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Review Article

On the Generalized q-Genocchi Numbers and Polynomials of Higher-Order

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We first consider the q-extension of the generating function for the higher-order generalized Genocchi numbers and polynomials attached to χ. The purpose of this paper is to present a systemic study of some families of higher-order generalized q-Genocchi numbers and polynomials attached to χ by using the generating function of those numbers and polynomials.

1. Introduction

As a well known definition, the Genocchi polynomials are defined by

$$\left(\frac{2t}{e^t+1}\right)e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad |t| < \pi, \tag{1.1}$$

where we use the technical method's notation by replacing $G^n(x)$ by $G_n(x)$, symbolically, (see [1, 2]). In the special case $x = 0$, $G_n = G_n(0)$ are called the nth Genocchi numbers. From the definition of Genocchi numbers, we note that $G_1 = 1, G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given by $G_{2n} = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}(0)$ (see [3]), where B_n is a Bernoulli number and $E_n(x)$ is an Euler polynomial. The first few Genocchi numbers for $2, 4, 6, \ldots$ are $-1, 1, -3, 17, -155, 2073, \ldots$. The first few prime Genocchi numbers are given by $G_6 = -3$ and $G_8 = 17$. It is known that there are no other prime Genocchi numbers with $n < 10^5$. For a real or complex parameter α, the higher-order Genocchi polynomials are defined by

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!} \tag{1.2}$$

(see [1, 4]). In the special case $x = 0$, $G_n^{(\alpha)} = G_n^{(\alpha)}(0)$ are called the nth Genocchi numbers of order α. From (1.1) and (1.2), we note that $G_n = G_n^{(1)}$. For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$, let χ be the Dirichlet character with conductor d. It is known that the generalized Genocchi polynomials attached to χ are defined by

$$\left(\frac{2t\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{at}}{e^{dt}+1}\right)e^{xt} = \sum_{n=0}^{\infty} G_{n,\chi}(x)\frac{t^n}{n!} \tag{1.3}$$

(see [1]). In the special case $x = 0$, $G_{n,\chi} = G_{n,\chi}(0)$ are called the nth generalized Genocchi numbers attached to χ (see [1, 4–6]).

For a real or complex parameter α, the generalized higher-order Genocchi polynomials attached to χ are also defined by

$$\left(\frac{2t\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{at}}{e^{dt}+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_{n,\chi}^{(\alpha)}(x)\frac{t^n}{n!} \tag{1.4}$$

(see [7]). In the special case $x = 0$, $G_{n,\chi}^{(\alpha)} = G_{n,\chi}^{(\alpha)}(0)$ are called the nth generalized Genocchi numbers attached to χ of order α (see [1, 4–9]). From (1.3) and (1.4), we derive $G_{n,\chi} = G_{n,\chi}^{(1)}$.

Let us assume that $q \in \mathbb{C}$ with $|q| < 1$ as an indeterminate. Then we, use the notation

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1.5}$$

The q-factorial is defined by

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \tag{1.6}$$

and the Gaussian binomial coefficient is also defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!} \tag{1.7}$$

(see [5, 10]). Note that

$$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{1.8}$$

It is known that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q = q^{n+1-k}\binom{n}{k-1}_q + \binom{n}{k}_q, \tag{1.9}$$

(see [5, 10]). The q-binomial formula are known that

$$(x-y)_q^n = (x-y)(x-qy)\cdots\left(x-q^{n-1}y\right) = \sum_{i=0}^{n}\binom{n}{i}_q q^{\binom{i}{2}}(-1)^i x^{n-i}y^i,$$
$$\frac{1}{(x-y)_q^n} = \frac{1}{(x-y)(x-qy)\cdots(x-q^{n-1}y)} = \sum_{l=0}^{\infty}\binom{n+l-1}{l}_q x^{n-l}y^l, \tag{1.10}$$

(see[10, 11]).

