

Research Article

Existence of Solutions to Fractional Mixed Integro-differential Equations with Nonlocal Initial Condition

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We study the existence and uniqueness theorem for the nonlinear fractional mixed Volterra-Fredholm integrodifferential equation with nonlocal initial condition $d^\alpha x(t)/dt^\alpha = f(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^1 h(t, s, x(s))ds)$, $x(0) = \int_0^1 g(s)x(s)ds$, where $t \in [0, 1]$, $0 < \alpha < 1$, and f is a given function. We point out that such a kind of initial conditions or nonlocal restrictions could play an interesting role in the applications of the mentioned model. The results obtained are applied to an example.

1. Introduction

Recently it has been proved that the differential models involving nonlocal derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth (see [1–6]). In fact, such models can be considered as an efficient alternative to the classical nonlinear differential models to simulate many complex processes (see [7]). For instance, fractional differential equations are an excellent tool to describe hereditary properties of viscoelastic materials and, in general, to simulate the dynamics of many processes on anomalous media. Theory of fractional differential equations has been extensively studied by several authors as Delbosco and Rodino [8], Kilbas et al. [6], Lakshmikantham et al. [9–11], and also see [2, 12–16].

Recently Mophou and N'Guérékata [17], studied the Cauchy problem with nonlocal conditions

$$\begin{aligned} D^q x(t) &= Ax(t) + t^n f(t, x(t), Bx(t)), \quad t \in [0, T], \quad n \in \mathbb{Z}^+, \\ x(0) + g(x) &= x_0, \end{aligned} \quad (1.1)$$

in general Banach space X with $0 < q < 1$, and A is the infinitesimal generator of a C_0 -semigroup of bounded linear operators. By means of the Krasnoselskii's Theorem, existence of solutions was also obtained.

Subsequently several authors have investigated the problem for different types of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces.

Very recently N'Guérékata [2, 18] discussed the existence of solutions of fractional abstract differential equations with nonlocal initial condition. The nonlocal Cauchy problem is discussed by authors in [15] using the fixed-point concepts. Tidke [19] studied the nonfractional mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions using Leray-Schauder Theorem.

Motivated by the above mentioned works in this manuscript we discuss the existence and the uniqueness of the solution for the following fractional integrodifferential equation with nonlocal integral initial condition in Banach Space:

$$\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} &= f\left(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^1 h(t, s, x(s))ds\right), \\ x(0) &= \int_0^1 g(s)x(s)ds, \end{aligned} \quad (1.2)$$

where $t \in J = [0, 1]$, $0 < \alpha < 1$, $g(t) \in (0, 1]$, $g \in L^1([0, 1], \mathbb{R}_+)$, $x \in Y = C(J, E)$ is a continuous function on J with values in the Banach space E and $\|x\|_Y = \max_{t \in J} \|x(t)\|_E$, and $f : J \times E \times E \times E \rightarrow E$, $k : D \times E \rightarrow E$, and $h : D_0 \times E \rightarrow E$ are continuous E -valued functions. Here $D = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq 1\}$, and $D_0 = J \times J$. The operator d^α/dt^α denotes the Caputo fractional derivative of order α .

For the sake of the shortness let

$$Kx(t) = \int_0^t k(t, s, x(s))ds, \quad Hx(t) = \int_0^1 h(t, s, x(s))ds. \quad (A)$$

The paper is organized as follows. In Section 2, some definitions, lemmas and preliminary results are introduced to be used in the sequel. Section 3 will involve the assumptions, main results and proofs of existence problem of (1.2), together with a nonlocal initial condition. Finally an example is presented.

2. Preliminaries

Let E be a real Banach space and θ the zero element of E . Let $L^1([0, 1], E)$ be the Banach space of measurable functions $x : [0, 1] \rightarrow E$ which are Lebesgue integrable, equipped with the

norm $\|x\|_{L^1} = \int_0^1 \|x(s)\| ds$. We will use the following notation $R^+ = (0, \infty)$, and $R_+ = [0, \infty)$. A function $x \in C([0, 1], E)$ is called a solution of (1.2) if it satisfies (1.2).

