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Research Article

Composition Theorems of Stepanov Almost Periodic Functions and Stepanov-Like Pseudo-Almost Periodic Functions

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We establish a composition theorem of Stepanov almost periodic functions, and, with its help, a composition theorem of Stepanov-like pseudo almost periodic functions is obtained. In addition, we apply our composition theorem to study the existence and uniqueness of pseudo-almost periodic solutions to a class of abstract semilinear evolution equation in a Banach space. Our results complement a recent work due to Diagana (2008).

1. Introduction

Recently, in [1, 2], Diagana introduced the concept of Stepanov-like pseudo-almost periodicity, which is a generalization of the classical notion of pseudo-almost periodicity, and established some properties for Stepanov-like pseudo-almost periodic functions. Moreover, Diagana studied the existence of pseudo-almost periodic solutions to the abstract semilinear evolution equation $u'(t) = A(t)u(t) + f(t, u(t))$. The existence theorems obtained in [1, 2] are interesting since $f(\cdot, u)$ is only Stepanov-like pseudo-almost periodic, which is different from earlier works. In addition, Diagana et al. [3] introduced and studied Stepanov-like weighted pseudo-almost periodic functions and their applications to abstract evolution equations.

On the other hand, due to the work of [4] by N'Guérékata and Pankov, Stepanov-like almost automorphic problems have widely been investigated. We refer the reader to [5–11] for some recent developments on this topic.

Since Stepanov-like almost-periodic (almost automorphic) type functions are not necessarily continuous, the study of such functions will be more difficult considering complexity and more interesting in terms of applications.

Very recently, in [12], Li and Zhang obtained a new composition theorem of Stepanov-like pseudo-almost periodic functions; the authors in [13] established a composition theorem of vector-valued Stepanov almost-periodic functions. Motivated by [2, 12, 13], in this paper, we will make further study on the composition theorems of Stepanov almost-periodic functions and Stepanov-like pseudo-almost periodic functions. As one will see, our main results extend and complement some results in [2, 13].

Throughout this paper, let $\mathbb{R}$ be the set of real numbers, let $\operatorname{mes}E$ be the Lebesgue measure for any subset $E \subset \mathbb{R}$, and X, Y be two arbitrary real Banach spaces. Moreover, we assume that $1 \leq p < +\infty$ if there is no special statement. First, let us recall some definitions and basic results of almost periodic functions, Stepanov almost periodic functions, pseudo-almost periodic functions, and Stepanov-like pseudo-almost periodic functions (for more details, see [2, 14, 15]).

Definition 1.1. A set $E \subset \mathbb{R}$ is called relatively dense if there exists a number $l > 0$ such that

$$(a, a+l) \cap E \neq \emptyset, \quad \forall a \in \mathbb{R}. \tag{1.1}$$

Definition 1.2. A continuous function $f : \mathbb{R} \to X$ is called almost periodic if for each $\varepsilon > 0$ there exists a relatively dense set $P(\varepsilon, f) \subset \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \| f(t+\tau) - f(t) \| < \varepsilon, \quad \forall \tau \in P(\varepsilon, f). \tag{1.2}$$

We denote the set of all such functions by $AP(\mathbb{R}, X)$ or $AP(X)$.

Definition 1.3. A continuous function $f : \mathbb{R} \times X \to Y$ is called almost periodic in t uniformly for $x \in X$ if, for each $\varepsilon > 0$ and each compact subset $K \subset X$, there exists a relatively dense set $P(\varepsilon, f, K) \subset \mathbb{R}$

$$\sup_{t\in\mathbb{R}} \| f(t+\tau, x) - f(t, x) \| < \varepsilon, \quad \forall \tau \in P(\varepsilon, f, K),\ \forall x \in K. \tag{1.3}$$

We denote by $AP(\mathbb{R} \times X, Y)$ the set of all such functions.

Definition 1.4. The Bochner transform $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, of a function $f(t)$ on $\mathbb{R}$, with values in X, is defined by

$$f^b(t, s) := f(t+s). \tag{1.4}$$

Definition 1.5. The space $BS^p(X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in X such that

$$\|f\|_{S^p} := \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \right)^{1/p} < +\infty \tag{1.5}$$

It is obvious that $L^p(\mathbb{R}; X) \subset BS^p(X) \subset L^p_{\text{loc}}(\mathbb{R}; X)$ and $BS^p(X) \subset BS^q(X)$ whenever $p \geq q \geq 1$.