There is an unexpected connection with q-analysis and quantum groups, and thus with noncommutative geometry q-analysis is a sort of q-deformation of the ordinary analysis. Spherical functions on quantum groups are q-special functions. Recently, many authors have studied the q-extension in various areas (see [1–15]). Govil and Gupta [10] have introduced a new type of q-integrated Meyer-König-Zeller-Durrmeyer operators, and their results are closely related to the study of q-Bernstein polynomials and q-Genocchi polynomials, which are treated in this paper. In this paper, we first consider the q-extension of the generating function for the higher-order generalized Genocchi numbers and polynomials attached to χ. The purpose of this paper is to present a systemic study of some families of higher-order generalized q-Genocchi numbers and polynomials attached to χ by using the generating function of those numbers and polynomials.

2. Generalized q-Genocchi Numbers and Polynomials

For $r \in \mathbb{N}$, let us consider the q-extension of the generalized Genocchi polynomials of order r attached to χ as follows:

$$F_{q,\chi}^{(r)}(t,x) = 2^r t^r \sum_{m_1,\ldots,m_r=0}^{\infty}\left(\prod_{j=1}^{r}\chi(m_j)\right)(-1)^{\sum_{j=1}^{r} m_j} e^{[x+m_1+\cdots+m_r]_q t} = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.1}$$

Note that

$$\lim_{q\to 1} F_{q,\chi}^{(r)}(t,x) = \left(\frac{2t\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{at}}{e^{dt}+1}\right)^r e^{xt}. \tag{2.2}$$

By (2.1) and (1.4), we can see that $\lim_{q\to 1} G_{n,\chi,q}^{(r)}(x) = G_{n,\chi}^{(r)}(x)$. From (2.1), we note that

$$G_{0,\chi,q}^{(r)}(x) = G_{1,\chi,q}^{(r)}(x) = \cdots = G_{r-1,\chi,q}^{(r)}(x) = 0,$$
$$\frac{G_{n+r,\chi,q}^{(r)}(x)}{\binom{n+r}{r}r!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty}\left(\prod_{j=1}^{r}\chi(m_j)\right)(-1)^{\sum_{j=1}^{r} m_j}[x+m_1+\cdots+m_r]_q^n. \tag{2.3}$$

In the special case $x = 0$, $G_{n,\chi,q}^{(r)} = G_{n,\chi,q}^{(r)}(0)$ are called the nth generalized q-Genocchi numbers of order r attached to χ. Therefore, we obtain the following theorem.

Theorem 2.1. *For $r \in \mathbb{N}$, one has*

$$\frac{G_{n+r,\chi,q}^{(r)}}{\binom{n+r}{r} r!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} [m_1 + \cdots + m_r]_q^n. \tag{2.4}$$

Note that

$$\begin{aligned} &2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} [m_1 + \cdots + m_r]_q^n \\ &\quad = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{j=1}^{r} \chi(a_j) \right) \frac{(-q^l)^{\sum_{i=1}^{r} a_i}}{(1+q^{ld})^r}. \end{aligned} \tag{2.5}$$

Thus we obtain the following corollary.

Corollary 2.2. *For $r \in \mathbb{N}$, we have*

$$\begin{aligned} \frac{G_{n+r,\chi,q}^{(r)}}{\binom{n+r}{r} r!} &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{j=1}^{r} \chi(a_j) \right) \frac{(-q^l)^{\sum_{i=1}^{r} a_i}}{(1+q^{ld})^r} \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \sum_{a_1,\ldots,a_r=0}^{d-1} (-1)^{\sum_{i=1}^{r} a_i} \left(\prod_{i=1}^{r} \chi(a_i) \right) \left[\sum_{i=1}^{r} a_i + md \right]_q^n. \end{aligned} \tag{2.6}$$

For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, one also considers the extended higher-order generalized (h, q)-Genocchi polynomials as follows:

$$\begin{aligned} F_{q,\chi}^{(h,r)}(t, x) &= 2^r t^r \sum_{m_1,\ldots,m_r=0}^{\infty} q^{\sum_{j=1}^{r} (h-j) m_j} \left(\prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} e^{[x + \sum_{j=1}^{r} m_j]_q t} \\ &= \sum_{n=0}^{\infty} G_{n,\chi,q}^{(h,r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

