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Research Article

Nonlocal Cauchy Problem for Nonautonomous Fractional Evolution Equations

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We investigate the mild solutions of a nonlocal Cauchy problem for nonautonomous fractional evolution equations $d^q u(t)/dt^q = -A(t)u(t) + f(t, (K_1u)(t), (K_2u)(t), \ldots, (K_nu)(t))$, $t \in I = [0, T]$, $u(0) = A^{-1}(0)g(u) + u_0$, in Banach spaces, where $T > 0$, $0 < q < 1$. New results are obtained by using Sadovskii's fixed point theorem and the Banach contraction mapping principle. An example is also given.

1. Introduction

During the past decades, the fractional differential equations have been proved to be valuable tools in the investigation of many phenomena in engineering and physics; they attracted many researchers (cf., e.g., [1–9]). On the other hand, the autonomous and nonautonomous evolution equations and related topics were studied in, for example, [6, 7, 10–20], and the nonlocal Cauchy problem was considered in, for example, [2, 5, 18, 21–26].

In this paper, we consider the following nonlocal Cauchy problem for nonautonomous fractional evolution equations

$$\frac{d^q u(t)}{dt^q} = -A(t)u(t) + f(t, (K_1u)(t), (K_2u)(t), \ldots, (K_nu)(t)), \quad t \in I = [0, T],$$
$$u(0) = A^{-1}(0)g(u) + u_0, \tag{1.1}$$

in Banach spaces, where $0 < q < 1$, $g : C(I; X) \to X$. The terms $(K_iu)(t)$, $i = 1, \ldots, n$ are defined by

$$(K_i u)(t) = \int_0^t k_i(t,s)u(s)ds, \tag{1.2}$$

the positive functions $k_i(t,s)$ are continuous on $D = \{(t,s) \in R^2 : 0 \le s \le t \le T\}$ and

$$K_i^* = \sup_{t\in[0,T]} \int_0^t k_i(t,s)ds < \infty. \tag{1.3}$$

Let us assume that $u \in L([0,T];X)$ and $A(t)$ is a family linear closed operator defined in a Banach space X. The fractional order integral of the function u is understood here in the Riemann-Liouville sense, that is,

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)ds. \tag{1.4}$$

In this paper, we denote that C is a positive constant and assume that a family of closed linear $\{A(t) : t \in [0,T]\}$ satisfying

(A1) the domain $D(A)$ of $\{A(t) : t \in [0,T]\}$ is dense in the Banach space X and independent of t,

(A2) the operator $[A(t)+\lambda]^{-1}$ exists in $L(X)$ for any λ with $\operatorname{Re}\lambda \le 0$ and

$$\left\| [A(t)+\lambda]^{-1} \right\| \le \frac{C}{|\lambda+1|}, \quad t \in [0,T]. \tag{1.5}$$

(A3) There exists constant $\gamma \in (0,1]$ and C such that

$$\left\| [A(t_1) - A(t_2)]A^{-1}(s) \right\| \le C|t_1 - t_2|^{\gamma}, \quad t_1, t_2, s \in [0,T]. \tag{1.6}$$

Under condition (A2), each operator $-A(s)$, $s \in [0,T]$ generates an analytic semigroup $\exp(-tA(s))$, $t > 0$, and there exists a constant C such that

$$\left\| A^n(s)\exp(-tA(s)) \right\| \le \frac{C}{t^n}, \tag{1.7}$$

where $n = 0, 1$, $t > 0$, $s \in [0,T]$ ([11]).

We study the existence of mild solution of (1.1) and obtain the existence theorem based on the measures of noncompactness. An example is given to show an application of the abstract results.

2. Preliminaries

Throughout this work, we set $I = [0, T]$. We denote by X a Banach space, $L(X)$ the space of all linear and bounded operators on X, and $C(I, X)$ the space of all X-valued continuous functions on I.

Lemma 2.1 (see [9]). (1) $I^q : L^1[0, T] \to L^1[0, T]$.

(2) For $g \in L^1[0, T]$, we have

$$\int_0^t \int_0^\eta (t-\eta)^{q-1} (\eta - s)^{\gamma-1} g(s) ds\, d\eta = B(q, \gamma) \int_0^t (t-s)^{q+\gamma-1} g(s) ds, \tag{2.1}$$

where $B(q, \gamma)$ is a Beta function.

Definition 2.2. Let B be a bounded set of seminormed linear space Y. The Kuratowski's measure of noncompactness (for brevity, α-measure) of B is defined as

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter} \le d\}. \tag{2.2}$$

From the definition, we can get some properties of α-measure immediately, see ([27]).

Lemma 2.3 (see [27]). *Let A and B be bounded sets of X. Then*

(1) $\alpha(A) \le \alpha(B)$, *if* $A \subseteq B$.

(2) $\alpha(A) = \alpha(A^{cl})$, *where* A^{cl} *denotes the closure of* A.

(3) $\alpha(A) = 0$ *if and only if* A *is precompact.*

(4) $\alpha(\lambda A) = |\lambda| \alpha(A)$, $\lambda \in R$.

(5) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(6) $\alpha(A + B) \le \alpha(A) + \alpha(B)$, *where* $A + B = \{x + y : x \in A, y \in B\}$.