Definition 1.6. A function $f \in BS^p(X)$ is called Stepanov almost periodic if $f^b \in AP(L^p(0,1;X))$; that is, for all $\varepsilon > 0$, there exists a relatively dense set $P(\varepsilon, f) \subset \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}\left(\int_0^1 \|f(t+s+\tau)-f(t+s)\|^p ds\right)^{1/p} < \varepsilon, \quad \forall \tau \in P(\varepsilon, f). \tag{1.6}$$

We denote the set of all such functions by $APS^p(\mathbb{R}, X)$ or $APS^p(X)$.

Remark 1.7. It is clear that $AP(X) \subset APS^p(X) \subset APS^q(X)$ for $p \geq q \geq 1$.

Definition 1.8. A function $f : \mathbb{R}\times X \to Y$, $(t,u) \mapsto f(t,u)$ with $f(\cdot, u) \in BS^p(Y)$, for each $u \in X$, is called Stepanov almost periodic in $t \in \mathbb{R}$ uniformly for $u \in X$ if, for each $\varepsilon > 0$ and each compact set $K \subset X$, there exists a relatively dense set $P(\varepsilon, f, K) \subset \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}\left(\int_0^1 \left\|f(t+s+\tau,u)-f(t+s,u)\right\|^p ds\right)^{1/p} < \varepsilon, \tag{1.7}$$

for each $\tau \in P(\varepsilon, f, K)$ and each $u \in K$. We denote by $APS^p(\mathbb{R}\times X, Y)$ the set of all such functions.

It is also easy to show that $APS^p(\mathbb{R}\times X, Y) \subset APS^q(\mathbb{R}\times X, Y)$ for $p \geq q \geq 1$.

Throughout the rest of this paper, let $C_b(\mathbb{R}, X)$ (resp., $C_b(\mathbb{R}\times X, Y)$) be the space of bounded continuous (resp., jointly bounded continuous) functions with supremum norm, and

$$PAP_0(\mathbb{R}, X) = \left\{\varphi \in C_b(\mathbb{R}, X) : \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left\|\varphi(t)\right\|dt = 0\right\}. \tag{1.8}$$

We also denote by $PAP_0(\mathbb{R}\times X, Y)$ the space of all functions $\varphi \in C_b(\mathbb{R}\times X, Y)$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left\|\varphi(t,x)\right\|dt = 0 \tag{1.9}$$

uniformly for x in any compact set $K \subset X$.

Definition 1.9. A function $f \in C_b(\mathbb{R}, X)$ $(C_b(\mathbb{R}\times X, Y))$ is called pseudo-almost periodic if

$$f = g + \varphi \tag{1.10}$$

with $g \in AP(X)(AP(\mathbb{R}\times X, Y))$ and $\varphi \in PAP_0(\mathbb{R}, X)(PAP_0(\mathbb{R}\times X, Y))$. We denote by $PAP(X)(PAP(\mathbb{R}\times X, Y))$ the set of all such functions.

It is well-known that $PAP(X)$ is a closed subspace of $C_b(\mathbb{R}, X)$, and thus $PAP(X)$ is a Banach space under the supremum norm.

Definition 1.10. A function $f \in BS^p(X)$ is called Stepanov-like pseudo-almost periodic if it can be decomposed as $f = g + h$ with $g^b \in AP(\mathbb{R}, L^p(0,1;X))$ and $h^b \in PAP_0(\mathbb{R}, L^p(0,1;X))$. We denote the set of all such functions by $PAPS^p(\mathbb{R}, X)$ or $PAPS^p(X)$.

It follows from [2] that $PAP(X) \subset PAPS^p(X)$ for all $1 \le p < +\infty$.

Definition 1.11. A function $F : \mathbb{R} \times X \to Y$, $(t,u) \mapsto f(t,u)$ with $f(\cdot,u) \in BS^p(Y)$, for each $u \in X$, is called Stepanov-like pseud-almost periodic in $t \in \mathbb{R}$ uniformly for $u \in X$ if it can be decomposed as $F = G+H$ with $G^b \in AP(\mathbb{R}\times X, L^p(0,1;Y))$ and $H^b \in PAP_0(\mathbb{R}\times X, L^p(0,1;Y))$. We denote by $PAPS^p(\mathbb{R} \times X, Y)$ the set of all such functions.

