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*Research Article*

# Generalized Zeros of $2 \times 2$ Symplectic Difference System and of Its Reciprocal System

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We establish a conjugacy criterion for a $2 \times 2$ symplectic difference system by means of the concept of a phase of any basis of this symplectic system. We also describe a construction of a $2 \times 2$ symplectic difference system whose recessive solution has the prescribed number of generalized zeros in $\mathbb{Z}$.

## 1. Introduction

The main aim of this paper is to establish a conjugacy criterion for the $2 \times 2$ symplectic difference system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \mathcal{S}_k \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad k \in \mathbb{Z}, \tag{S}$$

where $\mathcal{S}_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ with real-valued sequences $a$, $b$, $c$, and $d$ is such that $\det \mathcal{S}_k = a_k d_k - b_k c_k = 1$ for every $k \in \mathbb{Z}$. Recall that under this condition, the matrix $\mathcal{S}$ is *symplectic*. Generally, a $2n \times 2n$ matrix $\mathcal{S}$ is symplectic if

$$\mathcal{S}^T \mathcal{J} \mathcal{S} = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1.1}$$

$I$ being the $n\times n$ identity matrix, and this conditions reduces just to the condition $\det \mathcal{S} = 1$ for $2\times 2$ matrices. We introduce concepts of the first and second phase of any basis of system (S), and we study some of their properties. We generalize results introduced in [1–4] for a Sturm-Liouville difference equation, and we describe how to construct a $2\times 2$ symplectic difference system whose recessive solution has a prescribed number of generalized zeros. This result generalizes a construction for a Sturm-Liouville difference equation and so solves an open problem posed in [3, Section 4].

The paper is organized as follows. In Section 2, we introduce the definition of the first phase of any basis of the system (S), and we establish a formula for the forward difference of this phase. We apply this formula to study the relationship between (S) and its reciprocal system in Section 3, where the concept of the second phase is introduced. The forward difference of a first phase of (S) plays the crucial role in a conjugacy criterion for system (S), which is proved in Section 4. In Section 5, we show how to construct system (S) with prescribed oscillatory properties.

Definition of some concepts we need in our paper is now in order. A pair of linearly independent solutions $\binom{x}{u}$ and $\binom{y}{v}$ of (S) with the Casoratian $\omega$

$$\omega \equiv x_k v_k - y_k u_k = \text{const} \neq 0 \tag{1.2}$$

is said to be *a basis* of the system (S). If $\omega \equiv 1$, it is said to be *a normalized basis.* An interval $(m, m+1]$, $m\in\mathbb{Z}$, is said to contain *a generalized zero* of a solution $\binom{x}{u}$ of (S), if $x_m \neq 0$ and

$$x_{m+1} = 0 \quad \text{or} \quad b_m x_m x_{m+1} < 0. \tag{1.3}$$

A solution $\binom{x}{u}$ of (S) is said to be *oscillatory in* $\mathbb{Z}$ if it has infinitely many generalized zeros in $\mathbb{Z}$. In the opposite case, we say that $\binom{x}{u}$ is *nonoscillatory in* $\mathbb{Z}$. System (S) is said to be *nonoscillatory* (of *finite type*) in $\mathbb{Z}$ if every solution of (S) is nonoscillatory in $\mathbb{Z}$. A nonoscillatory system (S) is said to be 1-*general* in $\mathbb{Z}$ if it possesses two linearly independent solutions with no generalized zero, and it is said to be 1-*special* in $\mathbb{Z}$ if there is exactly one (up to the linear dependence) solution of (S) without any generalized zero in $\mathbb{Z}$. The definition of these concepts via recessive solutions of (S) is given later. System (S) is said to be *conjugate* in the interval $[M,N]$ ($[M,N]$ represents the discrete set $[M,N]\cap\mathbb{Z}$, $M,N\in\mathbb{Z}$, $N>M$), if there exists a solution of (S) which has at least two generalized zeros in $(M-1, N+1]$.

Note that the terminology conjugacy/1-general/1-special equation is borrowed from the theory of differential equations, see [5, 6], and it is closely related to the concepts of supercriticality/criticality/subcriticality of the Jacobi operators associated with the three-term recurrence relation

$$Tx := r_k x_{k+1} + q_k x_k + r_{k-1} x_{k-1} = 0, \tag{1.4}$$

see [7] and also [8].

At the end of this section, we recall the concept of the recessive solution of (S) and its relationship to conjugacy and other concepts defined above. Suppose that (S) is nonoscillatory. Then, there exists the unique (up to a multiplicative factor) solution $z^{[+]} = \binom{x^{[+]}}{u^{[+]}}$ with the property that $\lim_{k\to\infty} x_k^{[+]}/x_k = 0$ for any solution $z = \binom{x}{u}$ linearly independent of $z^{[+]}$. The solution $z^{[+]}$ is said to be *recessive at* $\infty$. The recessive solution $z^{[-]}$ at

$-\infty$ is defined analogously. System (S) is 1-special, respectively, 1-general if the recessive solutions $z^{[+]}, z^{[-]}$ have no generalized zero in $\mathbb{Z}$ and are linearly dependent, repectively, linearly independent. For more details concerning recessive solutions of discrete systems, we refer to [9, 10].

## 2. Phases and Their Properties

*Definition 2.1.* Let $\binom{x_k}{u_k}$ and $\binom{y_k}{v_k}$, $k \in \mathbb{Z}$, form a basis of (S) with the Casoratian $\omega$. By the *first phase* of this basis, we understand any real-valued sequence $\varphi = (\varphi_k)$, $k \in \mathbb{Z}$, such that

$$\varphi_k = \begin{cases} \arctan \dfrac{y_k}{x_k} & \text{if } x_k \neq 0, \\ \text{odd multiple of } \dfrac{\pi}{2} & \text{if } x_k = 0, \end{cases} \tag{2.1}$$

with $\Delta\varphi_k \in [0, \pi)$ if $\omega > 0$ and $\Delta\varphi_k \in (-\pi, 0]$ if $\omega < 0$.

Here, by arctan, we mean a particular value of the multivalued function which is inverse to the function tangent. By the requirement $\Delta\varphi \in [0, \pi)$, respectively, $\Delta\varphi \in (-\pi, 0]$, a first phase of $\binom{x}{u}, \binom{y}{v}$ is determined uniquely up to mod $\pi$.

The first phase (and the later introduced second phase) are sometimes called *zero-counting sequences*, since each jump of their value over an odd multiple of $\pi/2$ gives a generalized zero of a solution of (S) (or of its reciprocal system) as we will show later.

**Lemma 2.2.** *Let $\binom{x}{u}$ and $\binom{y}{v}$ form a basis of* (S) *with the Casoratian $\omega$. Then, there exist sequences $h$ and $g$, $h_k \neq 0$, such that the transformation*

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \mathcal{R}_k \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \tag{2.2}$$

$\mathcal{R}_k = \begin{pmatrix} h_k & 0 \\ g_k & \omega/h_k \end{pmatrix}$, *transforms system* (S) *into the so-called trigonometric system*

$$\begin{pmatrix} s_{k+1} \\ c_{k+1} \end{pmatrix} = \mathcal{T}_k \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \tag{T}$$

*where $\mathcal{T}$ is a symplectic matrix of the form $\mathcal{T}_k = \begin{pmatrix} p_k & q_k \\ -q_k & p_k \end{pmatrix}$ with*

$$p_k = \frac{a_k h_k + b_k g_k}{h_{k+1}}, \qquad q_k = \frac{\omega b_k}{h_k h_{k+1}}. \tag{2.3}$$

*Sequences $h$, $g$ are given by*

$$h_k^2 = x_k^2 + y_k^2, \qquad g_k = \frac{x_k u_k + y_k v_k}{h_k}. \tag{2.4}$$

*The values of the sequence $h$ can be chosen in such a way that $\omega q_k \geq 0$. In particular, if $b_k \neq 0$, then $h_k$ can be chosen in such a way that $\omega q_k > 0$ for $k \in \mathbb{Z}$.*

*Proof.* A similar statement is proved for general $2n \times 2n$ symplectic systems in [11]. However, in contrast to [11], our transformation matrix contains the Casoratian $\omega$, and the proof for scalar $2 \times 2$ systems can be simplified.