(7) $\alpha(A + x_0) = \alpha(A)$, *for any* $x_0 \in X$.

For $H \subset C(I, X)$ *we define*

$$\int_0^t H(s) ds = \left\{ \int_0^t u(s) ds : u \in H \right\}, \tag{2.3}$$

for $t \in I$, *where* $H(s) = \{u(s) \in X : u \in H\}$.

The following lemma will be needed.

Lemma 2.4 (see [27]). *If* $H \subset C(I, X)$ *is a bounded, equicontinuous set, then*

(1) $\alpha(H) = \sup_{t \in I} \alpha(H(t))$.

(2) $\alpha(\int_0^t H(s) ds) \le \int_0^t \alpha(H(s)) ds$, *for* $t \in I$.

Lemma 2.5 (see [28]). *If $\{u_n\}_{n=1}^{\infty} \subset L^1(I,X)$ and there exists a $m(\cdot) \in L^1(I,R^+)$ such that*

$$\|u_n(t)\| \leq m(t), \quad a.e\ t \in I, \tag{2.4}$$

then $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ is integrable and

$$\alpha\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.5}$$

We need to use the following Sadovskii's fixed point theorem.

Definition 2.6 (see [29]). Let P be an operator in Banach space X. If P is continuous and takes bounded, sets into bounded sets, and $\alpha(P(H)) < \alpha(H)$ for every bounded set H of X with $\alpha(H) > 0$, then P is said to be a condensing operator on X.

Lemma 2.7 (Sadovskii's fixed point theorem [29]). *Let P be a condensing operator on Banach space X. If $P(B) \subseteq B$ for a convex, closed, and bounded set B of X, then P has a fixed point in B.*

According to [4], a mild solution of (1.1) can be defined as follows.

Definition 2.8. A function $u \in C(I,X)$ satisfying the equation

$$\begin{aligned} u(t) = {}& A^{-1}(0)g(u) + u_0 + \int_0^t \psi(t-\eta,\eta)U(\eta)A(0)\left[A^{-1}(0)g(u) + u_0\right]d\eta \\ &+ \int_0^t \psi(t-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\dots,(K_nu)(\eta))d\eta \\ &+ \int_0^t\int_0^\eta \psi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\dots,(K_nu)(s))ds\,d\eta, \end{aligned} \tag{2.6}$$

is called a mild solution of (1.1), where

$$\psi(t,s) = q\int_0^\infty \theta t^{q-1}\xi_q(\theta)\exp(-t^q\theta A(s))d\theta, \tag{2.7}$$

and ξ_q is a probability density function defined on $[0,\infty)$ such that its Laplace transform is given by

$$\begin{aligned} &\int_0^\infty e^{-\sigma x}\xi_q(\sigma)d\sigma = \sum_{j=0}^{\infty}\frac{(-x)^j}{\Gamma(1+qj)}, \quad q \in (0,1],\ x > 0, \\ &\varphi(t,\tau) = \sum_{k=1}^{\infty}\varphi_k(t,\tau), \end{aligned} \tag{2.8}$$

where

$$\varphi_1(t,\tau) = [A(t) - A(\tau)]\psi(t-\tau,\tau),$$
$$\varphi_{k+1}(t,\tau) = \int_\tau^t \varphi_k(t,s)\varphi_1(s,\tau)ds, \quad k = 1,2\ldots, \tag{2.9}$$
$$U(t) = -A(t)A^{-1}(0) - \int_0^t \varphi(t,s)A(s)A^{-1}(0)ds.$$

To our purpose, the following conclusions will be needed. For the proofs refer to [4].

Lemma 2.9 (see [4]). *The operator-valued functions $\psi(t-\eta,\eta)$ and $A(t)\psi(t-\eta,\eta)$ are continuous in uniform topology in the variables t, η, where $0 \le \eta \le t-\varepsilon$, $0 \le t \le T$, for any $\varepsilon > 0$. Clearly,*

$$\|\psi(t-\eta,\eta)\| \le C(t-\eta)^{q-1}. \tag{2.10}$$

Moreover, we have

$$\|\varphi(t,\eta)\| \le C(t-\eta)^{\gamma-1}. \tag{2.11}$$

Remark 2.10. From the proof of Theorem 2.5 in [4], we can see

(1) $\|U(t)\| \le C + Ct^\gamma$.

(2) For $t \in I$, $\int_0^t \psi(t-\eta,\eta)U(\eta)d\eta$ is uniformly continuous in the norm of $L(X)$ and

$$\left\|\int_0^t \psi(t-\eta,\eta)U(\eta)d\eta\right\| \le C^2 t^q\left(\frac{1}{q} + t^\gamma B(q,\gamma+1)\right) := \widetilde{M(t)}. \tag{2.12}$$

3. Existence of Solution

Assume that

(B1) $f : I \times X \times X \times \cdots \times X \to X$ satisfies $f(\cdot, v_1, v_2, \ldots, v_n) : I \to X$ is measurable for all $v_i \in X$, $i = 1,2,\ldots,n$ and $f(t,\cdot,\cdot,\ldots,\cdot) : X \times X \times \cdots \times X \to X$ is continuous for a.e $t \in I$, and there exist a positive function $\mu(\cdot) \in L^p(I,R^+)$ $(p > 1/q > 1)$ and a continuous nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ such that

$$\|f(t,v_1,v_2,\ldots,v_n)\| \le \mu(t)\omega\left(\sum_{i=1}^n \|v_i\|\right), \quad (t,v_1,v_2,\ldots,v_n) \in I \times X \times X \times \cdots \times X, \tag{3.1}$$

and set $T_{p,q} = \max\{T^{q-1/p}, T^q\}$.