Next, let us recall some notations about evolution family and exponential dichotomy. For more details, we refer the reader to [16].

Definition 1.12. A set $\{U(t,s) : t \ge s,\ t,s \in \mathbb{R}\}$ of bounded linear operator on X is called an evolution family if

(a) $U(s,s) = I, U(t,s) = U(t,r)U(r,s)$ for t $\ge r \ge s$ and $t,r,s \in \mathbb{R}$,

(b) $\{(\tau,\sigma) \in \mathbb{R}^2 : \tau \ge \sigma\} \ni (t,s) \mapsto U(t,s)$ is strongly continuous.

Definition 1.13. An evolution family $U(t,s)$ is called hyperbolic (or has exponential dichotomy) if there are projections $P(t)$, $t \in \mathbb{R}$, being uniformly bounded and strongly continuous in t, and constants $M, \omega > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \ge s$,

(b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \ge s$ (and we set $U_Q(s,t) = U_Q(t,s)^{-1}$),

(c) $\|U(t,s)P(s)\| \le Me^{-\omega(t-s)}$ and $\|U_Q(s,t)Q(t)\| \le Me^{-\omega(t-s)}$ for all $t \ge s$,

where $Q := I - P$. We call that

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \ge s,\ t,s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R}, \end{cases} \tag{1.11}$$

is the Green's function corresponding to $U(t,s)$ and $P(\cdot)$.

Remark 1.14. Exponential dichotomy is a classical concept in the study of long-term behaviour of evolution equations; see, for example, [16]. It is easy to see that

$$\|\Gamma(t,s)\| \le \begin{cases} Me^{-\omega(t-s)}, & t \ge s, t,s \in \mathbb{R}, \\ Me^{-\omega(s-t)}, & t < s,\ t,s \in \mathbb{R}. \end{cases} \tag{1.12}$$

2. Main Results

Throughout the rest of this paper, for $r \ge 1$, we denote by $\mathcal{L}^r(\mathbb{R} \times X, X)$ the set of all the functions $f : \mathbb{R} \times X \to X$ satisfying that there exists a function $L_f \in BS^r(\mathbb{R})$ such that

$$\|f(t,u) - f(t,v)\| \le L_f(t)\|u - v\|, \quad \forall t \in \mathbb{R},\ \forall u,v \in X, \tag{2.1}$$

and, for any compact set $K \subset X$, we denote by $APS_K^p(\mathbb{R} \times X, Y)$ the set of all the functions $f \in APS^p(\mathbb{R} \times X, Y)$ such that (1.7) is replaced by

$$\sup_{t\in\mathbb{R}} \left[\int_0^1 \left(\sup_{u\in K} \| f(t+s+\tau, u) - f(t+s, u) \| \right)^p ds \right]^{1/p} < \varepsilon. \tag{2.2}$$

In addition, we denote by $\| \cdot \|_p$ the norm of $L^p(0,1;X)$ and $L^p(0,1;\mathbb{R})$.

Lemma 2.1. *Let $p \geq 1$, $K \subset X$ be compact, and $f \in APS^p(\mathbb{R} \times X, X) \bigcap \mathcal{L}^p(\mathbb{R} \times X, X)$. Then $f \in APS_K^p(\mathbb{R} \times X, X)$.*

Proof. For all $\varepsilon > 0$, there exist $x_1, \ldots, x_k \in K$ such that

$$K \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon). \tag{2.3}$$

Since $f \in APS^p(\mathbb{R} \times X, X)$, for the above $\varepsilon > 0$, there exists a relatively dense set $P(\varepsilon) \subset \mathbb{R}$ such that

$$\| f(t+\tau+\cdot, u) - f(t+\cdot, u) \|_p < \frac{\varepsilon}{k}, \tag{2.4}$$

for all $\tau \in P(\varepsilon)$, $t \in \mathbb{R}$, and $u \in K$. On the other hand, since $f \in \mathcal{L}^p(\mathbb{R} \times X, X)$, there exists a function $L_f \in BS^p(\mathbb{R})$ such that (2.1) holds.