Definition 2.1. A real function $f(t)$ is said to be in the space $C_\alpha, \alpha \in R$ if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$, where $g \in C[0, \infty)$, while $f(t)$ is said to be in the space C_α^m if and only if $f^{(m)} \in C_\alpha, m \in N$.

Definition 2.2. The fractional (arbitrary) order Riemann-Liouville integral (on the right and on the left) of the function $f \in L^1([a, b], R_+)$ of order $\alpha, a, b \in R_+$ is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{a^+}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds, \quad I_{b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds, \quad (2.1)$$

where Γ is the Gamma function of Euler.

When $a = 0$, we write $I^\alpha f(t) = I_{0^+}^\alpha f(t) = f * \phi_\alpha(t)$, where $\phi_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$, $\phi_\alpha(t) = 0$ for $t \leq 0$, and $*$ represents the Convolution of Laplace. Then, it is well known that $\phi_\alpha(t) \rightarrow \delta(t)$ as $\alpha \rightarrow 0$, where δ is the Delta function.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu, \mu \geq -1$ is defined as

$$I^\alpha f(t) = \phi_\alpha * f(t), \quad \alpha > 0, t > 0, \quad (2.2)$$

$$I^0 f(x) = f(x).$$

Definition 2.4. The Caputo's derivative of fractional order $\alpha > 0$ for a suitable function $f(t)$ is defined by

$$({}^c D^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \quad n-1 < \alpha < n, n = [\alpha] + 1, \quad (2.3)$$

where $[\alpha]$ denotes the integer part of real number α .

It is obvious that the Caputo's derivative of a constant is equal to 0.

Lemma 2.5. Let $\alpha > 0$ and $n = [\alpha] + 1$. Then

$$I^\alpha ({}^c D^\alpha f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k. \quad (2.4)$$

Lemma 2.6. If $Q(\tau, \alpha) = \Gamma(\alpha) I_{1^-}^\alpha g(\tau) = \int_\tau^1 g(s)(s-\tau)^{\alpha-1} ds$ for $\tau \in [0, 1]$, and if $g \in L^1([0, 1], R_+)$ satisfies $0 \leq g(s) \leq 1$ for $0 \leq s \leq 1$ and $\alpha > 0$, then

$$\frac{Q(\tau, \alpha)}{\Gamma(\alpha)} < e, \quad \frac{\int_0^t (t-s)^{\alpha-1} ds}{\Gamma(\alpha)} < e. \quad (2.5)$$

Proof. A direct computation shows

$$\begin{aligned} \frac{Q(\tau, \alpha)}{\Gamma(\alpha)} &= \frac{\int_{\tau}^1 g(s)(s-\tau)^{\alpha-1} ds}{\int_0^{\infty} s^{\alpha-1} e^{-s} ds} = \frac{\int_{\tau}^1 (s-\tau)^{\alpha-1} ds}{\int_0^{\infty} s^{\alpha-1} e^{-s} ds} \\ &= \frac{\int_0^{1-\tau} s^{\alpha-1} ds}{\int_0^{\infty} s^{\alpha-1} e^{-s} ds} \leq \frac{e \int_0^{1-\tau} s^{\alpha-1} e^{-s} ds}{\int_0^{\infty} s^{\alpha-1} e^{-s} ds} < e, \end{aligned} \quad (2.6)$$

$$\frac{\int_0^t (t-s)^{\alpha-1} ds}{\Gamma(\alpha)} = \frac{\int_0^t s^{\alpha-1} ds}{\int_0^{\infty} s^{\alpha-1} e^{-s} ds} \leq \frac{e \int_0^t s^{\alpha-1} e^{-s} ds}{\int_0^{\infty} s^{\alpha-1} e^{-s} ds} < e. \quad \square$$

Theorem 2.7 (Krasnoselkii). *Let X be a Banach space, let S be a bounded closed convex subset of X and let A, B be maps of S into X such that $Ax + By \in S$ for every pair $x, y \in S$. If A is completely continuous and B is a contraction then the equation $Ax + Bx = x$ has a solution on S .*

3. Main Results

We assume the following.