(B2) For any bounded sets $D, D_1, D_2, \ldots, D_n \subset X$, and $0 \le \tau \le s \le t \le T$,

$$\begin{aligned}
&\alpha(g(D)) \le \beta(t)\alpha(D), \\
&\alpha(\psi(t-s,s)f(s,D_1,D_2,\ldots,D_n)) \\
&\qquad \le \beta_1(t,s)\alpha(D_1) + \beta_2(t,s)\alpha(D_2) + \cdots + \beta_n(t,s)\alpha(D_n), \\
&\alpha(\psi(t-s,s)\varphi(s,\tau)f(\tau,D_1,D_2,\ldots,D_n)) \\
&\qquad \le \zeta_1(t,s,\tau)\alpha(D_1) + \zeta_2(t,s,\tau)\alpha(D_2) + \cdots + \zeta_n(t,s,\tau)\alpha(D_n),
\end{aligned} \tag{3.2}$$

where $\beta(t)$ is a nonnegative function, and $\sup_{t\in I}\beta(t) := \beta < \infty$,

$$\begin{gathered}
\sup_{t\in I}\int_0^t \beta_i(t,s)ds := \beta_i < \infty, \quad i = 1,2,\ldots,n, \\
\sup_{t\in I}\int_0^t\int_0^s \zeta_j(t,s,\tau)d\tau\, ds := \zeta_j < \infty, \quad j = 1,2,\ldots,n.
\end{gathered} \tag{3.3}$$

(B3) $g : C(I;X) \to X$ is continuous and there exists

$$0 < \alpha_1 < \left(C + \widetilde{M(T)}\right)^{-1}, \qquad \alpha_2 \ge 0 \tag{3.4}$$

such that

$$\|g(u)\| \le \alpha_1\|u\| + \alpha_2. \tag{3.5}$$

(B4) The functions μ and ω satisfy the following condition:

$$C(1 + CB(q,\gamma))T_{p,q}^{\gamma}\,\Omega_{p,q}\left(\sum_{i=1}^n K_i^*\right)\|\mu\|_{L^p}\liminf_{\tau\to\infty}\frac{\omega(\tau)}{\tau} < 1 - \alpha_1\left(C + \widetilde{M(T)}\right), \tag{3.6}$$

where $\Omega_{p,q} = ((p-1)/(pq-1))^{(p-1)/p}$, and $T_{p,q}^{\gamma} = \max\{T_{p,q}, T_{p,q+\gamma}\}$.

Theorem 3.1. *Suppose that (B1)–(B4) are satisfied, and if* $(C + \widetilde{M(T)})\beta + 4(\Sigma_{i=1}^n(\beta_i + 2\zeta_i)K_i^*) < 1$, *then* (1.1) *has a mild solution on* $[0,T]$.

Proof. Define the operator $F : C(I;X) \to C(I;X)$ by

$$F(u)(t) = A^{-1}(0)g(u) + u_0 + \int_0^t \varphi(t-\eta,\eta)U(\eta)A(0)\left[A^{-1}(0)g(u) + u_0\right]d\eta$$
$$+ \int_0^t \varphi(t-\eta,\eta) f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))d\eta$$
$$+ \int_0^t\int_0^\eta \varphi(t-\eta,\eta)\varphi(\eta,s) f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))ds\,d\eta, \quad t \in I. \tag{3.7}$$

Then we proceed in five steps.

Step 1. We show that F is continuous.

Let u_i be a sequence that $u_i \to u$ as $i \to \infty$. Since f satisfies (B1), we have

$$f(t,(K_1u_i)(t),(K_2u_i)(t),\ldots,(K_nu_i)(t)) \longrightarrow f(t,(K_1u)(t),(K_2u)(t),\ldots,(K_nu)(t)), \quad \text{as } i \longrightarrow \infty. \tag{3.8}$$

Then

$$\begin{aligned}
&\|F(u_i)(t) - F(u)(t)\| \\
&\quad\le \left\|A^{-1}(0)\right\| \|g(u_i) - g(u)\| + \int_0^t \|\varphi(t-\eta,\eta)U(\eta)\| \|g(u_i) - g(u)\| d\eta \\
&\qquad + \int_0^t \|\varphi(t-\eta,\eta)[f(\eta,(K_1u_i)(\eta),(K_2u_i)(\eta),\ldots,(K_nu_i)(\eta)) \\
&\qquad\qquad -f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))]\| d\eta \\
&\qquad + \int_0^t\int_0^\eta \|\varphi(t-\eta,\eta)\varphi(\eta,s)[f(s,(K_1u_i)(s),(K_2u_i)(s),\ldots,(K_nu_i)(s)) \\
&\qquad\qquad -f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))]\| ds\,d\eta.
\end{aligned} \tag{3.9}$$

