u_{n-1} - p|^2 - |u_{n+1} - p|^2\right] \\
&\leq \frac{1}{2}\sum_{n=1}^{m}\left[|u_n - p|^2 - |u_{n+1} - p|^2 + \alpha_{n-1}|u_{n-1} - p|^2 - \alpha_n|u_n - p|^2\right].
\end{aligned} \tag{2.13}$$

(The last inequality follows from Assumption (1.6)). Letting $m \to +\infty$, we get:

$$\sum_{n=1}^{+\infty}\lambda_{n+1}|u_{n+1} - p|^2 < +\infty. \tag{2.14}$$

So, $\liminf_{n\to+\infty}|u_n - p|^2 = 0$. This implies that $u_n \to p$ as $n \to +\infty$. □

In the following theorem, we assume that $A = \partial\varphi$, where φ is a proper, lower semicontinuous and convex function and $\operatorname{Argmin}\varphi \neq \phi$.

Theorem 2.7. *Let $A = \partial\varphi$, where φ is a proper, lower semicontinuous, and convex function. Assume that $A^{-1}(0)$ is nonempty (i.e., φ has at least one minimum point) and $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. If $\{\alpha_n\}$ satisfies* (1.6), *then $u_n \rightharpoonup p \in A^{-1}(0)$ as $n \to +\infty$.*

Proof. Since A is subdifferential of φ and $p \in A^{-1}(0)$, by Assumption (1.6), we have

$$\begin{aligned}\varphi(u_{n+1}) - \varphi(p) &\le \frac{1}{\lambda_{n+1}}((1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1}, u_{n+1} - p) \\ &\le \frac{1}{\lambda_{n+1}}\left(\frac{(1-\alpha_n)}{2}\left(|u_n - p|^2 - |u_{n+1} - p|^2\right) + \frac{\alpha_n}{2}\left(|u_{n-1} - p|^2 - |u_{n+1} - p|^2\right)\right) \\ &\le \frac{1}{\lambda_{n+1}}\left(\frac{1}{2}\left(|u_n - p|^2 - |u_{n+1} - p|^2\right) + \frac{1}{2}\left(\alpha_{n-1}|u_{n-1} - p|^2 - \alpha_n|u_n - p|^2\right)\right).\end{aligned} \tag{2.15}$$

Multiplying both sides of the above inequality by λ_{n+1} and summing from $n = 1$ to m and letting $m \to +\infty$, we get

$$\sum_{n=1}^{+\infty}\lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) < +\infty. \tag{2.16}$$

By assumption on $\{\lambda_n\}$, we deduce

$$\liminf_{n\to+\infty}\varphi(u_n) = \varphi(p). \tag{2.17}$$

By convexity of φ, we have

$$\begin{aligned}&\varphi(u_{n+1}) - (1-\alpha_n)\varphi(u_n) - \alpha_n\varphi(u_{n-1}) \\ &\quad\le \varphi(u_{n+1}) - \varphi((1-\alpha_n)u_n + \alpha_n(u_{n-1})) \\ &\quad\le \frac{1}{\lambda_{n+1}}((1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1}, u_{n+1} - (1-\alpha_n)u_n - \alpha_n u_{n-1}) \\ &\quad\le 0.\end{aligned} \tag{2.18}$$

Therefore,

$$\varphi(u_{n+1}) \le (1-\alpha_n)\varphi(u_n) + \alpha_n\varphi(u_{n-1}). \tag{2.19}$$

From (2.19), by Assumption (1.6), we get

$$\varphi(u_{n+1}) + \alpha_n\varphi(u_n) \le \varphi(u_n) + \alpha_{n-1}\varphi(u_{n-1}). \tag{2.20}$$

Again by (2.19), we get

$$\varphi(u_n) \leq \max\{\varphi(u_0), \varphi(u_1)\} \tag{2.21}$$

for all $n > 1$. By (2.20) and (2.21), we have that

$$\lim_{n\to+\infty}\left(\varphi(u_{n+1}) + \alpha_n \varphi(u_n)\right) \tag{2.22}$$

exists. From Assumptions (1.6), (2.17), and (2.21), we get

$$\lim_{n\to+\infty}\varphi(u_n) = \varphi(p). \tag{2.23}$$

If $u_{n_j} \rightharpoonup q$, then $\varphi(p) = \liminf_{j\to+\infty}\varphi(u_{n_j}) \geq \varphi(q)$. This implies that $q \in A^{-1}(0)$. On the other hand, for each $p \in A^{-1}(0)$ by (1.5), we get (2.7). The proof of Theorem 2.2 implies that there exists $\lim_{n\to+\infty}|u_n - p|$. Then the theorem is concluded by Opial's Lemma (see [13]). □

Acknowledgment

This research was in part supported by a Grant from IPM (no. 89470017).

References

[1] F. Alvarez, "On the minimizing property of a second order dissipative system in Hilbert spaces," *SIAM Journal on Control and Optimization*, vol. 38, no. 4, pp. 1102–1119, 2000.

[2] F. Alvarez and H. Attouch, "Convergence and asymptotic stabilization for some damped hyperbolic equations with non-isolated equilibria," *ESAIM. Control, Optimisation and Calculus of Variations*, vol. 6, pp. 539–552, 2001.

[3] F. Alvarez and H. Attouch, "An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping," *Set-Valued Analysis*, vol. 9, no. 1-2, pp. 3–11, 2001.

[4] H. Attouch, A. Cabot, and P. Redont, "The dynamics of elastic shocks via epigraphical regularization of a differential inclusion. Barrier and penalty approximations," *Advances in Mathematical Sciences and Applications*, vol. 12, no. 1, pp. 273–306, 2002.

[5] H. Attouch and F. Alvarez, "The heavy ball with friction dynamical system for convex constrained minimization problems," in *Optimization (Namur, 1998)*, vol. 481 of *Lecture Notes in Econom. and Math. Systems*, pp. 25–35, Springer, Berlin, 2000.

[6] H. Attouch, X. Goudou, and P. Redont, "The heavy ball with friction method. I. The continuous dynamical system: global exploration of the local minima of a real-valued function by asymptotic analysis of a dissipative dynamical system," *Communications in Contemporary Mathematics*, vol. 2, no. 1, pp. 1–34, 2000.

[7] B. Martinet, "Régularisation d'inéquations variationnelles par approximations successives," *Revue Française d'Informatique et de Recherche Opérationnelle*, vol. 4, no. Ser. R-3, pp. 154–158, 1970.

[8] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[9] H. Brézis and P.-L. Lions, "Produits infinis de résolvantes," *Israel Journal of Mathematics*, vol. 29, no. 4, pp. 329–345, 1978.

[10] B. Djafari Rouhani and H. Khatibzadeh, "On the proximal point algorithm," *Journal of Optimization Theory and Applications*, vol. 137, no. 2, pp. 411–417, 2008.

[11] P.-L. Lions, "Une méthode itérative de résolution d'une inéquation variationnelle," *Israel Journal of Mathematics*, vol. 31, no. 2, pp. 204–208, 1978.

[12] G. Moroşanu, *Nonlinear Evolution Equations and Applications*, vol. 26 of *Mathematics and its Applications (East European Series)*, D. Reidel Publishing, Dordrecht, The Netherlands, 1988.
[13] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.