Transformation (2.2) transforms the symplectic system (S) into the system

$$\begin{pmatrix} s_{k+1} \\ c_{k+1} \end{pmatrix} = \mathcal{T}_k \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \quad \mathcal{T}_k = \mathcal{R}_{k+1}^{-1} \mathcal{S}_k \mathcal{R}_k, \tag{2.5}$$

where $\mathcal{T} =: \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ with

$$\begin{gathered} \tilde{a}_k = \frac{a_k h_k + g_k b_k}{h_{k+1}}, \qquad \tilde{b}_k = \frac{\omega b_k}{h_k h_{k+1}}, \\ \tilde{c}_k = \frac{1}{\omega}\left[-g_{k+1}\left(a_k h_k + b_k g_k\right) + h_{k+1}\left(c_k h_k + d_k g_k\right)\right], \\ \tilde{d}_k = \frac{-b_k g_{k+1} + d_k h_{k+1}}{h_k}, \end{gathered} \tag{2.6}$$

as can be verified by a direct computation. Then

$$\det \mathcal{T}_k = \det\left(\mathcal{R}_{k+1}^{-1} \mathcal{S}_k \mathcal{R}_k\right) = \det \mathcal{R}_{k+1}^{-1} \det \mathcal{S}_k \det \mathcal{R}_k = \frac{1}{\omega} \cdot 1 \cdot \omega = 1, \tag{2.7}$$

which means that $\mathcal{T}$ is a symplectic matrix, even if the transformation matric $\mathcal{R}$ is not generally symplectic. This is due to the fact that we consider $2 \times 2$ systems where a matrix is symplectic if and only if its determinant equals 1.

We have (no index means index $k$)

$$\begin{aligned} h h_{k+1} \tilde{c} &= \frac{1}{\omega}\Big[-h\left(x_{k+1} u_{k+1} + y_{k+1} v_{k+1}\right)(ah + bg) + h h_{k+1}^2 (ch + dg)\Big] \\ &= \frac{1}{\omega}\Big[-\left(x_{k+1} u_{k+1} + y_{k+1} v_{k+1}\right)\Big(ah^2 + b(xu + yv)\Big) + h_{k+1}^2\Big(ch^2 + d(xu + yv)\Big)\Big] \\ &= \frac{1}{\omega}\Big[ - \left(x_{k+1} u_{k+1} + y_{k+1} v_{k+1}\right)\left(x(ax + bu) + y(ay + bv)\right) \\ &\quad + h_{k+1}^2 \left(x(cx + du) + y(cy + dv)\right)\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\omega}\left[-(x_{k+1}u_{k+1} + y_{k+1}v_{k+1})(xx_{k+1} + yy_{k+1}) + \left(x_{k+1}^2 + y_{k+1}^2\right)(xu_{k+1} + yv_{k+1})\right] \\
&= \frac{1}{\omega}\left[-xx_{k+1}y_{k+1}v_{k+1} - yx_{k+1}y_{k+1}u_{k+1} + yx_{k+1}^2 v_{k+1} + xy_{k+1}^2 u_{k+1}\right] \\
&= \frac{1}{\omega}\left[xy_{k+1}(-x_{k+1}v_{k+1} + y_{k+1}u_{k+1}) + yx_{k+1}(-y_{k+1}u_{k+1} + x_{k+1}v_{k+1})\right] \\
&= -xy_{k+1} + yx_{k+1} = -x(ay + bu) + y(ax + bu) = -\omega b.
\end{aligned}
\tag{2.8}
$$

Hence, $\widetilde{b} = -(\omega b/hh_{k+1}) = -\widetilde{c} =: q$. Similarly,

$$
\begin{aligned}
\widetilde{a} - \widetilde{d} &= \frac{1}{hh_{k+1}}\left[ah^2 + hgb + bg_{k+1}h_{k+1} - dh_{k+1}^2\right] \\
&= \frac{1}{hh_{k+1}}\left[ax^2 + ay^2 + b(xu + yv) + b(x_{k+1}u_{k+1} + y_{k+1}v_{k+1}) - dx_{k+1}^2 - dy_{k+1}^2\right] \\
&= \frac{1}{hh_{k+1}}\left[x(ax + bu) + y(ay + bv) + x_{k+1}(bu_{k+1} - dx_{k+1}) + y_{k+1}(bv_{k+1} - dy_{k+1})\right] \\
&= \frac{1}{hh_{k+1}}\left[xx_{k+1} + yy_{k+1} - x_{k+1}x - y_{k+1}y\right] = 0,
\end{aligned}
\tag{2.9}
$$

in the last line of this computation, we have used the fact that $\mathcal{S}^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, that is, $x_k = d_k x_{k+1} - b_k u_{k+1}$, $y_k = d_k y_{k+1} - b_k v_{k+1}$. Hence, $\widetilde{a} = \widetilde{d} =: p$.

Finally, concerning positivity of $\omega q$ if $b \neq 0$, we fix the sign of $h$ in a particular index, say $h_0 = \sqrt{x_0^2 + y_0^2}$ and the formula $\omega q = \omega^2 b/hh_{k+1}$ shows that the sign of $h$, that is, $h = \pm\sqrt{x^2 + y^2}$, at indices $k \neq 0$ can be "adjusted" in such a way that $\omega q_k > 0$ if $b_k \neq 0$. □

*Remark 2.3.* Transformation (2.2) preserves oscillatory properties of transformed systems in the following sense. If $b_k x_k x_{k+1} < 0$, that is, $b_k h_k s_k h_{k+1} s_{k+1} < 0$, then since $\operatorname{sgn}(b_k h_k h_{k+1}) = \operatorname{sgn}(\omega q_k)$, we have (using the positivity of the term $\omega q_k$) $b_k h_k s_k h_{k+1} s_{k+1} < 0$ if and only if $s_k s_{k+1} < 0$. Note also that $x_{k+1} = 0$ if and only if $s_{k+1} = 0$, since $h_k \neq 0$ for all $k$.

**Lemma 2.4** (see [12, Lemma 1]). *Let* (T) *be the trigonometric system. There exists the unique (up to* mod $2\pi$*) sequence* $\varphi_k \in [0, 2\pi)$ *such that*

$$\sin\varphi_k = q_k, \qquad \cos\varphi_k = p_k \tag{2.10}$$

*and the general solution* $\left(\begin{smallmatrix} s \\ c \end{smallmatrix}\right)$ *of* (T) *takes the form*

$$\begin{pmatrix} s_k \\ c_k \end{pmatrix} = \beta \begin{pmatrix} \sin(\xi_k + \alpha) \\ \cos(\xi_k + \alpha) \end{pmatrix}, \tag{2.11}$$

*where* $k \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{R}$ *and* $\xi$ *is any sequence such that* $\Delta\xi_k = \varphi_k$.

**Lemma 2.5.** *Let* $\binom{x^{[1]}}{u^{[1]}}$ *and* $\binom{x^{[2]}}{u^{[2]}}$ *be a basis of system* (S) *with the Casoratian* $\omega$, *and let* (T) *be the trigonometric system associated to* (S) *as formulated in Lemma 2.2. Then, there exists a solution* $\binom{s}{c}$ *of* (T) *such that*

$$\begin{pmatrix} x_k^{[1]} \\ u_k^{[1]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \dfrac{\omega}{h_k} \end{pmatrix} \begin{pmatrix} c_k \\ -s_k \end{pmatrix}, \qquad \begin{pmatrix} x_k^{[2]} \\ u_k^{[2]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \dfrac{\omega}{h_k} \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \tag{2.12}$$

*where* $k \in \mathbb{Z}$ *and* $h$, $g$ *are given by* (2.4). *Further, there exists a sequence* $\xi$ *such that*

$$s_k = \sin \xi_k, \qquad c_k = \cos \xi_k, \tag{2.13}$$

$\Delta \xi_k = \varphi_k$, *where the sequence* $\varphi$ *is given by* (2.10) *and* $\varphi_k \in [0, 2\pi)$ *for every* $k \in \mathbb{Z}$.

