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Research Article

Some Results on n-Times Integrated C-Regularized Semigroups

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We present a generation theorem of n-times integrated C-regularized semigroups and clarify the relation between differentiable $(n+1)$-times integrated C-regularized semigroups and singular n-times integrated C-regularized semigroups.

1. Introduction and Preliminaries

In 1987, Arendt [1] studied the n-times integrated semigroups, which are more general than C_0 semigroups (there exist many operators that generate n-times integrated semigroups but not C_0 semigroups).

In recent years, the n-times integrated C-regularized semigroups have received much attention because they can be used to deal with ill-posed abstract Cauchy problems and characterize the "weak" well-posedness of many important differential equations (cf., e.g., [2–18]).

Stimulated by the works in [2, 5–7, 9, 12–18], in this paper, we present a generation theorem of the n-times integrated C-regularized semigroups for the case that the domain of generator and the range of regularizing operator C are not necessarily dense, and prove that the subgenerator of an exponentially bounded, differentiable $(n+1)$-times integrated C-regularized semigroup is also a subgenerator of a singular n-times integrated C-regularized semigroup.

Throughout this paper, X is a Banach space; X^* denotes the dual space of X; $L(X,X)$ denotes the space of all linear and bounded operators from X to X, it will be abbreviated to $L(X)$; $L(X)^*$ denotes the dual space of $L(X)$. By $C^1((0,+\infty),X)$ we denote the space of all continuously differentiable X-valued functions on $(0,+\infty)$. $C((0,+\infty),X)$ is the space of all continuous X-valued functions on $(0,+\infty)$.

All operators are linear. For a closed linear operator A, we write $D(A)$, $R(A)$, $\rho(A)$ for the domain, the range, the resolvent set of A in a Banach space X, respectively.

We denote by $A_0 = A|_{\overline{D(A)}}$ the part of A in $\overline{D(A)}$, that is,

$$D(A_0) := \left\{x \in D(A); Ax \in \overline{D(A)}\right\}, \quad A_0 x = Ax, \text{ for } x \in D(A_0). \tag{1.1}$$

The C-resolvent set of A is defined as:

$$\rho_C(A) = \left\{\lambda \geq 0;\ (\lambda - A) \text{ is injective, } R(C) \subset R(\lambda - A) \text{ and } (\lambda - A)^{-1}C \in L(X)\right\}. \tag{1.2}$$

We abbreviate n-times integrated C-regularized semigroup to n-times integrated C-semigroup.

Definition 1.1. Let n be a nonnegative integer. Then A is the subgenerator of an exponentially bounded n-times integrated C-semigroup $\{S(t)\}_{t\geq 0}$ if $(\omega, \infty) \subset \rho_C(A)$ for some $\omega \geq 0$ and there exists a strongly continuous family $S(\cdot) : [0, \infty) \to L(X)$ with $\|S(t)\| \leq Me^{\omega t}$ for some $M > 0$ such that

$$(\lambda - A)^{-1}Cx = \lambda^n \int_0^\infty e^{-\lambda t} S(t)x\, dt \quad (\lambda > \omega, x \in X). \tag{1.3}$$

In this case, $\{S(t)\}_{t\geq 0}$ is called the exponentially bounded n-times integrated C-semigroup generated by $\widetilde{A} := C^{-1}AC$.

If $C = I$ (resp., $n = 0$), then A is called a generator of an exponentially bounded n-times integrated semigroup (resp., C-semigroup).

We recall some properties of n-times integrated C-semigroup.

Lemma 1.2 (see [10, Lemma 3.2]). *Assume that A is a subgenerator of an n-times integrated C-semigroup $\{S(t)\}_{t\geq 0}$. Then*

(i) $S(t)C = CS(t)$ $(t \geq 0)$,

(ii) $S(t)x \in D(A)$, *and* $AS(t)x = S(t)Ax$ $(t \geq 0, x \in D(A))$,

(iii) $S(t)x = (t^n/n!)Cx + A\int_0^t S(s)x\, ds$ $(t \geq 0, x \in X)$.

In particular, $S(0) = 0$.

Definition 1.3. Let $\omega \geq 0$. If $(\omega, \infty) \subset \rho_C(A)$ and there exists $\{S(t)\}_{t\geq 0} \subset L(X)$ such that

(i) $S(0) = 0$ and $S(\cdot) : (0, \infty) \to L(X)$ is strongly continuous,

(ii) for $\lambda > \omega$, $\int_0^\infty e^{-\lambda t}\|S(t)\|dt < \infty$,

(iii) $(\lambda - A)^{-1}Cx = \lambda^n \int_0^\infty e^{-\lambda t} S(t)x\, dt$, $\lambda > \omega$, $x \in X$,

then we say that $\{S(t)\}_{t\geq 0}$ is a singular n-times integrated C-semigroup with subgenerator A.