According to the condition (A2), (2.12), and the continuity of g, we have

$$\begin{gathered}
\left\|A^{-1}(0)\right\| \|g(u_i) - g(u)\| \longrightarrow 0, \quad \text{as } i \longrightarrow \infty; \\
\int_0^t \|\varphi(t-\eta,\eta)U(\eta)\| \|g(u_i) - g(u)\| d\eta \longrightarrow 0, \quad \text{as } i \longrightarrow \infty.
\end{gathered} \tag{3.10}$$

Noting that $u_i \to u$ in $C(I,X)$, there exists $\varepsilon > 0$ such that $\|u_i - u\| \leq \varepsilon$ for i sufficiently large. Therefore, we have

$$\begin{aligned} &\left\| \left[f(t,(K_1u_i)(t),(K_2u_i)(t),\ldots,(K_nu_i)(t)) - f(t,(K_1u)(t),(K_2u)(t),\ldots,(K_nu)(t)) \right] \right\| \\ &\quad \leq \mu(t)\left[\omega\left(\sum_{j=1}^{n}\left\|(K_ju_i)(t)\right\|\right) + \omega\sum_{j=1}^{n}\left\|(K_ju)(t)\right\|\right] \\ &\quad \leq \mu(t)\left[\omega\left(\sum_{j=1}^{n}K_j^*(\|u\|+\varepsilon)\right) + \omega\left(\sum_{j=1}^{n}K_j^*\|u\|\right)\right]. \end{aligned} \tag{3.11}$$

Using (2.10) and by means of the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} &\int_0^t \left\| \varphi(t-\eta,\eta)\left[f(\eta,(K_1u_i)(\eta),(K_2u_i)(\eta),\ldots,(K_nu_i)(\eta)) \right.\right. \\ &\qquad\qquad \left.\left. - f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta)) \right] \right\| d\eta \\ &\quad \leq C\int_0^t (t-\eta)^{q-1} \left\| \left[f(\eta,(K_1u_i)(\eta),(K_2u_i)(\eta),\ldots,(K_nu_i)(\eta)) \right.\right. \\ &\qquad\qquad \left.\left. - f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta)) \right] \right\| d\eta, \\ &\quad \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \end{aligned} \tag{3.12}$$

Similarly, by (2.10) and (2.11), we have

$$\begin{aligned} &\int_0^t\int_0^\eta \left\| \varphi(t-\eta,\eta)\varphi(\eta,s) \right. \\ &\qquad \times \left[f(s,(K_1u_i)(t),(K_2u_i)(t),\ldots,(K_nu_i)(t)) \right. \\ &\qquad\quad \left.\left. - f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s)) \right] \right\| ds\, d\eta \\ &\quad \leq C^2\int_0^t\int_0^\eta (t-\eta)^{q-1}(\eta-s)^{\gamma-1} \\ &\qquad \times \left\| f(s,(K_1u_i)(t),(K_2u_i)(t),\ldots,(K_nu_i)(t)) \right. \\ &\qquad\quad \left. - f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s)) \right\| ds\, d\eta \\ &\quad \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \end{aligned} \tag{3.13}$$

Therefore, we deduce that

$$\lim_{i\to\infty}\|F(u_i) - F(u)\| = 0. \tag{3.14}$$

Step 2. We show that F maps bounded sets of $C(I,X)$ into bounded sets in $C(I,X)$.

For any $r>0$, we set $B_r=\{u\in C(I,X):\|u\|\le r\}$. Now, for $u\in B_r$, by (B1), we can see

$$\|f(t,(K_1u)(t),(K_2u)(t),\dots,(K_nu)(t))\|\le\mu(t)\omega\left(\sum_{j=1}^{n}K_j^*r\right). \quad (3.15)$$

Based on (2.12), we denote that $S(t):=\int_0^t\varphi(t-\eta,\eta)U(\eta)d\eta$, we have

$$\|S(t)A(0)u_0\|\le C^2t^q\left(\frac{1}{q}+t^\gamma B(q,\gamma+1)\right)\|A(0)u_0\|=\widetilde{M(t)}\|A(0)u_0\|. \quad (3.16)$$

Then for any $u\in B_r$, by (A2), (2.10), (2.11), and Lemma 2.1, we have

$$\begin{aligned}\|(Fu)(t)\|\le{}&\left\|A^{-1}(0)g(u)\right\|+\|u_0\|+\left\|S(t)g(u)\right\|+\|S(t)A(0)u_0\|\\&+\int_0^t\|\varphi(t-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\dots,(K_nu)(\eta))\|d\eta\\&+\int_0^t\int_0^\eta\|\varphi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\dots,(K_nu)(s))\|ds\,d\eta\\\le{}&\left(C+\widetilde{M(t)}\right)\|g(u)\|+\|u_0\|+\widetilde{M(t)}\|A(0)u_0\|\\&+C\int_0^t(t-\eta)^{q-1}\mu(\eta)\omega\left(\sum_{j=1}^{n}K_j^*r\right)d\eta\\&+C^2\int_0^t\int_0^\eta(t-\eta)^{q-1}(\eta-s)^{\gamma-1}\mu(s)\omega\left(\sum_{j=1}^{n}K_j^*r\right)ds\,d\eta\\\le{}&\alpha_1\left(C+\widetilde{M(t)}\right)\|u\|+\alpha_2\left(C+\widetilde{M(t)}\right)+\|u_0\|+\widetilde{M(t)}\|A(0)u_0\|\\&+M_1\left[C\int_0^t(t-\eta)^{q-1}\mu(\eta)d\eta+C^2B(q,\gamma)\int_0^t(t-\eta)^{q+\gamma-1}\mu(\eta)d\eta\right],\end{aligned} \quad (3.17)$$