- (A1) If $f \in C([0, 1] \times E \times E \times E, E)$ and a nonnegative, bounded $p_f \in L^1([0, 1], \mathbb{R}^+)$, there exist $M > 0$, $p_f(t) \leq M$ for $t \in [0, 1]$ such that

$$\|f(t, x, Kx, Hx)\| \leq p_f(t)\|x\| \quad \text{for } x \in E. \quad (3.1)$$

- (A2) There exist positive constants L_1, L_2 , and L such that

$$\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \leq L_1(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|) \quad (3.2)$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in Y$, $L_2 = \max_{t \in J} \|f(t, 0, 0, 0)\|$, and $L = \max\{L_1, L_2\}$.

- (A3) There exist positive constants N_1, N_2 , and N such that

$$\|k(t, s, x_1) - k(t, s, x_2)\| \leq N_1\|x_1 - x_2\| \quad (3.3)$$

for all $x_1, x_2 \in Y$, $N_2 = \max_{(t,s) \in D_0} \|k(t, s, 0)\|$, and $N = \max\{N_1, N_2\}$.

- (A4) There exist positive constants C_1, C_2 , and C such that

$$\|h(t, s, x_1) - h(t, s, x_2)\| < C_1\|x_1 - x_2\| \quad (3.4)$$

for all $x_1, x_2 \in Y$, $C_2 = \max_{(t,s) \in D_0} \|h(t, s, 0)\|$, and $C = \max\{C_1, C_2\}$.

- (A5) $p = (L/\Gamma(\alpha + 1))(1 + C + N/(\alpha + 1))$ is such that $0 \leq p < 1$.

Firstly, we obtain the following lemmas to prove the main results on the existence of solutions to (1.2).

Lemma 3.1. *If (A1) holds with $\mu = \int_0^1 g(s)ds$, then the problem (1.2) is equivalent to the following equation:*

$$x(t) = \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) f(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau + \phi_\alpha(t) * f(t, x(t), Kx(t), Hx(t)). \tag{3.5}$$

Proof. By Lemma 2.5 and (1.2), we have

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Kx(s), Hx(s)) ds. \tag{3.6}$$

Therefore,

$$\begin{aligned} x(0) &= \int_0^1 g(s)x(s)ds \\ &= \int_0^1 g(s)dsx(0) + \frac{1}{\Gamma(\alpha)} \int_0^1 g(s) \int_0^s (s-\tau)^{\alpha-1} f(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau ds. \end{aligned} \tag{3.7}$$

So,

$$\begin{aligned} x(0) &= \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 f(\tau, x(\tau), Kx(\tau), Hx(\tau)) \left[\int_\tau^1 (s-\tau)^{\alpha-1} g(s) ds \right] d\tau \\ &= \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) f(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau, \end{aligned} \tag{3.8}$$

and then

$$x(t) = \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) f(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau + \phi_\alpha(t) * f(t, x(t), Kx(t), Hx(t)). \tag{3.9}$$

Conversely, if x is a solution of (3.5), then for every $t \in [0, 1]$, according to Definition 2.4 we have

$$\begin{aligned} {}^c D^\alpha x(t) &= {}^c D^\alpha \left[\frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) f(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau \right. \\ &\quad \left. + \phi_\alpha(t) * f(t, x(t), Kx(t), Hx(t)) \right] \\ &= \theta + {}^c D^\alpha (I^\alpha f(t, x(t), Kx(t), Hx(t))) \\ &= f(t, x(t), Kx(t), Hx(t)). \end{aligned} \tag{3.10}$$

It is obvious that $x(0) = \int_0^1 g(s)x(s)ds$. This completes the proof. □

Lemma 3.2. *If (A3) and (A4) are satisfied, K, H are defined in (A), then the conditions*

$$\begin{aligned}\|Kx(t)\| &\leq t(N_1\|x\| + N_2), \\ \|Kx_1(t) - Kx_2(t)\| &\leq N_1t\|x_1 - x_2\|, \\ \|Hx(t)\| &\leq (C_1\|x\| + C_2), \\ \|Hx_1(t) - Hx_2(t)\| &\leq C_1\|x_1 - x_2\|,\end{aligned}\tag{3.11}$$

are satisfied for any $t \in J$, and $x, x_1, x_2 \in Y$.