Fix $t \in \mathbb{R}$, $\tau \in P(\varepsilon)$. For each $u \in K$, there exists $i(u) \in \{1, 2, \ldots, k\}$ such that $\|u - x_{i(u)}\| < \varepsilon$. Thus, we have

$$\begin{aligned} &\| f(t+s+\tau, u) - f(t+s, u) \| \\ &\quad \leq L_f(t+s+\tau)\varepsilon + \left\| f\left(t+s+\tau, x_{i(u)}\right) - f\left(t+s, x_{i(u)}\right) \right\| + L_f(t+s)\varepsilon, \end{aligned} \tag{2.5}$$

for each $u \in K$ and $s \in [0,1]$, which gives that

$$\begin{aligned} &\sup_{u\in K} \| f(t+s+\tau, u) - f(t+s, u) \| \\ &\quad \leq \left[L_f(t+s+\tau) + L_f(t+s) \right]\varepsilon + \sum_{i=1}^{k} \| f(t+s+\tau, x_i) - f(t+s, x_i) \|, \quad \forall s \in [0,1]. \end{aligned} \tag{2.6}$$

Now, by Minkowski's inequality and (2.4), we get

$$\begin{aligned}
&\left[\int_0^1 \left(\sup_{u\in K}\| f(t+s+\tau,u)-f(t+s,u)\|\right)^p ds\right]^{1/p} \\
&\quad\le \left[\int_0^1 L_f^p(t+s+\tau)ds\right]^{1/p}\cdot\varepsilon+\left[\int_0^1 L_f^p(t+s)ds\right]^{1/p}\cdot\varepsilon \\
&\qquad+\sum_{i=1}^k\left[\int_0^1 \| f(t+s+\tau,x_i)-f(t+s,x_i)\|^p ds\right]^{1/p} \\
&\quad\le \left(2\|L_f\|_{S^p}+1\right)\varepsilon,
\end{aligned} \tag{2.7}$$

which means that $f\in APS_K^p(\mathbb{R}\times X,X)$. □

Theorem 2.2. *Assume that the following conditions hold:*

(a) $f\in APS^p(\mathbb{R}\times X,X)$ *with* $p>1$*, and* $f\in\mathcal{L}^r(\mathbb{R}\times X,X)$ *with* $r\ge\max\{p,p/(p-1)\}$*.*

(b) $x\in APS^p(X)$*, and there exists a set* $E\subset\mathbb{R}$ *with* $\operatorname{mes}E=0$ *such that*

$$K:=\overline{\{x(t):t\in\mathbb{R}\setminus E\}} \tag{2.8}$$

is compact in X.

Then there exists $q\in[1,p)$ *such that* $f(\cdot,x(\cdot))\in APS^q(X)$*.*

Proof. Since $r\ge p/(p-1)$, there exists $q\in[1,p)$ such that $r=pq/(p-q)$. Let

$$p'=\frac{p}{p-q},\qquad q'=\frac{p}{q}. \tag{2.9}$$

Then $p',q'>1$ and $1/p'+1/q'=1$. On the other hand, since $f\in\mathcal{L}^r(\mathbb{R}\times X,X)$, there is a function $L_f\in BS^r(\mathbb{R})$ such that (2.1) holds.

It is easy to see that $f(\cdot,x(\cdot))$ is measurable. By using (2.1), for each $t\in\mathbb{R}$, we have

$$\begin{aligned}
\left(\int_t^{t+1}\|f(s,x(s))\|^q ds\right)^{1/q} &\le \left(\int_t^{t+1}\|f(s,x(s))-f(s,0)\|^q ds\right)^{1/q}+\|f(\cdot,0)\|_{S^q} \\
&\le \left(\int_t^{t+1}L_f^q(s)\|x(s)\|^q ds\right)^{1/q}+\|f(\cdot,0)\|_{S^q} \\
&\le \left(\int_t^{t+1}L_f^r(s)ds\right)^{1/r}\cdot\left(\int_t^{t+1}\|x(s)\|^p dt\right)^{1/p}+\|f(\cdot,0)\|_{S^q} \\
&\le \|L_f\|_{S^r}\cdot\ \|x\|_{S^p}+\|f(\cdot,0)\|_{S^q}<+\infty.
\end{aligned} \tag{2.10}$$

Thus, $f(\cdot,x(\cdot))\in BS^q(X)$.