where $M_1=\omega(\sum_{j=1}^{n}K_j^*r)$.

By means of the Hölder inequality, we have

$$\begin{aligned}\int_0^t(t-\eta)^{q-1}\mu(\eta)d\eta&=t^{(pq-1)/p}M_{p,q}\|\mu\|_{L^p}\le T_{p,q}\Omega_{p,q}\|\mu\|_{L^p},\\\int_0^t(t-\eta)^{\gamma+q-1}\mu(\eta)d\eta&\le T_{p,q+\gamma}\Omega_{p,q+\gamma}\|\mu\|_{L^p}.\end{aligned} \quad (3.18)$$

Thus

$$\|(Fu)(t)\| \leq \alpha_1\Big(C+\widetilde{M(T)}\Big)r+\alpha_2\Big(C+\widetilde{M(T)}\Big)+\|u_0\|+\widetilde{M(T)}\|A(0)u_0\| \\ +M_1\Omega_{p,q}T_{p,q}^{\gamma}\Big[C+C^2B(q,\gamma)\Big]\|\mu\|_{L^p}:=\tilde{r}. \tag{3.19}$$

This means $F(B_r)\subset B_{\tilde{r}}$.

Step 3. We show that there exists $m\in N$ such that $F(B_m)\subset B_m$.

Suppose the contrary, that for every $m\in N$, there exists $u_m\in B_m$ and $t_m\in I$, such that $\|(Fu_m)(t_m)\|>m$. However, on the other hand

$$\|f(t,(K_1u_m)(t),(K_2u_m)(t),\ldots,(K_nu_m)(t))\|\leq \mu(t)\omega\left(\sum_{j=1}^{n}K_j^*m\right), \tag{3.20}$$

we have

$$\begin{aligned} m<\|(Fu_m)(t_m)\| &\leq \alpha_1\Big(C+\widetilde{M(T)}\Big)\|u_m\|+\alpha_2\Big(C+\widetilde{M(T)}\Big)+\|u_0\| \\ &\quad+\widetilde{M(T)}\|A(0)u_0\|+M_1\left[C\int_0^{t_m}(t_m-\eta)^{q-1}\mu(\eta)d\eta+C^2B(q,\gamma)\int_0^{t_m}(t_m-\eta)^{q+\gamma-1}\mu(\eta)d\eta\right] \\ &\leq \alpha_1\Big(C+\widetilde{M(T)}\Big)\|u_m\|+\alpha_2\Big(C+\widetilde{M(T)}\Big)+\|u_0\| \\ &\quad+\widetilde{M(T)}\|A(0)u_0\|+M_1\Omega_{p,q}T_{p,q}^{\gamma}\Big[C+C^2B(q,\gamma)\Big]\|\mu\|_{L^p} \\ &\leq \alpha_1\Big(C+\widetilde{M(T)}\Big)m+\alpha_2\Big(C+\widetilde{M(T)}\Big)+\|u_0\| \\ &\quad+\widetilde{M(T)}\|A(0)u_0\|+M_1\Omega_{p,q}T_{p,q}^{\gamma}\Big[C+C^2B(q,\gamma)\Big]\|\mu\|_{L^p}. \end{aligned} \tag{3.21}$$

Dividing both sides by m and taking the lower limit as $m\to\infty$, we obtain

$$C(1+CB(q,\gamma))T_{p,q}^{\gamma}\,\Omega_{p,q}\sum_{j=1}^{n}K_j^*\|\mu\|_{L^p}\liminf_{m\to\infty}\frac{\omega(m)}{m}\geq 1-\alpha_1\Big(C+\widetilde{M(T)}\Big) \tag{3.22}$$

which contradicts (B4).

Step 4. Denote

$$F(u)(t)=A^{-1}(0)g(u)+u_0+\int_0^t\varphi(t-\eta,\eta)U(\eta)A(0)\Big[A^{-1}(0)g(u)+u_0\Big]d\eta+G(u)(t), \tag{3.23}$$

where

$$G(u)(t)=\int_0^t \psi(t-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))d\eta$$
$$+\int_0^t\int_0^\eta \psi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))ds\,d\eta. \tag{3.24}$$

We show that $G(u)(\cdot)$ is equicontinuous.