Proof. By (A3), we have

$$\begin{aligned}\|Kx(t)\| &\leq \int_0^t \|k(t, s, x(s))\| ds \\ &= \int_0^t \|k(t, s, x(s)) - k(t, s, 0) + k(t, s, 0)\| ds \\ &\leq \int_0^t \|k(t, s, x(s)) - k(t, s, 0)\| ds + \int_0^t \|k(t, s, 0)\| ds \\ &\leq N_1t\|x\| + N_2t \leq (N_1\|x\| + N_2).\end{aligned}\tag{3.12}$$

On the other hand,

$$\begin{aligned}\|Kx_1(t) - Kx_2(t)\| &\leq \int_0^t \|k(t, s, x_1(s)) - k(t, s, x_2(s))\| ds \\ &\leq N_1 \int_0^t \|x_1(s) - x_2(s)\| ds \\ &\leq N_1t\|x_1 - x_2\|.\end{aligned}\tag{3.13}$$

Similarly, for the other conditions, we use assumption (A4), to get

$$\begin{aligned}\|Hx(t)\| &\leq \int_0^1 \|h(t, s, x(s))\| ds \leq (C_1\|x\| + C_2), \\ \|Kx_1(t) - Kx_2(t)\| &\leq C_1\|x_1 - x_2\|.\end{aligned}\tag{3.14}$$

□

Theorem 3.3. *If (A1)–(A5) are satisfied, then the fractional integrodifferential equation (1.2) has a unique solution continuous in J .*

Proof. We use the Banach contraction principle to prove the existence and uniqueness of the solution to (1.2). Let $B_r = \{x \in Y : \|x\| \leq r\} \subseteq Y$, where $r \geq (eM/(1 - \mu) + (L/\Gamma(\alpha + 1))(1 + C + N/(\alpha + 1)))$ and define the operator Ψ on the Banach space Y by

$$\begin{aligned}\Psi(x(t)) &= \frac{1}{(1 - \mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) f(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s), Kx(s), Hx(s)) ds.\end{aligned}\tag{3.15}$$

Firstly, we show that the operator Ψ maps B_r into itself. By using (A1) and triangle inequality, we have

$$\begin{aligned}
 \|\Psi(x(t))\| &\leq \frac{eM}{1-\mu}\|x\| + \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}f(s,x(s),Kx(s),Hx(s))ds\right\| \\
 &\leq \frac{eM}{1-\mu}\|x\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s),Kx(s),Hx(s))\|ds \\
 &\leq \frac{eM}{1-\mu}\|x\| \\
 &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s),Kx(s),Hx(s)) - f(s,0,0,0) + f(s,0,0,0)\|ds \\
 &\leq \frac{eM}{1-\mu}\|x\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s),Kx(s),Hx(s)) - f(s,0,0,0)\|ds \\
 &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,0,0,0)\|ds.
 \end{aligned} \tag{3.16}$$

Now, if (A2) is satisfied, then

$$\begin{aligned}
 \|\Psi(x(t))\| &\leq \frac{eM}{1-\mu}\|x\| + \frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\|x(s)\| + \|Kx(s)\| + \|Hx(s)\|)ds \\
 &\quad + \frac{L_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
 &\leq \frac{eM}{1-\mu}\|x\| + \frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|ds + \frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|Kx(s)\|ds \\
 &\quad + \frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|Hx(s)\|ds + \frac{L_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds.
 \end{aligned} \tag{3.17}$$

Using Lemma 3.2 and (A3), we have

$$\begin{aligned}
 \|\Psi(x(t))\| &\leq \frac{eM}{1-\mu}\|x\| + \frac{L_1}{\Gamma(\alpha+1)}t^\alpha\|x\| + \frac{L_1}{\Gamma(\alpha)}(N_1\|x\| + N_2)\int_0^t (t-s)^{\alpha-1}(s)ds \\
 &\quad + \frac{L_1}{\Gamma(\alpha+1)}(C_1\|x\| + C_2)t^\alpha + \frac{L_2}{\Gamma(\alpha+1)}t^\alpha \\
 &\leq \frac{eM}{1-\mu}\|x\| + \frac{L_1}{\Gamma(\alpha+1)}t^\alpha\|x\| + \frac{L_1}{\Gamma(\alpha+2)}t^{\alpha+1}(N_1\|x\| + N_2) \\
 &\quad + \frac{L_1}{\Gamma(\alpha+1)}(C_1\|x\| + C_2)t^\alpha + \frac{L_2}{\Gamma(\alpha+1)}t^\alpha \\
 &= \frac{eM}{1-\mu}\|x\| + \frac{L_1N_2}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{L_1C_2}{\Gamma(\alpha+1)}t^\alpha + \frac{L_2}{\Gamma(\alpha+1)}t^\alpha \\
 &\quad + \frac{L_1}{\Gamma(\alpha+1)}t^\alpha\left(1 + \frac{N_1}{\alpha+1}t + C_1\right)\|x\|,
 \end{aligned} \tag{3.18}$$