*Proof.* By Lemma 2.2, there exist solutions $\binom{s^{[i]}}{c^{[i]}}$, $i = 1, 2$, of (T) such that

$$\begin{pmatrix} x_k^{[i]} \\ u_k^{[i]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \dfrac{\omega}{h_k} \end{pmatrix} \begin{pmatrix} s_k^{[i]} \\ c_k^{[i]} \end{pmatrix}, \tag{2.14}$$

that is,

$$s^{[i]} = h^{-1} x^{[i]}, \qquad c^{[i]} = \frac{-g x^{[i]} + h u^{[i]}}{\omega}. \tag{2.15}$$

By a direct computation, we have

$$s^{[1]} c^{[2]} - c^{[1]} s^{[2]} = 1 \tag{2.16}$$

and after a few steps

$$\left(s^{[i]}\right)^2 + \left(c^{[i]}\right)^2 = 1. \tag{2.17}$$

By Lemma 2.4, there exist real constants $\alpha^{[i]}$, $\beta^{[i]}$ such that

$$\begin{pmatrix} s_k^{[i]} \\ c_k^{[i]} \end{pmatrix} = \beta^{[i]} \begin{pmatrix} \sin\left(\xi_k + \alpha^{[i]}\right) \\ \cos\left(\xi_k + \alpha^{[i]}\right) \end{pmatrix}, \tag{2.18}$$

where $\xi$ is an arbitrary sequence such that $\Delta\xi_k = \varphi_k$ and $\varphi$ is given by (2.10). By (2.17), we have $\beta^{[i]} = 1$, and by (2.16), we obtain

$$\begin{aligned} s_k^{[1]} c_k^{[2]} - c_k^{[1]} s_k^{[2]} &= \sin\left(\xi_k + \alpha^{[1]}\right) \cos\left(\xi_k + \alpha^{[2]}\right) - \sin\left(\xi_k + \alpha^{[2]}\right) \cos\left(\xi_k + \alpha^{[1]}\right) \\ &= \sin\left(\alpha^{[1]} - \alpha^{[2]}\right) = 1, \end{aligned} \tag{2.19}$$

that is, $\alpha^{[1]} - \alpha^{[2]} = (\pi/2)(\operatorname{mod} 2\pi)$. Hence, $s^{[1]} = c^{[2]}$ and $c^{[1]} = -s^{[2]}$ what implies (2.12). Since $(\xi_k)$ was an arbitrary sequence such that $\Delta\xi_k = \varphi_k$, changing $\xi_k$ to $\xi_k - \alpha^{[2]}$, we get (2.13). □

*Notation.* In the following, by Arctan and Arccot, we mean the principal branches of the multivalued functions arctan and arccot with the values in $(-\pi/2, \pi/2)$ and $(0, \pi)$, respectively.

**Theorem 2.6.** *Let $z^{[1]} = \binom{x}{u}$ and $z^{[2]} = \binom{y}{v}$ form a basis of* (S) *with the Casoratian $\omega$, and let $\varphi$ be a first phase of this basis. If $b_k \neq 0$, then*

$$\Delta\varphi_k = \begin{cases} \operatorname{Arccot} \dfrac{x_k x_{k+1} + y_k y_{k+1}}{\omega b_k} & \text{if } \omega > 0, \\[2ex] \operatorname{Arccot} \dfrac{x_k x_{k+1} + y_k y_{k+1}}{\omega b_k} - \pi & \text{if } \omega < 0. \end{cases} \tag{2.20}$$

*If $b_k = 0$, then $\Delta\varphi_k = 0$.*

*Proof.* Let (T) be a trigonometric system associated to (S) with the basis $z^{[1]}, z^{[2]}$ and with $p$, $q$ satisfying (2.3). Let $\varphi$ be a first phase of this basis. By Lemma 2.5, there exists a solution $\binom{s}{c}$ of (T) such that $s_k = \sin\xi_k$, $c_k = \cos\xi_k$ and $z^{[1]} = \binom{x}{u}$, $z^{[2]} = \binom{y}{v}$ satisfy

$$x_k = h_k \cos\xi_k, \qquad y_k = h_k \sin\xi_k, \tag{2.21}$$

where $h$ is given by (2.4), $\Delta\xi_k = \varphi_k$ and $\varphi_k$ is given by (2.10). Hence, for $x_k \neq 0$,

$$\tan\xi_k = \frac{y_k}{x_k} \tag{2.22}$$

and if $x_k = 0$, then $\xi_k$ is equal to an odd multiple of $\pi/2$. On the other hand, by Definition 2.1 for $x_k \neq 0$

$$\tan\varphi_k = \frac{y_k}{x_k}, \tag{2.23}$$

and if $x_k = 0$, then $\varphi_k$ is equal to an odd multiple of $\pi/2$. Consequently,

$$\varphi_k \equiv \xi_k (\operatorname{mod} \pi), \tag{2.24}$$

and it implies (since the additive multiple of $\pi$ to get equality in (2.24) is independent of $k$)

$$\Delta\psi_k \equiv \Delta\xi_k. \tag{2.25}$$

For $\omega > 0$, we defined in Definition 2.1 that $\Delta\psi_k \in [0,\pi)$. Suppose that $b_k \neq 0$. According to Lemma 2.2, we can choose $q_k > 0$, and then by Lemma 2.4, we can take $\varphi_k \in (0,\pi)$. Using (2.25), we have $\varphi_k = \Delta\psi_k$, and thus $\cot\Delta\psi_k = p_k/q_k$, and hence

$$\Delta\psi_k \text{Arccot} = \frac{p_k}{q_k}. \tag{2.26}$$

Let $\omega < 0$. Then, we defined $\Delta\psi_k \in (-\pi,0]$ and based on Lemma 2.2, under the assumption $b_k \neq 0$, we can choose $q_k < 0$ and then $\varphi_k \in (\pi,2\pi)$ defined in (2.10). Using (2.25), we have $\varphi_k = \Delta\psi_k + 2\pi$, and consequently $\cot(\Delta\psi_k + 2\pi) = \cot\Delta\psi_k = p_k/q_k$ and

$$\Delta\psi_k = \text{Arccot}\frac{p_k}{q_k} - \pi, \tag{2.27}$$

in this case. Finally, if $b_k = 0$, then $q_k = 0$, and we put $\varphi_k = 0$. Hence, by (2.25), $\Delta\psi_k \equiv \varphi_k (\text{mod } \pi)$ and since by Definition 2.1 $\Delta\psi_k \in (-\pi,\pi)$, then we have $\Delta\psi_k = 0$.

Summarizing, by a direct computation

$$\frac{p_k}{q_k} = \frac{h_k}{\omega b_k}(a_k h_k + b_k g_k) = \frac{1}{\omega b_k}\left[a_k h_k^2 + b_k(x_k u_k + y_k v_k)\right]. \tag{2.28}$$

Since $x_{k+1} = a_k x_k + b_k u_k$ and $y_{k+1} = a_k y_k + b_k v_k$,

$$x_k x_{k+1} + y_k y_{k+1} = a_k h_k^2 + b_k(x_k u_k + y_k v_k), \tag{2.29}$$

and this gives, together with (2.26) and (2.27), the conclusion (2.20). □

We continue in this section with a statement which justifies why phases are sometimes called zero-counting sequences. We formulate the statement for a first phase, for a second phase the statement is similar.

**Theorem 2.7.** *Let $\psi$ be the first phase of* (S) *determined by the basis* $\binom{x}{u}, \binom{y}{v}$. *Then,* $\binom{x}{u}$ *has a generalized zero in* $(k, k+1)$ *if and only if $\psi$ skips over an odd multiple of $\pi/2$ between $k$ and $k+1$.*

*Proof.* Suppose that $\binom{x}{u}$ has a generalized zero in $(k, k+1)$, that is, $x_k x_{k+1} b_k < 0$. By Lemma 2.5 $x_k = h_k c_k$, $y_k = h_k s_k$, where $\binom{c}{-s}$ is a solution of trigonometric (T) with $\omega q_k > 0$ (Lemma 2.2). Suppose that $\omega > 0$, that is, $\Delta\psi_k \in (0,\pi)$ (for $\omega < 0$ the proof is analogical). Then, $\binom{c}{-s}$ has a generalized zero in $(k, k+1)$ which means that $c_k$ and $c_{k+1}$ have different sign. Since $c_k = \cos\xi_k$, $c_{k+1} = \cos\xi_{k+1}$, where $\xi$ is a sequence with $\Delta\xi_k = \Delta\psi_k$ (compare (2.25)), $\xi_k$ and $\xi_{k+1}$ lay in different intervals whose endpoints form odd multiples of $\pi/2$. Conversely, if $\psi$ skips an odd multiple of $\pi/2$ between $k$ and $k+1$, $\xi$ does also, and reasoning in the same way as above, we see that $\binom{x}{u}$ has a generalized zero in $(k, k+1)$. □

*Remark 2.8.* A slightly modified statement we have in the case when $\binom{x}{u}$ has a zero at $k+1$, that is, $x_k \neq 0$ and $x_{k+1} = 0$. More precisely, by the definition of the first phase $\varphi_{k+1} = (2m+1)(\pi/2)$ for some integer $m$, and, if $\varphi$ is increasing, then $\varphi_k \in ((2m-1)(\pi/2), (2m+1)(\pi/2))$.