Let $0<t_2<t_1<T$ and $u\in B_m$. Then

$$\|(Gu)(t_1)-(Gu)(t_2)\|\le I_1+I_2+I_3+I_4, \tag{3.25}$$

where

$$I_1=\int_0^{t_2}\|[\psi(t_1-\eta,\eta)-\psi(t_2-\eta,\eta)]f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))\|d\eta,$$
$$I_2=\int_{t_2}^{t_1}\|\psi(t_1-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))\|d\eta,$$
$$I_3=\int_0^{t_2}\int_0^\eta\|[\psi(\mathrm{t}_1-\eta,\eta)-\psi(t_2-\eta,\eta)]\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))\|ds\,d\eta,$$
$$I_4=\int_{t_2}^{t_1}\int_0^\eta\|\psi(t_1-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))\|ds\,d\eta. \tag{3.26}$$

It follows from Lemma 2.9, (B1), and (3.20) that $I_1,I_3\to 0$, as $t_2\to t_1$.

For I_2, from (2.10), (3.20), and (B1), we have

$$I_2=\int_{t_2}^{t_1}\|\psi(t_1-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))\|d\eta$$
$$\le CM_1\int_{t_2}^{t_1}(t_1-\eta)^{q-1}\mu(\eta)d\eta\longrightarrow 0,\quad \text{as } t_2\longrightarrow t_1. \tag{3.27}$$

Similarly, by (2.10), (2.11), (B1), and Lemma 2.1, we have

$$I_4=\int_{t_2}^{t_1}\int_0^\eta\|\psi(t_1-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))\|ds\,d\eta$$
$$\le C^2M_1\int_{t_2}^{t_1}(t_1-\eta)^{q-1}\int_0^\eta(\eta-s)^{\gamma-1}\mu(s)ds\,d\eta\longrightarrow 0,\quad \text{as } t_2\longrightarrow t_1. \tag{3.28}$$

Step 5. We show that $\alpha(F(H)) < \alpha(H)$ for every bounded set $H \subset B_m$. For any $\varepsilon > 0$, we can take a sequence $\{h_v\}_{v=1}^{\infty} \subset H$ such that

$$\alpha(H) \leq 2\alpha(\{h_v\}) + \varepsilon, \tag{3.29}$$

(cf. [30]). So it follows from Lemmas 2.3–2.5, 2.9, (2) in Remark 2.10, and (B2) that

$$\begin{aligned}
\alpha(F(H)) &\leq C\alpha\big(g(H)\big) + \widetilde{M(T)}\alpha\big(g(H)\big) + 2\alpha(G\{h_v\}) + \varepsilon \\
&\leq C\alpha\big(g(H)\big) + \widetilde{M(T)}\alpha\big(g(H)\big) \\
&\quad + 2\sup_{t\in I}\alpha\Bigg(\left\{\int_0^t \psi(t-\eta,\eta) f(\eta,(K_1h_v)(\eta),(K_2h_v)(\eta),\dots,(K_nh_v)(\eta))d\eta\right\} \\
&\qquad + \left\{\int_0^t\int_0^\eta \psi(t-\eta,\eta)\varphi(\eta,s)\right. \\
&\qquad\qquad \left.\times f(s,(K_1h_v)(s),(K_2h_v)(s),\dots,(K_nh_v)(s))ds\,d\eta\right\}\Bigg) + \varepsilon \\
&\leq C\beta\alpha(H) + \widetilde{M(T)}\beta\alpha(H) \\
&\quad +4\sup_{t\in I}\left(\int_0^t \alpha(\{\psi(t-\eta,\eta) f(\eta,(K_1h_v)(\eta),(K_2h_v)(\eta),\dots,(K_nh_v)(\eta))\})d\eta\right) \\
&\quad +8\sup_{t\in I}\left(\int_0^t\int_0^\eta \alpha(\{\psi(t-\eta,\eta)\varphi(\eta,s) f(s,(K_1h_v)(s),(K_2h_v)(s),\dots,(K_nh_v)(s))\})\right) \\
&\quad + \varepsilon \leq C\beta\alpha(H) + \widetilde{M(T)}\beta\alpha(H) + 4\sup_{t\in I}\left(\int_0^t\left(\sum_{i=1}^n \beta_i(t,\eta)K_i^*\right)\alpha(\{h_v\})d\eta\right) \\
&\quad + 8\sup_{t\in I}\left(\int_0^t\int_0^\eta\left(\sum_{i=1}^n \zeta_i(t,\eta,s)K_i^*\right)\alpha(\{h_v\})ds\,d\eta\right) + \varepsilon \\
&\leq C\beta\alpha(H) + \widetilde{M(T)}\beta\alpha(H) + \left(4\sum_{i=1}^n \beta_iK_i^* + 8\sum_{i=1}^n \zeta_iK_i^*\right)\alpha(\{h_v\}) + \varepsilon \\
&= \left[\left(C+\widetilde{M(T)}\right)\beta + 4\left(\sum_{i=1}^n(\beta_i+2\zeta_i)K_i^*\right)\right]\alpha(H) + \varepsilon.
\end{aligned} \tag{3.30}$$

Since ε is arbitrary, we can obtain

$$\alpha(F(H)) \leq \left[\left(C+\widetilde{M(T)}\right)\beta + 4\left(\Sigma_{i=1}^n(\beta_i+2\zeta_i)K_i^*\right)\right]\alpha(H) < \alpha(H). \tag{3.31}$$

In summary, we have proven that F has a fixed point $\widetilde{u} \in B_m$. Consequently, (1.1) has at least one mild solution. □

Our next result is based on the Banach's fixed point theorem.