if $x \in B_r$, we have

$$\begin{aligned} \|\Psi(x(t))\| &\leq \frac{eM}{1-\mu}\|x\| + \frac{L}{\Gamma(\alpha+1)}\left(1 + \frac{N}{\alpha+1} + C\right) \\ &\quad + \frac{Lr}{\Gamma(\alpha+1)}\left(1 + \frac{N}{\alpha+1} + C\right) \\ &\leq r. \end{aligned} \quad (3.19)$$

Thus $\Psi B_r \subset B_r$. Next, we prove that Ψ is a contraction mapping. For this, let $x_1, x_2 \in Y$. Applying (A2), we have

$$\begin{aligned} &\|\Psi x_1(t) - \Psi x_2(t)\| \\ &= \left\| \frac{Q(\tau, \alpha)}{(1-\mu)\Gamma(\alpha)} \int_0^1 [f(s, x_1(\tau), Kx_1(\tau), Hx_1(\tau)) - f(s, x_2(\tau), Kx_2(\tau), Hx_2(\tau))] d\tau \right\| \\ &\quad + \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_1(s), Kx_1(s), Hx_1(s)) ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_2(s), Kx_2(s), Hx_2(s)) ds \right\| \\ &\leq \frac{e}{(1-\mu)} \int_0^1 [L_1 \|x_1(\tau) - x_2(\tau)\| + \|Kx_1(\tau) - Kx_2(\tau)\| + \|Hx_1(\tau) - Hx_2(\tau)\|] d\tau \\ &\quad + \frac{L_1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} \|x_1(s) - x_2(s)\| ds \right. \\ &\quad \left. + \int_0^t (t-s)^{\alpha-1} \|Kx_1(s) - Kx_2(s)\| ds + \int_0^t (t-s)^{\alpha-1} \|Hx_1(s) - Hx_2(s)\| ds \right) \end{aligned} \quad (3.20)$$

then using (A3), (A4) and Lemma 3.2, one gets

$$\begin{aligned} &\|\Psi x_1(t) - \Psi x_2(t)\| \\ &\leq \frac{L_1 e}{(1-\mu)} \|x_1 - x_2\| \left(\int_0^1 d\tau + N_1 \int_0^1 \tau d\tau + C_1 \int_0^1 d\tau \right) \\ &\quad + \frac{L_1}{\Gamma(\alpha)} \|x_1 - x_2\| \left(\int_0^t (t-s)^{\alpha-1} ds + N_1 \int_0^t (t-s)^{\alpha-1}(s) ds + C_1 \int_0^t (t-s)^{\alpha-1} ds \right) \\ &\leq \frac{L_1 e}{(1-\mu)} \left(1 + \frac{N_1}{2} + C_1 \right) \|x_1 - x_2\| + \frac{L_1}{\Gamma(\alpha)} \left(\frac{t^\alpha}{\alpha} + \frac{N_1 \Gamma(\alpha) t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{C_1 t^\alpha}{\alpha} \right) \|x_1 - x_2\| \\ &= \left[\frac{L_1 e}{(1-\mu)} \left(1 + \frac{N_1}{2} + C_1 \right) + \frac{L_1}{\Gamma(\alpha+1)} \left(1 + C_1 + \frac{N_1}{\alpha+1} (t) t^\alpha \right) \right] \|x_1 - x_2\| \\ &\leq \left[\frac{L e}{(1-\mu)} \left(1 + \frac{N}{2} + C_1 \right) + \frac{L}{\Gamma(\alpha+1)} \left(1 + C + \frac{N}{\alpha+1} \right) \right] \|x_1 - x_2\| \\ &\leq \left[\frac{L e}{(1-\mu)} \left(1 + \frac{N}{2} + C \right) + p \right] \|x_1 - x_2\| \\ &\leq \Omega_{C, N, \alpha, \mu, p} \|x_1 - x_2\|, \end{aligned} \quad (3.21)$$