From (2.7), *one notes that*

$$\begin{aligned} G^{(h,r)}_{0,\chi,q}(x) &= G^{(h,r)}_{1,\chi,q}(x) = \cdots = G^{(h,r)}_{r-1,\chi,q}(x) = 0, \\ \frac{G^{(h,r)}_{n+r,\chi,q}(x)}{\binom{n+r}{r} r!} &= 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} q^{\sum_{j=1}^{r}(h-j)m_j} \left(\prod_{i=1}^{r} \chi(m_i)\right) (-1)^{\sum_{j=1}^{r} m_j} [x+m_1+\cdots+m_r]_q^n \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx} (-1)^l \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{j=1}^{r} \chi(a_j)\right) q^{\sum_{j=1}^{r}(h-j)a_j} (-1)^{a_1+\cdots+a_r} q^{l(a_1+\cdots+a_r)} \\ &\quad \times \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} q^{d(m_1+\cdots+m_r)+d(\sum_{j=1}^{r}(h-j)m_j)} \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l} q^{lx} (-1)^l \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{j=1}^{r} \chi(a_j)\right) q^{\sum_{j=1}^{r}(h-j)a_j} (-q^l)^{\sum_{j=1}^{r} a_i}}{(-q^{d(h-r+l)};q)_r}, \end{aligned} \tag{2.8}$$

where $(-x;q)_r = (1+x)(1+xq)\cdots(1+xq^{r-1})$.

Therefore, we obtain the following theorem.

Theorem 2.3. *For* $h \in \mathbb{Z}, r \in \mathbb{N}$, *one has*

$$\begin{aligned} \frac{G^{(h,r)}_{n+r,\chi,q}(x)}{\binom{n+r}{r} r!} &= 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} q^{\sum_{j=1}^{r}(h-j)m_j} \left(\prod_{i=1}^{r} \chi(m_i)\right) (-1)^{\sum_{j=1}^{r} m_j} [x+m_1+\cdots+m_r]_q^n \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l} (-q^x)^l \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{j=1}^{r} \chi(a_j)\right) q^{\sum_{j=1}^{r}(h-j)a_j} (-q^l)^{\sum_{j=1}^{r} a_i}}{(-q^{d(h-r+l)};q)_r}, \\ G^{(h,r)}_{0,\chi,q}(x) &= G^{(h,r)}_{1,\chi,q}(x) = \cdots = G^{(h,r)}_{r-1,\chi,q}(x) = 0. \end{aligned} \tag{2.9}$$

Note that

$$\frac{1}{(-q^{d(h-r+l)};q)_r} = \frac{1}{(1+q^{d(h-r+l)})} = \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{d(h-r+l)m}. \tag{2.10}$$

By (2.10), *one sees that*

$$\begin{aligned} &\frac{1}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l} (-1)^l q^{l(x+\sum_{i=1}^{r} a_i)}}{(-q^{d(h-r+l)};q)_r} \\ &\quad = \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{d(h-r)m} \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l(x+\sum_{i=1}^{r} a_i+dm)} \\ &\quad = \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{d(h-r)m} \left[x+\sum_{i=1}^{r} a_i+dm\right]_q^n. \end{aligned} \tag{2.11}$$

By (2.10) and (2.11), we obtain the following corollary.

Corollary 2.4. *For $h \in \mathbb{Z}, r \in \mathbb{N}$, we have*

$$\frac{G_{n+r,\chi,q}^{(h,r)}(x)}{\binom{n+r}{r} r!} = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{d(h-r)m} \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{j=1}^{r} \chi(a_j) \right) q^{\sum_{j=1}^{r}(h-j)a_j} \left[x + \sum_{i=1}^{r} a_i + dm \right]_q^n \tag{2.12}$$

By (2.7), we can derive the following corollary.