Remark 1.4. Clearly, an exponentially bounded n-times integrated C-semigroup is a singular n-times integrated C-semigroup. But the converse is not true.

2. The Main Results

Theorem 2.1. *Let $M > 0$, $\omega \geq 0$ be constants, and let A be a closed operator satisfying $(\omega, \infty) \subset \rho_C(A)$. Assume that $\varphi(t)$ is the nonnegative measurable function on $[0, \infty)$. A necessary and sufficient condition for A is the subgenerator of an $(n+1)$-times integrated C-semigroup $\{S(t)\}_{t\geq 0}$ satisfying*

(A1) $\limsup_{\lambda\to\infty} \|\lambda^{n+2} \int_0^\infty e^{-\lambda t} S(t) dt\| \leq M$,

(A2) $\|S(t) - S(s)\| \leq \int_t^s \varphi(u) e^{\omega u} du$, $0 \leq t \leq s$, *is that for* $\lambda > \omega$,

(i) $\limsup_{\lambda\to\infty} \|\lambda(\lambda - A)^{-1} C\| \leq M$,

(ii) $\|[(\lambda - A)^{-1} C/\lambda^n]^{(m)}\| \leq \int_0^\infty e^{-(\lambda-\omega)t} t^m \varphi(t) dt$, $m = 1, 2, \ldots$.

Proof. Sufficiency. Let $\psi(t) = e^{\omega t}\varphi(t)$. Set

$$f(\lambda) = \int_0^\infty e^{-\lambda t} \psi(t) dt = \int_0^\infty e^{-(\lambda-\omega)t} \varphi(t) dt, \quad \lambda > \omega. \tag{2.1}$$

For $x^* \in X^*$, we have

$$\begin{aligned} \left| \left\langle \left[\frac{(\lambda - A)^{-1} C}{\lambda^n} x \right]^{(m)}, x^* \right\rangle \right| &\leq \|x\| \cdot \|x^*\| \int_0^\infty e^{-\lambda t} t^m \psi(t) dt \\ &\leq \left| \left(\|x\| \cdot \|x^*\| \cdot f(\lambda) \right)^{(m)} \right|, \quad m = 1, 2, \ldots. \end{aligned} \tag{2.2}$$

Using this fact together with Widder's classical theorem, it is not difficult to see that the existence of a measurable function $h(\cdot, x, x^*)$ with $|h(t, x, x^*)| \leq \|x^*\| \|x\| \psi(t)$, a.e., $(t \geq 0)$ such that

$$\left\langle \frac{(\lambda - A)^{-1} C}{\lambda^n} x, x^* \right\rangle = \int_0^\infty e^{-\lambda t} h(t, x, x^*) dt, \quad \lambda > \omega. \tag{2.3}$$

Let $H(t, x, x^*) = \int_0^t h(s, x, x^*) ds$, $t \geq 0$, $x^* \in X^*$. In view of the convolution theorem for Laplace transforms and from (2.3), we have

$$\left\langle \frac{(\lambda - A)^{-1} C}{\lambda^n} x, x^* \right\rangle = \lambda \int_0^\infty e^{-\lambda t} H(t, x, x^*) dt, \quad \lambda > \omega, \ x^* \in X^*. \tag{2.4}$$

Using the uniqueness of Laplace transforms and the linearity of $h(\cdot, x, x^*)$ for each $x^* \in X^*$, $x \in X$, we can see that for each $t \geq 0$, $H(t, x, x^*)$ is linear and

$$|H(t + h, x, x^*) - H(t, x, x^*)| \leq \int_t^{t+h} |h(s, x, x^*)| ds \leq \|x\| \cdot \|x^*\| \int_t^{t+h} \psi(s) ds. \tag{2.5}$$

Hence for all $t \geq 0$, there exists $S(t) \in L(X)^{**}$ such that

$$H(t, x, x^*) = \langle S(t)x, x^* \rangle, \quad x \in X, \ x^* \in X^*, \tag{2.6}$$

$$\|S(t+h) - S(t)\| \leq \int_t^{t+h} \varphi(s) ds, \quad t \geq 0, \ h \geq 0, \tag{2.7}$$

$$\frac{(\lambda - A)^{-1} C}{\lambda^n} = \lambda \int_0^\infty e^{-\lambda t} S(t) dt. \tag{2.8}$$

Denote by $q : \ L(x)^{**} \ \to \ L(x)^{**}/L(X)$ the quotient mapping. Since $(\lambda - A)^{-1} C \in L(X)$, we deduce

$$0 = q\left(\frac{(\lambda - A)^{-1} C}{\lambda^n} \right) = \lambda \int_0^\infty e^{-\lambda t} q(S(t)) dt. \tag{2.9}$$

It follows from the uniqueness theorem for Laplace transforms that $q(S(t)) = 0$, that is, $S(t) \in L(X)$.