where $\Omega_{C,N,\alpha,\mu,p} = [(Le/(1-\mu))(1+(N/2)+C)+p] < 1$ depends on the parameter of the problem. Therefore Ψ has a unique fixed-point $x = \Psi(x) \in B_r$, which is a solution of (3.5), and hence is a solution of (1.2). \square

Theorem 3.4. *Assume (A1)–(A4) holds. If $eM < 1 - \mu$, then (1.2) has at least one solution on I .*

Proof. Choose $r \geq eM/(1-\mu) + \|p_f\|_{L^1}/\Gamma(\alpha+1)$ and consider $B_r : \{x \in C : \|x\| \leq r\}$. Now define on B_r the operators A, B by

$$\begin{aligned} (Ax)(t) &:= \phi_\alpha(t) * f(t, x(t), Kx(t), Hx(t)), \\ (Bx)(t) &:= \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) f(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau. \end{aligned} \tag{3.22}$$

Let us observe that if $x, y \in B_r$ then $Ax + By \in B_r$. Indeed it is easy to check the inequality

$$\|Ax + By\| \leq \frac{eM}{(1-\mu)} + \frac{\|p_f\|_{L^1}}{\Gamma(\alpha+1)} \leq r. \tag{3.23}$$

We can easily show that that B is a contraction mapping. Let $u, v \in B_r$. Then

$$\begin{aligned} &\|(Bu)(t) - (Bv)(t)\| \\ &:= \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) \|f(\tau, u(\tau), Ku(\tau), Hu(\tau)) - f(\tau, v(\tau), Kv(\tau), Hv(\tau))\| d\tau \\ &\leq \frac{e}{(1-\mu)} \int_0^1 [L_1 \|u(\tau) - v(\tau)\| + \|Ku(\tau) - Kv(\tau)\| + \|Hu(\tau) - Hv(\tau)\|] d\tau \\ &\leq \frac{L_1 e}{(1-\mu)} \left(1 + \frac{N_1}{2} + C_1\right) \|u - v\| \\ &\leq \Lambda_{N_1, C_1, L_1} \|u - v\|, \end{aligned} \tag{3.24}$$

where $\Lambda_{N_1, C_1, L_1} = (L_1 e/(1-\mu))(1+(N_1/2)+C_1) < 1$ depends only on the parameter of the problem and hence B is contraction. Since x is continuous, then $(Ax)(t)$ is continuous in view of (A1). Let us now note that A is uniformly bounded on B_r . This follows from the inequality

$$\|(Ax)(t)\| \leq \frac{\|p_f\|_{L^1}}{\Gamma(\alpha+1)}. \tag{3.25}$$

Now let us prove that $(Ax)(t)$ is equicontinuous.

Let $t_1, t_2 \in J$ and $x \in B_r$. Using the fact that f is bounded on the compact set $J \times B_r$ (thus $\sup_{(t,s) \in J \times B_r} \|f(t, x(s), Kx(s), Hx(s))\| := c_0 < \infty$), we will get

$$\begin{aligned}
\|Ax(t_1) - Ax(t_2)\| &= \frac{1}{\Gamma(q)} \left\| \int_0^{t_1} (t_1 - s)^{q-1} \int_0^s f(s, x(s), Kx(s), Hx(s)) ds \right. \\
&\quad \left. - \int_0^{t_2} (t_2 - s)^{q-1} \int_0^s f(s, x(s), Kx(s), Hx(s)) ds \right\| \\
&= \frac{1}{\Gamma(q)} \left\| \int_{t_2}^{t_1} (t_1 - s)^{q-1} \int_0^s f(s, x(s), Kx(s), Hx(s)) ds \right. \\
&\quad \left. - \int_0^{t_2} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \int_0^s f(s, x(s), Kx(s), Hx(s)) ds \right\| \\
&\leq \frac{1}{\Gamma(q)} \left(\left\| \int_{t_2}^{t_1} (t_1 - s)^{q-1} \int_0^s f(s, x(s), Kx(s), Hx(s)) ds \right\| \right) \\
&\quad + \frac{1}{\Gamma(q)} \left(\left\| \int_0^{t_2} (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \int_0^s f(s, x(s), Kx(s), Hx(s)) ds \right\| \right) \\
&\leq \frac{c_0}{\Gamma(q+1)} |2(t_1 - t_2)^q + t_2^q - t_1^q| \\
&\leq \frac{2c_0}{\Gamma(q+1)} |t_1 - t_2|^q,
\end{aligned} \tag{3.26}$$