(G1) There exists a positive function $l(\cdot) \in L^1(I, R^+)$ and a constant $\mu > 0$ such that

$$\begin{aligned} &\|g(u) - g(u^*)\| \le \mu\|u - u^*\|, \\ &\|f(t, v_1, v_2, \dots, v_n) - f(t, w_1, w_2, \dots, w_n)\| \\ &\quad \le l(t)\left(\sum_{i=1}^{n}\|v_i - w_i\|\right), \quad (v_i, w_i) \in X^2,\ i = 1, 2, \dots, n. \end{aligned} \tag{3.32}$$

(G2) There exists a constant $0 < \delta < 1$ such that the function $\Lambda : I \to R^+$ defined by

$$\Lambda(t) = \mu\left(C + \widetilde{M(T)}\right) + C\left(\sum_{i=1}^{n}K_i^*\right)\Gamma(q)I^q l(t) + C^2\left(\sum_{i=1}^{n}K_i^*\right)\Gamma(q)\Gamma(\gamma)I^{q+\gamma}l(t) \le \delta, \quad t \in I. \tag{3.33}$$

Theorem 3.2. *Assume that (G1), (G2) are satisfied, then* (1.1) *has a unique mild solution.*

Proof. Let F be defined as in Theorem 3.1. For any $u, u^* \in C(I, X)$, we have

$$\begin{aligned} &\|f(t, (K_1u)(t), (K_2u)(t), \dots, (K_nu)(t)) - f(t, (K_1u^*)(t), (K_2u^*)(t), \dots, (K_nu^*)(t))\| \\ &\quad \le l(t)\left(\sum_{i=1}^{n}\|(K_iu)(t) - (K_iu^*)(t)\|\right) \\ &\quad \le l(t)\sum_{i=1}^{n}K_i^*\|u - u^*\|. \end{aligned} \tag{3.34}$$

Thus, from (A2), (2.10), (2.11), Lemma 2.1, we have

$$\begin{aligned} &\|(Fu)(t) - (Fu^*)(t)\| \\ &\le \mu C\|u - u^*\| + \mu\int_0^t\|\varphi(t-\eta, \eta)U(\eta)\|\|u - u^*\|d\eta \\ &\quad + \int_0^t\|\varphi(t-\eta, \eta)\|\|f(\eta, (K_1u)(\eta), (K_2u)(\eta), \dots, (K_nu)(\eta)) \\ &\qquad\qquad - f(\eta, (K_1u^*)(\eta), (K_2u^*)(\eta), \dots, (K_nu^*)(\eta))\|d\eta \\ &\quad + \int_0^t\int_0^\eta\|\varphi(t-\eta, \eta)\varphi(\eta, s)\|\|f(s, (Ku)(s), (Hu)(s)) - f(s, (Ku^*)(s), (Hu^*)(s))\|ds\, d\eta \end{aligned}$$

$$\le \|u-u^*\|\left[\mu\left(C+\widetilde{M(T)}\right)+C\left(\sum_{i=1}^{n}K_i^*\right)\int_0^t (t-\eta)^{q-1}l(\eta)d\eta\right.$$
$$\left.+C^2\left(\sum_{i=1}^{n}K_i^*\right)\int_0^t\int_0^\eta (t-\eta)^{q-1}(\eta-s)^{\gamma-1}l(s)ds\,d\eta\right]$$
$$=\left[\mu\left(C+\widetilde{M(T)}\right)+C\left(\sum_{i=1}^{n}K_i^*\right)\Gamma(q)I^q l(t)+C^2\left(\sum_{i=1}^{n}K_i^*\right)\Gamma(q)\Gamma(\gamma)I^{q+\gamma}l(t)\right]\|u-u^*\|$$
$$=\Lambda(t)\|u-u^*\|. \tag{3.35}$$

We get

$$\|F(u)-F(u^*)\| \le \delta\|u-u^*\|. \tag{3.36}$$

By the Banach contraction mapping principle, F has a unique fixed point, which is a mild solution of (1.1). □

4. An Example

To illustrate the usefulness of our main result, we consider the following fractional differential equation:

$$\frac{\partial^q}{\partial t^q}u(t,\xi)=b(t,\xi)\frac{\partial^2}{\partial\xi^2}u(t,\xi)+\frac{t^n}{n}\int_0^t(t-s)u(s,\xi)ds+\frac{t^n}{n}\int_0^t e^{-(t+s)}u(s,\xi)ds, \quad \xi\in[0,1],$$
$$u(t,0)=u(t,1)=0, \tag{4.1}$$
$$u(0,\xi)=-\int_0^\xi\int_0^y b^{-1}(0,x)\sin\left|\frac{u}{\lambda}\right|dx\,dy,$$

where $0<q<1$, $0\le t\le 1$, $\lambda > C+\widetilde{M(1)}$, $n\in N$, $b(t,\xi)$ is continuous function and is uniformly Hölder continuous in t, that is, there exist $C>0$ and $\gamma\in(0,1)$ such that

$$\|b(t_1,\xi)-b(t_2,\xi)\|\le C|t_1-t_2|^\gamma, \quad 0\le t_1\le t_2\le 1. \tag{4.2}$$

Let $X=L^2([0,1],R)$ and define $A(t)$ by

$$D(A(t))=H^2(0,1)\cap H_0^1(0,1)=\left\{H^2(0,1):z(0)=z(1)=0\right\},$$
$$-A(t)(z)=b(t,\xi)z''. \tag{4.3}$$

Then $-A(s)$ generates an analytic semigroup $\exp(-tA(s))$.