We illustrate the above statements concerning properties of the first phase by the following example.

*Example 2.9.* Consider the Fibonacci recurrence relation

$$x_{k+2} = x_{k+1} + x_k, \quad k \in \mathbb{Z}, \tag{2.30}$$

that is,

$$\Delta\left((-1)^k \Delta x_k\right) + (-1)^k x_{k+1} = 0, \tag{2.31}$$

which can be viewed as symplectic system (S) with the matrix

$$\mathcal{S}_k = \begin{pmatrix} 1 & (-1)^k \\ (-1)^{k+1} & 0 \end{pmatrix}, \tag{2.32}$$

that is, the entry corresponding to $b_k$ changes its sign. A basis $\binom{x}{u}, \binom{y}{v}$ of (S) corresponding to (2.31) has the first components given by

$$x_k = \left(\frac{1-\sqrt{5}}{2}\right)^k, \qquad y_k = \left(\frac{1+\sqrt{5}}{2}\right)^k, \tag{2.33}$$

with the positive Casoratian $\omega = \sqrt{5}$. By Definition 2.1, $\Delta\varphi_k \in [0, \pi)$ and

$$\varphi_k = \begin{cases} \operatorname{Arctan}\left(\dfrac{1+\sqrt{5}}{1-\sqrt{5}}\right)^k + \dfrac{k}{2}\pi, & k \text{ even}, \\ \operatorname{Arctan}\left(\dfrac{1+\sqrt{5}}{1-\sqrt{5}}\right)^k + \dfrac{k+1}{2}\pi, & k \text{ odd}. \end{cases} \tag{2.34}$$

Notice that every jump of the value $\varphi_k$ over an odd multiple of $\pi/2$ corresponds to a generalized zero of $x$ in $(k, k+1)$. A corresponding trigonometric system (T) to symplectic system (S) has by Lemma 2.5 two linearly independent solutions $\binom{c}{-s}$ and $\binom{s}{c}$, where the sequences $c$ and $s$ are given by (2.13). Since by (2.24) $\xi_k = \varphi_k + m\pi$ for some $m \in \mathbb{Z}$, the first components of a basis of (T) can be (up to the sign) uniquely determined by

$$c_k = \cos\varphi_k, \qquad s_k = \sin\varphi_k. \tag{2.35}$$

It means that the components $x$, respectively, $y$ of solutions of (S) have generalized zeros in $(k, k+1]$ if and only if components $c$, respectively, $s$ of solutions of (T) have a generalized zero in $(k, k+1]$. By (2.21), together with (2.24), we express the first components of the basis of (S) as

$$x_k = h_k \cos \varphi_k, \qquad y_k = h_k \sin \varphi_k. \tag{2.36}$$

By Lemma 2.2, we choose the sign of the sequence $h$ in such a way that

$$\dots, h_0 > 0, h_1 > 0, h_2 < 0, h_3 < 0, h_4 > 0, \dots, \tag{2.37}$$

so the term $\omega q_k$ (i.e., $b_k h_k h_{k+1}$) is positive. Such a choice of the sign of the members of $(h_k)$ must agree with the sign of sequences $(x_k)$ and $(y_k)$. In fact, then by (2.36), $y_k$ is positive for any $k$ and $x_k$ is positive for every even and negative for every odd integer $k$.

Next, we describe the behavior of the phase $\varphi$ and corresponding trigonometric sequences $c$ and $s$ in case when $\omega < 0$. Consider (2.31), that is, the corresponding symplectic system with the basis $\binom{x}{u}, \binom{y}{v}$ having the first components

$$x_k = \left(\frac{1+\sqrt{5}}{2}\right)^k, \qquad y_k = \left(\frac{1-\sqrt{5}}{2}\right)^k, \tag{2.38}$$

with the negative Casoratian $\omega = -\sqrt{5}$. By Definition 2.1,

$$\varphi_k = \begin{cases} \operatorname{Arctan}\left(\dfrac{1-\sqrt{5}}{1+\sqrt{5}}\right)^k - \dfrac{k}{2}\pi, & k \text{ even}, \\ \operatorname{Arctan}\left(\dfrac{1-\sqrt{5}}{1+\sqrt{5}}\right)^k - \dfrac{k-1}{2}\pi, & k \text{ odd}. \end{cases} \tag{2.39}$$

The first components of a basis of the trigonometric system (T) corresponding to (S) is of the form

$$c_k = \cos \varphi_k, \qquad s_k = \sin \varphi_k. \tag{2.40}$$

Choosing the sign of the sequence $h$ as follows

$$\dots, h_0 > 0, h_1 > 0, h_2 < 0, h_3 < 0, h_4 > 0, \dots, \tag{2.41}$$

we get $x_k$ positive for any $k$ and $y_k$ positive for every even and negative for every odd integer $k$.

## 3. Reciprocal System

*A reciprocal system* to (S) is the symplectic system

$$\begin{pmatrix} \overline{x}_{k+1} \\ \overline{u}_{k+1} \end{pmatrix} = \mathcal{S}_k^r \begin{pmatrix} \overline{x}_k \\ \overline{u}_k \end{pmatrix}, \quad k \in \mathbb{Z}, \tag{S$^r$}$$

where

$$\mathcal{S}_k^r = \mathcal{J}\mathcal{S}_k\mathcal{J}^{-1} = \begin{pmatrix} d_k & -c_k \\ -b_k & a_k \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.1}$$

related to (S) by the substitution $\binom{\overline{x}}{\overline{u}} = \mathcal{J}\binom{x}{u}$. From definition of the symplectic system (S) and its reciprocal system (S$^r$), it follows that if $\binom{x}{u}$ is a solution of (S), then $\binom{u}{-x}$ is a solution of its reciprocal system (S$^r$).

*Definition 3.1.* By the *second phase* of the basis $\binom{x}{u}, \binom{y}{v}$ of system (S), we understand any first phase of the basis $\binom{u}{-x}, \binom{v}{-y}$ of its reciprocal system (S$^r$), that is, any real-valued sequence $\varrho = (\varrho_k)$, $k \in \mathbb{Z}$, such that

$$\varrho_k = \begin{cases} \arctan \dfrac{v_k}{u_k} & \text{if } u_k \neq 0, \\ \text{odd multiple of } \dfrac{\pi}{2} & \text{if } u_k = 0, \end{cases} \tag{3.2}$$

with $\Delta\varrho_k \in [0, \pi)$ if $\omega > 0$ and $\Delta\varrho_k \in (-\pi, 0]$ if $\omega < 0$.

The proofs of the next statement and of its corollary are the same as those of Lemma 2.2 and Theorem 2.6, respectively.

**Lemma 3.2.** *Let* $\binom{x}{u}$ *and* $\binom{y}{v}$ *form a basis of* (S) *with the Casoratian* $\omega$, *that is,* $\binom{u}{-x}$ *and* $\binom{v}{-y}$ *is a basis of* (S$^r$) *with the Casoratian* $\overline{\omega} = \omega = -u_k y_k + x_k v_k$. *Then, there exist sequences* $\overline{h}$ *and* $\overline{g}$, $\overline{h}_k \neq 0$ *for* $k \in \mathbb{Z}$, *such that the transformation*

$$\begin{pmatrix} \overline{x}_k \\ \overline{u}_k \end{pmatrix} = \begin{pmatrix} \overline{h}_k & 0 \\ \overline{g}_k & \dfrac{\omega}{\overline{h}_k} \end{pmatrix} \begin{pmatrix} \overline{s}_k \\ \overline{c}_k \end{pmatrix}, \tag{3.3}$$

*transforms system* (S$^r$) *into the trigonometric system*

$$\begin{pmatrix} \overline{s}_{k+1} \\ \overline{c}_{k+1} \end{pmatrix} = \begin{pmatrix} \overline{p}_k & \overline{q}_k \\ -\overline{q}_k & \overline{p}_k \end{pmatrix} \begin{pmatrix} \overline{s}_k \\ \overline{c}_k \end{pmatrix} \tag{T$^r$}$$

*which is symplectic with the sequences $\overline{p}$, $\overline{q}$ given by*

$$\overline{p}_k = \frac{d_k\overline{h}_k - c_k\overline{g}_k}{\overline{h}_{k+1}}, \qquad \overline{q}_k = -\frac{c_k\omega}{\overline{h}_k\overline{h}_{k+1}}, \tag{3.4}$$

*where*

$$\overline{h}_k^2 = u_k^2 + v_k^2, \qquad \overline{g}_k = -\frac{x_k u_k + y_k v_k}{\overline{h}_k}. \tag{3.5}$$