which does not depend on x . So $A(B_r)$ is relatively compact. By the Arzela-Ascoli Theorem, A is compact. We now conclude the result of the theorem based on the Krasnoselkii's theorem above. \square

4. Example

Consider the following fractional integrodifferential equation:

$$\begin{aligned}
{}^c D^\alpha x(t) &= \frac{e^{-t} x(t)}{(9 + e^t)(1 + x(t))} + \frac{1}{10} \int_0^t e^{-(1/2)x(s)} ds + \frac{1}{10} \int_0^t e^{-(1/49)x(s)} ds, \quad t \in J = [0, 1], \\
x(0) &= \int_0^1 \frac{1}{2} x(s) ds,
\end{aligned} \tag{4.1}$$

where $0 < \alpha \leq 1$. Take $E = R^+$. Set $Kx(t) = \int_0^t e^{-(1/2)x(s)} ds$, $Hx(t) = \int_0^t e^{-(1/49)x(s)} ds$, $f(t, x, Kx, Hx) = e^{-t}x/(9 + e^t)(1 + x) + Kx + Hx$, $g(s) = 1/2$, $p_f(t) = e^{-t}x(t)/(9 + e^t)$. Then it is clear that

$$\begin{aligned} f &\in C([0, 1] \times E \times E \times E, E), \quad p_f(t) \leq \frac{1}{10} = M, \\ p_f &\in L([0, 1], R^+), \quad \|f(t, x, Kx, Hx)\| \leq p_f(x). \end{aligned} \tag{4.2}$$

So, (A1) is satisfied. Let $x, y \in E$ and $t \in J$. Then we have

$$\begin{aligned} \|Kx - Ky\| &= \left| \int_0^t e^{-(1/2)x(s)} ds - \int_0^t e^{-(1/2)y(s)} ds \right| \leq \frac{1}{2}|x - y|, \\ \|Hx - Hy\| &= \left| \int_0^t e^{-(1/49)x(s)} ds - \int_0^t e^{-(1/49)y(s)} ds \right| \leq \frac{1}{49}|x - y|, \\ \|f(t, x, Kx, Hx) - f(t, y, y, Hy)\| &= \left| \frac{e^{-t}x}{(9 + e^t)(1 + x)} - \frac{e^{-t}y}{(9 + e^t)(1 + y)} + \frac{1}{10}((Kx - Ky) + (Hx - Hy)) \right| \\ &\leq \frac{e^{-t}|x - y|}{(9 + e^t)(1 + x)(1 + y)} + \frac{1}{10}(\|Kx - Ky\| + \|Hx - Hy\|) \\ &\leq \frac{e^{-t}}{9 + e^t}|x - y| + \frac{1}{10}(\|Kx - Ky\| + \|Hx - Hy\|) \\ &\leq \frac{1}{10}(|x - y| + \|Kx - Ky\| + \|Hx - Hy\|). \end{aligned} \tag{4.3}$$

Hence the conditions (A1)–(A4) hold with $M = 1/10$, $L_1 = 1/10$, $N_1 = 1/2$, $C_1 = 1/49$. Choose $r = 1$ and $\mu = 1/2$. Indeed

$$\begin{aligned} \frac{L_1 e}{(1 - \mu)} \left(1 + \frac{N_1}{2} + C_1 \right) &= \frac{149e}{490} < 1, \\ \frac{eM}{1 - \mu} &= \frac{e}{5} < 1. \end{aligned} \tag{4.4}$$

Further (A5) is satisfied by a suitable choice of α . Then by Lemma 3.2 the problem (1.2) has a unique solution on $[0, 1]$.

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