Next, let us show that $f(\cdot, x(\cdot)) \in APS^q(X)$. By Lemma 2.1, $f \in APS^p_K(\mathbb{R} \times X, X)$. In addition, we have $x \in APS^p(X)$. Thus, for all $\varepsilon > 0$, there exists a relatively dense set $P(\varepsilon) \subset \mathbb{R}$ such that

$$\left[\int_0^1 \left(\sup_{u\in K}\left\| f(t+s+\tau, u) - f(t+s, u)\right\|\right)^p ds\right]^{1/p} < \varepsilon, \qquad \|x(t+\tau+\cdot) - x(t+\cdot)\|_p < \varepsilon \tag{2.11}$$

for all $\tau \in P(\varepsilon)$ and $t \in \mathbb{R}$. By using (2.11), we deduce that

$$\begin{aligned}
&\left(\int_0^1 \left\| f(t+s+\tau, x(t+s+\tau)) - f(t+s, x(t+s))\right\|^q\right)^{1/q} \\
&\quad \le \left(\int_0^1 L_f^q(t+s+\tau)\left\|x(t+s+\tau) - x(t+s)\right\|^q\right)^{1/q} \\
&\qquad + \left(\int_0^1 \left\| f(t+s+\tau, x(t+s)) - f(t+s, x(t+s))\right\|^q\right)^{1/q} \\
&\quad \le \left(\int_0^1 L_f^r(t+s+\tau)dt\right)^{1/r} \cdot \left(\int_0^1 \left\|x(t+s+\tau) - x(t+s)\right\|^p dt\right)^{1/p} \\
&\qquad + \left(\int_0^1 \| f(t+s+\tau, x(t+s)) - f(t+s, x(t+s))\|^p\right)^{1/p} \\
&\quad \le \|L_f\|_{S^r} \cdot \|x(t+\tau+\cdot) - x(t+\cdot)\|_p + \left[\int_0^1 \left(\sup_{u\in K}\left\| f(t+s+\tau, u) - f(t+s, u)\right\|\right)^p ds\right]^{1/p} \\
&\quad \le \left(\|L_f\|_{S^r} + 1\right)\varepsilon
\end{aligned} \tag{2.12}$$

for all $\tau \in P(\varepsilon)$ and $t \in \mathbb{R}$. Thus, $f(\cdot, x(\cdot)) \in APS^q(X)$. □

Lemma 2.3. *Let $K \subset X$ be compact, $f \in \mathcal{L}^p(\mathbb{R} \times X, X)$, and $f^b \in PAP_0(\mathbb{R} \times X, L^p(0,1;X))$. Then $\widetilde{f} \in PAP_0(\mathbb{R}, \mathbb{R})$, where*

$$\widetilde{f}(t) = \left\|\sup_{u\in K}\left\| f(t+\cdot, u)\right\|\right\|_p, \quad t \in \mathbb{R}. \tag{2.13}$$

Proof. Noticing that K is a compact set, for all $\varepsilon > 0$, there exist $x_1, \ldots, x_k \in K$ such that

$$K \subset \bigcup_{i=1}^k B(x_i, \varepsilon). \tag{2.14}$$

Combining this with $f \in \mathcal{L}^p(\mathbb{R} \times X, X)$, for all $u \in K$, there exists x_i such that

$$\|f(t+s,u)\| \le \|f(t+s,u) - f(t+s,x_i)\| + \|f(t+s,x_i)\| \le L_f(t+s)\varepsilon + \|f(t+s,x_i)\| \tag{2.15}$$

for all $t \in \mathbb{R}$ and $s \in [0,1]$. Thus, we get

$$\sup_{u \in K}\|f(t+s,u)\| \le L_f(t+s)\varepsilon + \sum_{i=1}^{k}\|f(t+s,x_i)\|, \quad \forall t \in \mathbb{R},\ \forall s \in [0,1], \tag{2.16}$$

which yields that

$$\widetilde{f}(t) = \left\|\sup_{u \in K}\|f(t+\cdot,u)\|\right\|_p \le \|L\|_{S^p} \cdot \varepsilon + \sum_{i=1}^{k}\|f^b(t,x_i)\|_p, \quad \forall t \in \mathbb{R}. \tag{2.17}$$

On the other hand, since $f^b \in PAP_0(\mathbb{R} \times X, L^p(0,1;X))$, for the above $\varepsilon > 0$, there exists $T_0 > 0$ such that, for all $T > T_0$,

$$\frac{1}{2T}\int_{-T}^{T}\|f^b(t,x_i)\|_p dt < \frac{\varepsilon}{k}, \quad i = 1,2,\ldots,k. \tag{2.18}$$