For $t \in [0,1]$, $\xi \in [0,1]$, we set

$$\begin{aligned} u(t)(\xi) &= u(t,\xi), \\ g(u) &= \sin\left|\frac{u}{\lambda}\right|, \\ A^{-1}(0)g(u) &= -\int_0^{\xi}\int_0^{y} b^{-1}(0,x)\sin\left|\frac{u}{\lambda}\right| dx\,dy, \\ f(t,(K_1u)(t),(K_2u)(t))(\xi) &= \frac{t^n}{n}\int_0^t (t-s)u(s,\xi)ds + \frac{t^n}{n}\int_0^t e^{-(t+s)}u(s,\xi)ds, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} (K_1u(t))(\xi) &= \int_0^t (t-s)u(s,\xi)ds, \\ (K_2u(t))(\xi) &= \int_0^t e^{-(t+s)}u(s,\xi)ds, \\ K_1^* &= \sup_{t\in I}\int_0^t (t-s)ds < \frac{1}{2} < \infty, \\ K_2^* &= \sup_{t\in I}\int_0^t e^{-(t+s)}ds = \frac{1}{4} < \infty. \end{aligned} \tag{4.5}$$

Moreover, we can get

$$\begin{aligned} \|g(u)\| &\le \frac{1}{\lambda}\|u\|, \\ \alpha(g(D)) &\le \frac{1}{\lambda}\alpha(D) \end{aligned} \tag{4.6}$$

for any $D \subset X$. Then the above equation (4.1) can be written in the abstract form as (1.1). On the other hand,

$$\begin{aligned} \|f(t,(Ku)(t),(Hu)(t))(\xi)\| &\le \frac{t^n}{n}\|(K_1u)(t,\xi)\| + \|(K_2u)(t,\xi)\| \\ &\le \frac{t^n}{n}\left(K_1^*\|u\| + K_2^*\|u\|\right) \\ &= \mu(t)\omega\left(K_1^*\|u\| + K_2^*\|u\|\right), \end{aligned} \tag{4.7}$$

where $\mu(t) = t^n$, $\omega(z) = z/n$ satisfying (B1). For any $u_1, u_2 \in X$,

$$\begin{aligned}&\left\| \psi(t-s,s) f(s,(K_1u_1)(s),(K_2u_1)(s))(\xi) - \psi(t-s,s) f(s,(K_1u_2)(s),(K_2u_2)(s))(\xi) \right\| \\ &\quad \le \frac{Cs^n}{n}(t-s)^{q-1}\left(\|(K_1u_1)(s)(\xi) - (K_1u_2)(s)(\xi)\| + \|(K_2u_1)(s)(\xi) - (K_2u_2)(s)(\xi)\|\right).\end{aligned} \tag{4.8}$$

Therefore, for any bounded sets $D_1, D_2 \subset X$, we have

$$\alpha\big(\psi(t-s,s) f(s,D_1,D_2)\big) \le \frac{Cs^n}{n}(t-s)^{q-1}(\alpha(D_1) + \alpha(D_2)). \tag{4.9}$$

Moreover,

$$\frac{C}{n}\sup_{t\in[0,1]}\int_0^t (t-s)^{q-1}s^n ds = \frac{C}{n}\sup_{t\in[0,1]} t^{n+q}B\big(q,n+1\big) = \frac{C}{n}B\big(q,n+1\big) := \beta_1 = \beta_2. \tag{4.10}$$

Similarly, we obtain

$$\begin{aligned}&\alpha\big(\psi(t-s,s)\varphi(s,\tau) f(\tau,D_1,D_2)\big) \le \frac{C^2}{n}(t-s)^{q-1}(s-\tau)^{\gamma-1}\tau^n(\alpha(D_1) + \alpha(D_2)), \\ &\frac{C^2}{n}\sup_{t\in[0,1]}\int_0^t\int_0^s (t-s)^{q-1}(s-\tau)^{\gamma-1}\tau^n d\tau\, ds \le \frac{C^2}{n}B(q,\gamma)B\big(q+\gamma,n+1\big) := \zeta_1 = \zeta_2.\end{aligned} \tag{4.11}$$

Suppose further that

(1) $(3/4n)C(1 + CB(q,\gamma))((p-1)/(pq-1))^{(p-1)/p}\|\mu\|_{L^p} < 1 - (C + \widetilde{M(1)})/\lambda$,

(2) $(1/\lambda)(C + \widetilde{M(1)}) + 3(\beta_1 + 2\zeta_1) < 1$.

Then (4.1) has a mild solution by Theorem 3.1.

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