*Moreover, transformation* (3.3) *preserves oscillatory properties of* $(\mathrm{S}^r)$*, and the sequence* $(\overline{h}_k)$*,* $k \in \mathbb{Z}$*, can be chosen in such a way that* $\omega\overline{q}_k \geq 0$ *and if* $c_k \neq 0$ *in such a way that* $\omega\overline{q}_k > 0$*.*

**Corollary 3.3.** *Let* $\binom{x}{u}$ *and* $\binom{y}{v}$ *form a basis of* (S) *with the Casoratian* $\omega$*; that is,* $\binom{u}{-x}$ *and* $\binom{v}{-y}$ *form the basis of* $(\mathrm{S}^r)$ *with the same Casoratian* $\omega$*. Let* $(\varrho_k)$ *be the second phase of the basis* $\binom{x}{u}, \binom{y}{v}$ *of* (S)*. If* $c_k \neq 0$*,* $k \in \mathbb{Z}$*, then*

$$\Delta\varrho_k = \begin{cases} \operatorname{Arccot}\dfrac{u_k u_{k+1} + v_k v_{k+1}}{-\omega c_k} & \text{if } \omega > 0, \\[2ex] \operatorname{Arccot}\dfrac{u_k u_{k+1} + v_k v_{k+1}}{-\omega c_k} - \pi & \text{if } \omega < 0. \end{cases} \tag{3.6}$$

*If* $c_k = 0$*, then* $\Delta\varrho_k = 0$*.*

In the next statement, we use the relationship between the first phase $\varphi$ and the second phase $\varrho$ of the basis $\binom{x}{u}, \binom{y}{v}$ of symplectic system (S) and the fact that the behavior of the first and second phases of system (S) plays the crucial role in counting generalized zeros of solutions of symplectic system (S) and of its reciprocal system $(\mathrm{S}^r)$.

**Theorem 3.4.** *If system* (S) *with the sequences* $b_k \neq 0$ *and* $c_k \neq 0$ *which do not change their sign has a solution with two consecutive generalized zeros in* $(l-1, l]$*, and let* $(m-1, m]$*,* $l < m$*,* $l, m \in \mathbb{Z}$*, then its reciprocal system* $(\mathrm{S}^r)$ *is either conjugate in* $[l-1, m]$ *with a solution having a generalized zero in* $(l-1, l]$ *or* $(m-1, m]$*, or there exists a solution of* $(\mathrm{S}^r)$ *with exactly one generalized zero in* $[l, m]$*.*

*Proof.* Let $\binom{x}{u}$ be the solution of (S) having consecutive generalized zeros in $(l-1, l]$ and $(m-1, m]$ and $\binom{y}{v}$ be a solution which together with $\binom{x}{u}$ form the basis of the solution space of (S). Denote by $\varphi$ and $\varrho$ the first and second phase of this basis. Then, by Lemma 2.5,

$$\begin{aligned} x_k &= h_k \cos\varphi_k, \qquad & u_k &= g_k \cos\varphi_k - \frac{\omega}{h_k}\sin\varphi_k, \\ y_k &= h_k \sin\varphi_k, \qquad & v_k &= g_k \sin\varphi_k + \frac{\omega}{h_k}\cos\varphi_k, \end{aligned} \tag{3.7}$$

and by Lemma 3.2,

$$u_k = \overline{h}_k \cos\varrho_k, \qquad v_k = \overline{h}_k \sin\varrho_k. \tag{3.8}$$

Hence,

$$\overline{h}_k \cos \varrho_k = g_k \cos \varphi_k - \frac{\omega}{h_k} \sin \varphi_k,$$
$$\overline{h}_k \sin \varrho_k = g_k \sin \varphi_k + \frac{\omega}{h_k} \cos \varphi_k. \tag{3.9}$$

Multiplying the first equation by $-\sin \varphi_k$, the second one by $\cos \varphi_k$, and adding the resulting equations, we obtain

$$\overline{h}_k \sin(\varrho_k - \varphi_k) = \frac{\omega}{h_k}. \tag{3.10}$$

Since we assume that the sequences $b, c$ are of constant sign, the last part of Lemma 2.2 together with the second formulas in (2.3), (3.4) imply that $h$ and $\overline{h}$ have constant sign as well and by (3.10) the same holds for the sequence $\sin(\varrho_k - \varphi_k)$. Suppose, to be specific, that $\sin(\varrho_k - \varphi_k) < 0$ (if this sequence is positive, the proof is similar) then there exists an odd integer $n$ such that

$$n\pi < \varrho_k - \varphi_k < (n+1)\pi. \tag{3.11}$$

Recall that by Definition 2.1, the first phase $\varphi$ and the second phase $\varrho$ are defined as the monotone sequences on $\mathbb{Z}$. In addition, by Lemma 3.2, the Casoratian $\omega$ of (S) equals to the Casoratian $\overline{\omega}$ of $(\mathrm{S}^r)$, and thus, again by Definition 2.1, both phases $\varphi$ and $\varrho$ of (S) are either nondecreasing or nonincreasing. Moreover, if $\omega = \overline{\omega} \neq 0$, $b_k \neq 0$ and $c_k \neq 0$, $k \in \mathbb{Z}$, then by Theorem 2.6 and Corollary 3.3, $\Delta\varphi_k \neq 0$ and $\Delta\varrho_k \neq 0$ for $k \in \mathbb{Z}$.

Suppose that the first phase $\varphi_k$ of the basis $\binom{x}{u}, \binom{y}{v}$ of (S) given by Definition 2.1 is increasing; that is, for every integer $k$, we have $\Delta\varphi_k \in (0, \pi)$. If we suppose a decreasing sequence $\varphi$, the proof is analogous. Since the phases are determined up to mod $\pi$, without loss of generality, we may suppose that $n = -1$ in (3.11), that is,

$$0 < \varphi_k - \varrho_k < \pi. \tag{3.12}$$

Moreover, we can also suppose that $\varphi_{l-1} \in (-\pi/2, \pi/2)$. Since $x$ has consecutive generalized zeros in $(l-1, l]$ and $(m-1, m]$, we have

$$\varphi_l \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad \varphi_j \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad j = l+1, \dots, m-1, \quad \varphi_m \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right), \tag{3.13}$$

that is, $\varphi_k$ skips $\pi/2$ between $l-1$ and $l$ and $3\pi/2$ between $m-1$ and $m$ and stays in the strip $(\pi/2, 3\pi/2)$ between $l$ and $m$. Formula (3.12) admits the following behavior of the sequence $\varrho$ (to draw a picture may help to visualize the situation).

(i) $\varrho_{l-1} < -\pi/2$, $\varrho_l \in [-\pi/2, \pi/2)$, there exists $r$, $l < r < m$, such that

$$\varrho_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad i = l+1, \dots, r-1, \quad \varrho_j \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad j = r, \dots, m-1, \quad \varrho_m \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right). \tag{3.14}$$

(ii) The sequence $\varrho$ has the same behavior as in (i) up to $m$, where $\varrho_m < 3\pi/2$.

(iii) We have $\varrho_{l-1} \in [-\pi/2, \pi/2)$, $\varrho_l \in (-\pi/2, \pi/2)$ and for $k > l$ the sequence $\varrho$ behaves as in (i).

(iv) We have

$$\varrho_{l-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \varrho_l \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad \varrho_j \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad j = l+1, \dots, m-1, \quad \varrho_m \geq \frac{3\pi}{2}. \tag{3.15}$$

(v) Finally,

$$\begin{gathered} \varrho_{l-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \varrho_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad j = l, \dots, r-1, \\ \varrho_r \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad \varrho_j \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad j = r+1, \dots, m. \end{gathered} \tag{3.16}$$

The cases (i)–(iv) correspond to conjugacy of $(S^r)$, while the last case corresponds to the existence of a solution with exactly one generalized zero in $[l, m-1]$. □

## 4. A Conjugacy Criterion

In this section, we establish a conjugacy criterion for system (S) by means of the first phase $\psi$ and the associated Riccati equation.

The *conjugacy* of (S) in $[M, N]$ means that there exists a solution of (S) with at least two generalized zeros in $(M-1, N+1]$, that is, there exists a solution $\binom{x}{u}$ and two intervals $(l-1, l]$, $(m, m+1]$, where $M \leq l < m \leq N$, such that $x_{l-1} \neq 0$ and either $x_l x_{l-1} b_{l-1} < 0$ or $x_l = 0$, and $x_m \neq 0$ and either $x_m x_{m+1} b_m < 0$ or $x_{m+1} = 0$. Conversely, we say that system (S) is *disconjugate* in $[M, N]$ if every solution of (S) has at most one generalized zero in $(M-1, N+1]$.