This together with (2.17) implies that

$$\frac{1}{2T}\int_{-T}^{T}\widetilde{f}(t)dt \le \left(\|L_f\|_{S^p} + 1\right)\varepsilon. \tag{2.19}$$

Hence, $\widetilde{f} \in PAP_0(\mathbb{R},\mathbb{R})$. □

Theorem 2.4. *Assume that $p > 1$ and the following conditions hold:*

(a) $f = g + h \in PAPS^p(\mathbb{R} \times X, X)$ *with* $g^b \in AP(\mathbb{R} \times X, L^p(0,1;X))$ *and* $h^b \in PAP_0(\mathbb{R} \times X, L^p(0,1;X))$. *Moreover,* $f, g \in \mathcal{L}^r(\mathbb{R} \times X, X)$ *with* $r \ge \max\{p, p/(p-1)\}$;

(b) $x = y + z \in PAPS^p(X)$ *with* $y^b \in AP(\mathbb{R}, L^p(0,1;X))$ *and* $z^b \in PAP_0(\mathbb{R}, L^p(0,1;X))$, *and there exists a set* $E \subset \mathbb{R}$ *with* $\operatorname{mes} E = 0$ *such that*

$$K := \overline{\{y(t) : t \in \mathbb{R} \setminus E\}} \tag{2.20}$$

is compact in X.

Then there exists $q \in [1,p)$ *such that* $f(\cdot, x(\cdot)) \in PAPS^q(X)$.

Proof. Let $p,\ p',$ and q' be as in the proof of Theorem 2.2. In addition, let $f(t,x(t)) = H(t) + I(t) + J(t)$, where

$$H(t) = g(t, y(t)), \qquad I(t) = f(t,x(t)) - f(t,y(t)), \qquad J(t) = h(t,y(t)). \tag{2.21}$$

It follows from Theorem 2.2 that $H \in APS^q(X)$, that is, $H^b \in AP(\mathbb{R}, L^q(0,1;X))$.

Next, let us show that $I^b, J^b \in PAP_0(\mathbb{R}, L^q(0,1;X))$. For I^b, we have

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \|I^b(t)\|_q dt &= \frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \|I(t+s)\|^q ds\right)^{1/q} dt \\
&\le \frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 L_f^q(t+s)\|z(t+s)\|^q ds\right)^{1/q} dt \\
&\le \|L_f\|_{S^r}\frac{1}{2T}\int_{-T}^{T}\|z^b(t)\|_p dt \longrightarrow 0, \quad (T \longrightarrow +\infty),
\end{aligned} \tag{2.22}$$

where $z^b \in PAP_0(\mathbb{R}, L^p(0,1;X))$ was used. For J^b, since $h = f - g \in \mathcal{L}^r(\mathbb{R}\times X, X) \subset \mathcal{L}^p(\mathbb{R}\times X, X)$, by Lemma 2.3, we know that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left\|\sup_{u\in K}\|h(t+\cdot,u)\|\right\|_p dt = 0, \tag{2.23}$$

which yields

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\|J^b(t)\|_q dt &\le \frac{1}{2T}\int_{-T}^{T}\|J^b(t)\|_p dt \\
&= \frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \|h(t+s, y(t+s))\|^p ds\right)^{1/p} dt \\
&\le \frac{1}{2T}\int_{-T}^{T}\left[\int_0^1\left(\sup_{u\in K}\|h(t+s,u)\|\right)^p ds\right]^{1/p} dt \longrightarrow 0 \quad (T \longrightarrow +\infty),
\end{aligned} \tag{2.24}$$

that is, $J^b \in PAP_0(\mathbb{R}, L^q(0,1;X))$. Now, we get $f(\cdot, x(\cdot)) \in PAPS^q(X)$. □

Next, let us discuss the existence and uniqueness of pseudo-almost periodic solutions for the following abstract semilinear evolution equation in X:

$$u'(t) = A(t)u(t) + f(t, u(t)). \tag{2.25}$$

Theorem 2.5. *Assume that $p > 1$ and the following conditions hold:*

(a) $f = g + h \in PAPS^p(\mathbb{R} \times X, X)$ *with* $g^b \in AP(\mathbb{R} \times X, L^p(0,1;X))$ *and* $h^b \in PAP_0(\mathbb{R} \times X, L^p(0,1;X))$. *Moreover,* $f, g \in \mathcal{L}^r(\mathbb{R} \times X, X)$ *with*

$$r \geq \max\left\{p, \frac{p}{p-1}\right\}, \qquad r > \frac{p}{p-1}; \tag{2.26}$$