Corollary 2.5. *For $h \in \mathbb{Z}, r, d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have*

$$\begin{gathered} q^{d(h-1)} \frac{G_{n+r,\chi,q}^{(h,r)}(x+d)}{\binom{n+r}{r} r!} + \frac{G_{n+r,\chi,q}^{(h,r)}(x)}{\binom{n+r}{r} r!} = 2 \sum_{l=0}^{d-1} \chi(l)(-1)^l \frac{G_{n+r-1,\chi,q}^{(h-1,r-1)}}{\binom{n+r-1}{r-1}(r-1)!}, \\ q^x \frac{G_{n+r,\chi,q}^{(h+1,r)}(x)}{\binom{n+r}{r} r!} = (q-1) \frac{G_{n+r+1,\chi,q}^{(h,r)}(x)}{\binom{n+r+1}{r} r!} + \frac{G_{n+r,\chi,q}^{(h,r)}(x)}{\binom{n+r}{r} r!}. \end{gathered} \tag{2.13}$$

For $h = r$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.6. *For $r \in \mathbb{N}$, one has*

$$\begin{aligned} G_{n+r,\chi,q}^{(r,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-q^x)^l \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{j=1}^{r} \chi(a_j) \right) \frac{q^{\sum_{j=1}^{r}((r-j)a_j + la_j)} (-1)^{a_1+\cdots+a_r}}{(-q^{dl};q)_r} \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{j=1}^{r} \chi(a_j) \right) q^{\sum_{j=1}^{r}(r-j)a_j} \left[x + \sum_{i=1}^{r} a_i + dm \right]_q^n. \end{aligned} \tag{2.14}$$

In particular,

$$\frac{G_{n+r,\chi,q^{-1}}^{(r,r)}(r-x)}{\binom{n+r}{r} r!} = (-1)^n q^{n+\binom{r}{2}} \frac{G_{n+r,\chi,q}^{(r,r)}(x)}{\binom{n+r}{r} r!}. \tag{2.15}$$

Let $x = r$ in Corollary 2.6. Then one has

$$\frac{G_{n+r,\chi,q^{-1}}^{(r,r)}}{\binom{n+r}{r} r!} = (-1)^n q^{n+\binom{r}{2}} \frac{G_{n+r,\chi,q}^{(r,r)}(r)}{\binom{n+r}{r} r!}. \tag{2.16}$$

Let $w_1, w_2, \ldots, w_r \in \mathbb{Q}_+$. Then, one has defines Barnes' type generalized q-Genocchi polynomials attached to χ as follows:

$$\begin{aligned} F_{q,\chi}^{(r)}(t, x \mid w_1, w_2, \ldots, w_r) &= 2^r t^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r} e^{[x+\sum_{j=1}^{r} w_j m_j]_q t} \\ &= \sum_{n=0}^{\infty} G_{n,\chi,q}^{(r)}(x \mid w_1, w_2, \ldots, w_r)\frac{t^n}{n!}. \end{aligned} \tag{2.17}$$

By (2.17), *one sees that*

$$\frac{G_{n+r,\chi,q}^{(r)}(x \mid w_1, \ldots, w_r)}{\binom{n+r}{r} r!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{\sum_{j=1}^{r} m_j} \left[x + \sum_{j=1}^{r} w_j m_j\right]_q^n. \tag{2.18}$$

It is easy to see that

$$\begin{aligned} &2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r} \left[x + \sum_{j=1}^{r} w_j m_j\right]_q^n \\ &\quad = \frac{2^r}{(1-q)^n}\sum_{l=0}^{n} \frac{\binom{n}{l}(-q^x)^l \sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{j=1}^{r}\chi(a_j)\right)(-1)^{\sum_{j=1}^{r} a_j} q^{l\sum_{i=1}^{r} w_i a_i}}{(1+q^{dlw_1})\cdots(1+q^{dlw_r})}. \end{aligned} \tag{2.19}$$

Therefore, we obtain the following theorem.

Theorem 2.7. *For $r \in \mathbb{N}, w_1, w_2, \ldots, w_r \in \mathbb{Q}_+$, one has*

$$\begin{aligned} \frac{G_{n+r,\chi,q}^{(r)}(x \mid xw_1, w_2, \ldots, w_r)}{\binom{n+r}{r} r!} &= 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{\sum_{j=1}^{r} m_j} [x + w_1 m_1 + \cdots + w_r m_r]_q^n \\ &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n} \frac{\binom{n}{l}(-q^x)^l \sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{j=1}^{r}\chi(a_j)\right)(-1)^{\sum_{j=1}^{r} a_j} q^{l\sum_{i=1}^{r} w_i a_i}}{(1+q^{dlw_1})\cdots(1+q^{dlw_r})}. \end{aligned} \tag{2.20}$$

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