**Theorem A** (see [9, Chapter 3]). *If* $\binom{x_k}{u_k}$, $x_k \neq 0$, *is a solution of* (S) *on the interval* $[0, N+1]$, *then the sequence* $w_k = u_k / x_k$ *is a solution of the Riccati difference equation*

$$w_{k+1} = \frac{c_k + d_k w_k}{a_k + b_k w_k}, \tag{4.1}$$

*defined for* $k \in [0, N]$. *Also, if* $\binom{x}{u}$ *has no generalized zero in the interval* $[0, N+1]$ *and* $b_k > 0$, *then* $a_k + b_k w_k > 0$ *for* $k \in [0, N]$.

**Theorem B** (see [9, Theorem 5.30], see also [13]). *Suppose that system* (S) *possesses a solution with no generalized zero in* $[M, N+1]$. *Then, every nontrivial solution* $\binom{x}{u}$ *of this system has at most one generalized zero in this interval.*

In this section, as usual, we put $\sum_{i=m}^{n}(\cdot) = 0$ if $m > n$ and $\prod_{i=k}^{l}(\cdot) = 1$ if $k > l$.

**Theorem 4.1.** *Let the sequence* $b_k$ *in* (S) *be positive. Suppose that there exist positive real numbers* $\delta_1$ *and* $\delta_2$ *such that*

$$\sum_{k=0}^{N} \operatorname{Arccot} \mathcal{A}_k \geq \frac{\pi}{4}, \tag{4.2}$$

$$\sum_{k=M+1}^{0} \operatorname{Arccot} \mathcal{B}_k \geq \frac{\pi}{4}, \tag{4.3}$$

*where* $M \leq -1$ *and* $N \geq 1$ *are arbitrary fixed integers,*

$$\begin{aligned} \mathcal{A}_k &= \frac{2}{\delta_1 b_k}\left[1 + b_k\left(\delta_1 + \sum_{j=0}^{k-1} F_j\right)\right] \prod_{j=0}^{k-1}\left[1 + b_j\left(\delta_1 + \sum_{i=0}^{j-1} F_i\right)\right]^2, \\ \mathcal{B}_k &= \frac{2}{\delta_2 b_k}\left[1 + b_k\left(\delta_2 + \sum_{j=k}^{-1} F_j\right)\right] \prod_{j=k+1}^{-1}\left[1 + b_j\left(\delta_2 + \sum_{i=j}^{-1} F_i\right)\right]^2, \end{aligned} \tag{4.4}$$

$$F_k = \frac{a_{k+1} - 1}{b_{k+1}} + \frac{d_k - 1}{b_k}. \tag{4.5}$$

*Then, system* (S) *is conjugate in* $[M, N]$.

*Proof.* In the first part of the proof, we show that the solution $\binom{x}{u}$ of (S) given by the condition

$$x_0 = 1, \qquad x_1 = 1 \tag{4.6}$$

has a generalized zero in $(0, N+1]$. Let $\binom{y}{v}$ be another linearly independent solution of (S) given by the condition

$$y_0 = 1, \qquad y_1 = 1 + \delta_1 b_0. \tag{4.7}$$

Since $x_{k+1} = a_k x_k + b_k u_k$ and $y_{k+1} = a_k y_k + b_k v_k$ for every integer $k$, this holds especially for $k = 0$, and hence

$$\begin{aligned} u_0 &= \frac{1}{b_0}(x_1 - a_0 x_0) = \frac{1}{b_0}(1 - a_0), \\ v_0 &= \frac{1}{b_0}(y_1 - a_0 y_0) = \frac{1}{b_0}(1 - a_0) + \delta_1. \end{aligned} \tag{4.8}$$

The Casoratian $\omega$ satisfies

$$\omega = x_0 v_0 - y_0 u_0 = \delta_1 > 0. \tag{4.9}$$

Suppose, by contradiction, that $\binom{x}{u}$ has no generalized zero in $(0, N+1]$, that is, due to the fact that $x_0 = 1$ and $b_k > 0$, we have $x_k > 0$ for every $k = 1, 2, \ldots, N$. Then, by Theorem B, we get

$$y_k > x_k, \tag{4.10}$$

for $k = 1, \ldots, N+1$, because otherwise the solution $\binom{x}{u} - \binom{y}{v}$ has generalized zeros at $k = 0$ and in the interval $(m, m+1]$, $m$ being the integer where (4.10) is violated.

Let $\varphi$ be the first phase of solutions $\binom{x}{u}$ and $\binom{y}{v}$, that is, by Definition 2.1,

$$\varphi_k = \arctan \frac{y_k}{x_k}, \quad \Delta\varphi_k \in [0, \pi). \tag{4.11}$$

By Theorem 2.6, we have

$$\Delta\varphi_k = \operatorname{Arccot} \frac{x_k x_{k+1} + y_k y_{k+1}}{\delta_1 b_k}, \tag{4.12}$$

taking account that $\varphi_0 = \pi/4$ and using (4.10), we get for $k = 1, \ldots, N+1$

$$\varphi_k = \sum_{j=0}^{k-1} \Delta\varphi_j + \varphi_0 = \sum_{j=0}^{k-1} \operatorname{Arccot} \frac{x_j x_{j+1} + y_j y_{j+1}}{\delta_1 b_j} + \frac{\pi}{4} > \sum_{j=0}^{k-1} \operatorname{Arccot} \frac{2 y_j y_{j+1}}{\delta_1 b_j} + \frac{\pi}{4}. \tag{4.13}$$

Let $w_k = v_k / y_k$. Then, from the first equation in (S)

$$w_k = \frac{1}{b_k} \left( \frac{y_{k+1}}{y_k} - a_k \right), \tag{4.14}$$

and $w$ is a solution of the Riccati equation (4.1). Denote $\widetilde{w}_k = w_k + (a_k - 1)/b_k$. Then,

$$\begin{aligned}
\widetilde{w}_{k+1} &= \frac{a_{k+1} - 1}{b_{k+1}} + \frac{c_k + d_k(\widetilde{w}_k - (a_k - 1)/b_k)}{a_k + b_k(\widetilde{w}_k - (a_k - 1)/b_k)} \\
&= \frac{a_{k+1} - 1}{b_{k+1}} + \frac{b_k c_k + b_k d_k \widetilde{w}_k - a_k d_k + d_k}{b_k(b_k \widetilde{w}_k + 1)} \\
&= \frac{a_{k+1} - 1}{b_{k+1}} + \frac{-1 + d_k(1 + b_k \widetilde{w}_k) - b_k \widetilde{w}_k + b_k \widetilde{w}_k}{b_k(b_k \widetilde{w}_k + 1)} \\
&= \frac{a_{k+1} - 1}{b_{k+1}} + \frac{d_k - 1}{b_k} + \frac{\widetilde{w}_k}{1 + b_k \widetilde{w}_k}.
\end{aligned} \tag{4.15}$$

Further denote $F_k = (a_{k+1}-1)/b_{k+1}+(d_k-1)/b_k$. Then, since $1+b_k\tilde{w}_k = b_k w_k + a_k = y_{k+1}/y_k > 0$,

$$\Delta\tilde{w}_k = F_k - \frac{b_k\tilde{w}_k^2}{1+b_k\tilde{w}_k} \le F_k, \tag{4.16}$$

which means that

$$\tilde{w}_k \le \tilde{w}_0 + \sum_{j=0}^{k-1} F_j = \delta_1 + \sum_{j=0}^{k-1} F_j. \tag{4.17}$$

Hence, (4.14) implies

$$y_k = y_0 \prod_{j=0}^{k-1}(w_j b_j + a_j) = \prod_{j=0}^{k-1}(b_j\tilde{w}_j + 1), \tag{4.18}$$

and using (4.17),

$$y_k \le \prod_{j=0}^{k-1}\left[1 + b_j\left(\delta_1 + \sum_{i=0}^{j-1} F_i\right)\right]. \tag{4.19}$$

Now,

$$\begin{aligned} \frac{2y_k y_{k+1}}{\delta_1 b_k} &\le \frac{2}{\delta_1 b_k}\prod_{j=0}^{k-1}\left[1 + b_j\left(\delta_1 + \sum_{i=0}^{j-1} F_i\right)\right]\prod_{j=0}^{k}\left[1 + b_j\left(\delta_1 + \sum_{i=0}^{j-1} F_i\right)\right] \\ &= \frac{2}{\delta_1 b_k}\left[1 + b_k\left(\delta_1 + \sum_{j=0}^{k-1} F_j\right)\right]\prod_{j=0}^{k-1}\left[1 + b_j\left(\delta_1 + \sum_{i=0}^{j-1} F_i\right)\right]^2 \\ &= \mathcal{A}_k. \end{aligned} \tag{4.20}$$

Let $k = N+1$ in (4.13). Then, together with assumption (4.2),

$$\varphi_{N+1} > \sum_{k=0}^{N} \operatorname{Arccot} \mathcal{A}_k + \frac{\pi}{4} \ge \frac{\pi}{2}. \tag{4.21}$$

On the other hand, since $\binom{x}{u}$ has no generalized zero in $(0, N+1]$, it follows that $\varphi_k < \pi/2$ for every $k = 0, \dots, N+1$, a contradiction with (4.21). It means that the solution $\binom{x}{u}$ has a generalized zero in $(0, N+1]$.