(b) *the evolution family $U(t,s)$ generated by $A(t)$ has an exponential dichotomy with constants $M, \omega > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function Γ;*

(c) *for all $\varepsilon > 0$, for all $h > 0$, and for all $F \in APS^1(X)$ there exists a relatively dense set $P(\varepsilon) \subset \mathbb{R}$ such that* $\sup_{r\in\mathbb{R}} \|F(r + \cdot + \tau) - f(r + \cdot)\| < \varepsilon$ *and*

$$\sup_{r\in\mathbb{R}} \left\|\Gamma(t + r + \tau, s + r + \tau) - \Gamma(t + r, s + r)\right\| < \varepsilon, \tag{2.27}$$

for all $\tau \in P(\varepsilon)$ and $t, s \in \mathbb{R}$ with $|t - s| \geq h$.

Then (2.25) *has a unique pseudo-almost periodic mild solution provided that*

$$\left\|L_f\right\|_{S^r} < \frac{1 - e^{-\omega}}{2M} \cdot \left(\frac{\omega r'}{1 - e^{-\omega r'}}\right)^{1/r'}, \quad \text{where } (1/r) + (1/r') = 1. \tag{2.28}$$

Proof. Let $u = v + w \in PAP(X)$, where $v \in AP(X)$ and $w \in PAP_0(X)$. Then $u \in PAPS^p(X)$ and $K := \overline{\{v(t) : t \in \mathbb{R}\}}$ is compact in X. By the proof of Theorem 2.4, there exists $q \in (1, p)$ such that $f(\cdot, u(\cdot)) \in PAPS^q(X)$.

Let

$$f(t, u(t)) = f_1(t) + f_2(t), \qquad t \in \mathbb{R}, \tag{2.29}$$

where $f_1^b \in AP(\mathbb{R}, L^q(0,1;X))$ and $f_2^b \in PAP_0(\mathbb{R}, L^q(0,1;X))$. Denote

$$F(u)(t) := \int_{\mathbb{R}} \Gamma(t,s) f(s, u(s)) ds = F_1(u)(t) + F_2(u)(t), \quad t \in \mathbb{R}, \tag{2.30}$$

where

$$F_1(u)(t) = \int_{\mathbb{R}} \Gamma(t,s) f_1(s) ds, \qquad F_2(u)(t) = \int_{\mathbb{R}} \Gamma(t,s) f_2(s) ds. \tag{2.31}$$

By [13, Theorem 2.3] we have $F_1(u) \in AP(\mathbb{X})$. In addition, by a similar proof to that of [2, Theorem 3.2], one can obtain that $F_2(u) \in PAP_0(\mathbb{X})$. So F maps $PAP(\mathbb{X})$ into $PAP(\mathbb{X})$. For $u, v \in PAP(\mathbb{X})$, by using the Hölder's inequality, we obtain

$$\begin{aligned}
\|F(u)(t) - F(v)(t)\| &\le \int_{\mathbb{R}} \|\Gamma(t,s)\| \cdot \|f(s,u(s)) - f(s,v(s))\| ds \\
&\le \int_{-\infty}^{t} M e^{-\omega(t-s)} L_f(s) ds \cdot \|u - v\| + \int_{t}^{+\infty} M e^{-\omega(s-t)} L_f(s) ds \cdot \|u - v\| \\
&\le \frac{2M}{1 - e^{-\omega}} \left(\frac{1 - e^{-\omega r'}}{\omega r'} \right)^{1/r'} \|L_f\|_{S^r} \cdot \|u - v\|,
\end{aligned} \tag{2.32}$$

for all $t \in \mathbb{R}$, which yields that F has a unique fixed point $u \in PAP(\mathbb{X})$ and

$$u(t) = \int_{\mathbb{R}} \Gamma(t,s) f(s,u(s)) ds, \quad t \in \mathbb{R}. \tag{2.33}$$

This completes the proof. □

Remark 2.6. For some general conditions which can ensure that the assumption (c) in Theorem 2.5 holds, we refer the reader to [17, Theorem 4.5]. In addition, in the case of $A(t) \equiv A$ and A generating an exponential stable semigroup $T(t)$, the assumption (c) obviously holds.

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