In the second part of the proof, we show that the solution $\binom{x}{u}$ of (S) given by condition (4.6) has also a generalized zero in $(M-1, 0]$. Since $\mathcal{S}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, we have $x_k = d_k x_{k+1} - b_k u_{k+1}$ and $u_k = -c_k x_{k+1} + a_k u_{k+1}$, in particular,

$$x_{-1} = d_{-1}x_0 - b_{-1}u_0 = d_{-1} - \frac{b_{-1}}{b_0}(1 - a_0). \tag{4.22}$$

Let $\binom{\overline{y}}{\overline{v}}$ be another linearly independent solution of (S) given by the condition

$$\overline{y}_0 = 1, \qquad \overline{y}_{-1} = \delta_2 b_{-1} + d_{-1} - \frac{b_{-1}}{b_0}(1 - a_0), \tag{4.23}$$

with the corresponding second component $\overline{v}_0$ expressed by

$$\overline{v}_0 = \frac{1}{b_{-1}}\left(d_{-1}\overline{y}_0 - \overline{y}_{-1}\right) = \frac{1}{b_0}(1 - a_0) - \delta_2. \tag{4.24}$$

The Casoratian $\overline{\omega}$ of $\binom{x}{u}$, $\binom{\overline{y}}{\overline{v}}$ satisfies

$$\overline{\omega} = x_0\overline{v}_0 - \overline{y}_0 u_0 = -\delta_2 < 0. \tag{4.25}$$

Suppose, by contradiction, that the solution $\binom{x}{u}$ has no generalized zero in the interval $(M-1, 0]$, that is, $x_k > 0$ for $k = M, \dots, 0$. Then, by Theorem B (using the same argument as in the first part of the proof)

$$\overline{y}_k > x_k, \tag{4.26}$$

for every $k = M, \dots, -1$. Let $\overline{\varphi}$ be the first phase of $\binom{x}{u}$ and $\binom{\overline{y}}{\overline{v}}$ with the Casoratian $\overline{\omega} < 0$. By Definition 2.1,

$$\overline{\varphi}_k = \arctan \frac{\overline{y}_k}{x_k}, \quad \Delta\overline{\varphi}_k \in (-\pi, 0], \tag{4.27}$$

and by Theorem 2.6

$$\Delta\overline{\varphi}_k = \operatorname{Arccot}\frac{x_k x_{k+1} + \overline{y}_k \overline{y}_{k+1}}{-\delta_2 b_k} - \pi. \tag{4.28}$$

Taking into account that $\overline{\varphi}_0 = \pi/4$, (4.26), and that the function $\operatorname{Arccot}(\cdot) - \pi/2$ is odd, we get for every $k = M, \dots, -1$

$$\begin{aligned} -\overline{\varphi}_k + \overline{\varphi}_0 &= \sum_{j=k}^{-1} \Delta\overline{\varphi}_j = \sum_{j=k}^{-1}\left[\operatorname{Arccot}\frac{x_{j+1}x_j + \overline{y}_{j+1}\overline{y}_j}{-\delta_2 b_j} - \frac{\pi}{2} - \frac{\pi}{2}\right] \\ &= -\sum_{j=k}^{-1}\operatorname{Arccot}\frac{x_{j+1}x_j + \overline{y}_{j+1}\overline{y}_j}{\delta_2 b_j} < -\sum_{j=k}^{-1}\operatorname{Arccot}\frac{2\overline{y}_{j+1}\overline{y}_j}{\delta_2 b_j}. \end{aligned} \tag{4.29}$$

Hence,

$$\overline{\varphi}_k > \frac{\pi}{4} + \sum_{j=k}^{-1} \operatorname{Arccot} \frac{2\overline{y}_j \overline{y}_{j+1}}{\delta_2 b_j}. \tag{4.30}$$

Let us estimate the term $(2\overline{y}_{k+1}\overline{y}_k)/b_k$ by means of the Riccati equation. Let $\overline{w}_k = \overline{v}_k/\overline{y}_k$. Then, $\overline{y}_k = d_k\overline{y}_{k+1} - b_k\overline{v}_{k+1}$, that is,

$$\frac{\overline{y}_k}{\overline{y}_{k+1}} = d_k - b_k\overline{w}_{k+1}, \tag{4.31}$$

and from the backward Riccati equation for $\overline{w}$ (which follows from (4.1)),

$$\overline{w}_k = \frac{-c_k + a_k\overline{w}_{k+1}}{d_k - b_k\overline{w}_{k+1}}. \tag{4.32}$$

Put $\tilde{w}_k = \overline{w}_k - (d_{k-1} - 1)/b_{k-1}$. Then, substituting into (4.32), we have (no index mean the index $k$ here and also in later computations)

$$\begin{aligned} \tilde{w} + \frac{d_{k-1} - 1}{b_{k-1}} &= \frac{-c + a(\tilde{w}_{k+1} + (d-1)/b)}{d - b(\tilde{w}_{k+1} + (d-1)/b)} = \frac{a\tilde{w}_{k+1} + (-cb + ad - a)/b}{1 - b\tilde{w}_{k+1}} \\ &= \frac{1/b + (a/b)(b\tilde{w}_{k+1} - 1)}{1 - b\tilde{w}_{k+1}} = -\frac{a}{b} + \frac{1/b - \tilde{w}_{k+1} + \tilde{w}_{k+1}}{1 - b\tilde{w}_{k+1}} \\ &= -\frac{a-1}{b} + \frac{\tilde{w}_{k+1}}{1 - b\tilde{w}_{k+1}}, \end{aligned} \tag{4.33}$$

and hence,

$$\tilde{w}_k - \tilde{w}_{k+1} = -\frac{a_k - 1}{b_k} - \frac{d_{k-1} - 1}{b_{k-1}} + \frac{b_k\tilde{w}_{k+1}^2}{1 - b_k\tilde{w}_{k+1}}. \tag{4.34}$$

Since,

$$\begin{aligned} 1 - b\tilde{w}_{k+1} &= 1 - b\left(\overline{w}_{k+1} - \frac{d-1}{b}\right) = d - b\overline{w}_{k+1} \\ &= d - b\frac{c + d\overline{w}}{a + b\overline{w}} = \frac{ad - cb}{a + b\overline{w}} = \frac{1}{a + b\overline{w}} = \frac{\overline{y}_k}{\overline{y}_{k+1}} > 0, \end{aligned} \tag{4.35}$$

we have $-\Delta\tilde{w}_k - \geq F_{k-1}$, where $F_k$ is given by (4.5). Summing the last inequality from $k+1$ to $-1$, we obtain

$$-\tilde{w}_{k+1} + \tilde{w}_0 \leq \sum_{j=k+1}^{-1} F_{j-1} \tag{4.36}$$

and this means that

$$1 - b_k \widetilde{w}_{k+1} \le 1 + b_k \left( -\widetilde{w}_0 + \sum_{j=k+1}^{-1} F_{j-1} \right). \tag{4.37}$$

From (4.31),

$$\frac{\overline{y}_k}{\overline{y}_{k+1}} = d_k - b_k \left( \widetilde{w}_{k+1} + \frac{d_k - 1}{b_k} \right) = 1 - b_k \widetilde{w}_{k+1}, \tag{4.38}$$

hence $\overline{y}_k = (1 - b_k \widetilde{w}_{k+1}) \overline{y}_{k+1}$, that is, from (4.37),

$$\overline{y}_k = \overline{y}_0 \prod_{j=k}^{-1} (1 - b_j \widetilde{w}_{j+1}) \le \prod_{j=k}^{-1} \left[ 1 + b_j \left( -\widetilde{w}_0 + \sum_{i=j+1}^{-1} F_{i-1} \right) \right]. \tag{4.39}$$

Finally,

$$\widetilde{w}_0 = \overline{w}_0 - \frac{d_{-1} - 1}{b_{-1}} = -\delta_2 + \frac{1 - a_0}{b_0} - \frac{d_{-1} - 1}{b_{-1}} = -\delta_2 - F_{-1}, \tag{4.40}$$

this implies

$$\overline{y}_k \le \prod_{j=k}^{-1} \left[ 1 + b_j \left( \delta_2 + \sum_{i=j}^{-1} F_i \right) \right]. \tag{4.41}$$

Substituting from (4.30),

$$\overline{\varphi}_M > \frac{\pi}{4} + \sum_{k=M}^{-1} \operatorname{Arccot} \frac{2}{\delta_2 b_k} \left[ 1 + b_k \left( \delta_2 + \sum_{j=k}^{-1} F_j \right) \right] \prod_{j=k+1}^{-1} \left[ 1 + b_j \left( \delta_2 + \sum_{i=j}^{-1} F_i \right) \right]^2, \tag{4.42}$$

and hence

$$\overline{\varphi}_M > \sum_{k=M}^{-1} \operatorname{Arccot} \mathcal{B}_k + \frac{\pi}{4} \ge \frac{\pi}{2}. \tag{4.43}$$

On the other hand, since we suppose that $\binom{x}{u}$ has no generalized zero in $(M-1, 0]$, it holds $\overline{\varphi}_M < \pi/2$, a contradiction with (4.43).

Summarizing, we have proved that the solution $\binom{x}{u}$ has at least one generalized zero in $(M-1, 0]$ and one generalized zero in $(0, N+1]$. The proof is complete. □

*Remark 4.2.* The conjugacy criterion for the Sturm-Liouville equation

$$\Delta(r_k \Delta x_k) + q_k x_{k+1} = 0, \quad r_k > 0, \tag{4.44}$$

formulated in [1, Theorem 2] is the corollary of the above criterion for $a_k = 1$, $b_k = 1/r_k$, $c_k = -q_k$ and $d_k = 1 - q_k/r_k$. Theorem 4.1 also extends the results proved in [1, 2, 12, 14].

## 5. Systems with Prescribed Oscillatory Properties

In this concluding section, we present a method of constructing a symplectic system (S) whose recessive solution has the prescribed number of generalized zeros in $\mathbb{Z}$.

**Theorem 5.1.** *Suppose that* $\binom{x}{u}, \binom{y}{v} \in \mathbb{R}^2$, $k \in \mathbb{Z}$, *are sequences such that the Casoratian* $\omega = \det\begin{pmatrix} x_k & y_k \\ u_k & v_k \end{pmatrix} = 1$ *for any* $k \in \mathbb{Z}$. *Then, these sequences form a normalized basis of symplectic system* (S) *with*

$$\begin{aligned} a_k &= x_{k+1} v_k - y_{k+1} u_k, \\ b_k &= -x_{k+1} y_k + y_{k+1} x_k, \\ c_k &= u_{k+1} v_k - v_{k+1} u_k, \\ d_k &= -u_{k+1} y_k + v_{k+1} x_k. \end{aligned} \tag{5.1}$$

*Moreover, if* $b_k \neq 0$ *for* $k \in \mathbb{Z}$,

$$\lim_{k \to \pm\infty} \frac{x_k}{y_k} = 0, \tag{5.2}$$

*and* $\binom{x}{u}$ *has* $(m-1)$ *generalized zeros in* $\mathbb{Z}$, *then the first phase determined by* $\binom{x}{u}$, $\binom{y}{v}$ *satisfies*

$$\lim_{k \to \infty} \varphi_k - \lim_{k \to -\infty} \varphi_k = \sum_{k \in \mathbb{Z}, b_k \neq 0} \operatorname{Arccot} \frac{x_k x_{k+1} + y_k y_{k+1}}{b_k} = m\pi. \tag{5.3}$$

*Proof.* Let $\binom{x}{u}$ and $\binom{y}{v}$ be sequences with the Casoratian equal to 1, and let $a, b, c$, and $d$ be given by (5.1). Then, by the Cramer rule, we obtain

$$\begin{aligned} x_{k+1} &= a_k x_k + b_k u_k, \\ u_{k+1} &= c_k x_k + d_k u_k, \\ y_{k+1} &= a_k y_k + b_k v_k, \\ v_{k+1} &= c_k y_k + d_k v_k, \end{aligned} \tag{5.4}$$

so $\binom{x}{u}$ and $\binom{y}{v}$ are solutions of (S) with $a, b, c, d$ given by (5.1). It is easy to verify that $a_k d_k - b_k c_k = 1$ holds for any integer $k$; that is, system (S) is symplectic. Now, suppose that assumptions of the second part of the theorem are satisfied. Then, (S) is nonoscillatory in $\mathbb{Z}$ and $\binom{x}{u}$ is its recessive solution both in $\infty$ and $-\infty$. Since

$$\Delta\left(\frac{y_k}{x_k}\right) = \frac{\Delta y_k x_k - \Delta x_k y_k}{x_k x_{k+1}} = \frac{y_{k+1}x_k - x_{k+1}y_k}{x_k x_{k+1}}$$
$$= \frac{(a_k y_k + b_k v_k)x_k - (a_k x_k + b_k u_k)y_k}{x_k x_{k+1}} = \frac{b_k}{x_k x_{k+1}} > 0, \tag{5.5}$$

for large and small $k$, the limits $\lim_{k\to\pm\infty} y_k/x_k$ exist and by (5.2) $\lim_{k\to\infty} y_k/x_k = \infty$ $\lim_{k\to-\infty} y_k/x_k = -\infty$. It follows, by the definition of the first phase, that limits $\lim_{k\to\infty}\varphi_k > \lim_{k\to-\infty}\varphi_k$ also exist and equal to odd multiples of $\pi/2$. This, coupled with the fact that $\binom{x}{u}$ has exactly $m-1$ generalized zeros in $\mathbb{Z}$; that is, $\varphi_k$ equals $(m-1)$ times an odd multiple of $\pi/2$ or skips over this multiple, gives (5.3). □

We finish the paper with an example illustrating the previous theorem.

*Example 5.2.* Consider a pair of two-dimensional sequences $\binom{x}{u}, \binom{y}{v}$ with

$$x_k = \left(k+\frac{1}{2}\right)^{(n-1)}, \qquad u_k = \frac{n-1}{(k+3/2)^{(n)}},$$
$$y_k = \left(k+\frac{1}{2}\right)^{(n)}, \qquad v_k = \frac{n}{(k+3/2)^{(n-1)}}, \tag{5.6}$$

where $n \geq 1$ and $(k+\alpha)^{(n)} := (k+\alpha)\cdots(k+\alpha-n+1)$. By a direct computation, one can verify that $x_k v_k - y_k u_k = 1$ and that (5.1) read

$$a_k = 1,$$
$$b_k = \left(k+\frac{3}{2}\right)\left(k+\frac{1}{2}\right)^2\cdots\left(k-n+\frac{7}{2}\right)^2\left(k-n+\frac{5}{2}\right),$$
$$c_k = \frac{-n(n-1)}{(k+5/2)(k+3/2)^2\cdots(k-n+7/2)^2(k-n+5/2)}, \tag{5.7}$$
$$d_k = \frac{(k+5/2)(k+3/2)-n(n-1)}{(k+5/2)(k+3/2)}.$$

Obviously, $\lim_{k\to\pm\infty} x_k/y_k = 0$, so the assumptions of the previous theorem are satisfied, and since

$$x_k x_{k+1} b_k = \left(k+\frac{3}{2}\right)^2\left(k+\frac{1}{2}\right)^4\cdots\left(k-n+\frac{7}{2}\right)^4\left(k-n+\frac{5}{2}\right)^2 > 0, \tag{5.8}$$

for $k \in \mathbb{Z}$, the solution $\binom{x}{u}$ has no generalized zero in $\mathbb{Z}$. Consequently, (5.3) reads (as can be again verified by a direct computation)

$$\sum_{k\in\mathbb{Z}} \operatorname{Arccot}\left[k^2 + (4-2n)k + n^2 - 4n + \frac{19}{4}\right] = \pi. \quad (5.9)$$

By a similar method, one can find the explicit formula for the sum of various infinite series involving the function Arccot.

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