Combining (2.7) and (2.8) yields that $S(t) : [0, \infty) \ \to \ L(X)$ is strongly continuous and

$$\int_0^\infty e^{-\lambda t} \|S(t)\| dt \leq \int_0^\infty e^{-\lambda t} \int_0^t \varphi(s) ds \, dt = \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} \varphi(t) dt < \infty. \tag{2.10}$$

Now, we conclude that $\{S(t)\}_{t \geq 0}$ is an $(n+1)$-times integrated C-semigroup satisfying (A2). Assertion (A1) is immediate, by (2.8) and (i).

Necessity. Let $\psi(t) = e^{\omega t} \varphi(t)$. Since $\{S(t)\}_{t \geq 0}$ is an $(n+1)$-times integrated C-semigroup on X, we have

$$(\lambda - A)^{-1} C = \lambda^{n+1} \int_0^\infty e^{-\lambda t} S(t) dt \tag{2.11}$$

for $\lambda > \omega$. Noting that $\|S(t+h) - S(t)\| \leq \int_t^{t+h} \psi(s) \, ds \ (h \geq 0)$ and $S(0) = 0$, we find

$$\|S(t)\| \leq \int_0^t \psi(s) ds. \tag{2.12}$$

Then for any $y^* \in L(X)^*$ and $\lambda > \omega$, we obtain

$$\begin{aligned} \left\langle \frac{(\lambda - A)^{-1} C}{\lambda^n}, y^* \right\rangle &= \left\langle \lambda \int_0^\infty e^{-\lambda t} S(t) dt, y^* \right\rangle \\ &\leq \lambda \int_0^\infty e^{-\lambda t} \|S(t)\| \cdot \|y^*\| dt \leq \|y^*\| \int_0^\infty e^{-\lambda t} \psi(t) dt. \end{aligned} \tag{2.13}$$

Therefore, there exists a measurable function $\eta(t)$ on $[0,\infty)$ with $|\eta(t)| \le \psi(t)$ (a.e.) such that

$$\left\| \frac{(\lambda - A)^{-1}C}{\lambda^n} \right\| = \int_0^\infty e^{-\lambda t}\eta(t)dt. \tag{2.14}$$

Furthermore, by calculation, we have

$$\left\| \left[\frac{(\lambda - A)^{-1}C}{\lambda^n} \right]^{(m)} \right\| \le \int_0^\infty e^{-\lambda t} t^m \psi(t)dt = \int_0^\infty e^{-(\lambda-\omega)t} t^m \varphi(t)dt, \quad m = 1,2,\dots. \tag{2.15}$$

Assertion (i) is an immediate consequence of (2.11) and (A1). □

Remark 2.2. If $n = 0$ and $C = I$, then $\{S(t)\}_{t\ge 0}$ is an integrated semigroup in the sense of Bobrowski [2].

Theorem 2.3. *Let $M > 0$, $\omega \ge 0$ be constants, and let A be a closed operator satisfying $(\omega,\infty) \subset \rho(A)$. Assume that A is a subgenerator of an $(n+1)$-times integrated C-semigroup $\{S(t)\}_{t\ge 0}$ and satisfies (ii) of Theorem 2.1 and $\limsup_{\lambda\to\infty}\|\lambda(\lambda - A)^{-1}\| \le M$. If $A_0 = A|_{\overline{D(A)}}$ is a subgenerator of an n-times integrated C-semigroup $\{S_0(t)\}_{t\ge 0}$ on $\overline{D(A)}$, then for $\mu \in \rho(A)$, $x \in X$,*

$$S(t)x = (\mu - A_0)\int_0^t S_0(s)(\mu - A)^{-1}x\,ds, \tag{2.16}$$

$$S(t)x = \lim_{\mu\to\infty} \mu \int_0^t S_0(s)(\mu - A)^{-1}x\,ds. \tag{2.17}$$

Proof. For $\mu \in \rho(A)$, $x \in X$, set $\{\widehat{S}(t)\}_{t\ge 0}$ as follows:

$$\widehat{S}(t)x = \mu\int_0^t S_0(s)(\mu - A)^{-1}x\,ds - S_0(t)(\mu - A)^{-1}x + \frac{t^n}{n!}(\mu - A)^{-1}Cx. \tag{2.18}$$

Since $S_0(t)$ is strongly continuous on $\overline{D(A)}$, $\widehat{S}(t)$ is strongly continuous on X.

Fixing $\lambda > \omega$, we have

$$\begin{aligned} \lambda^{n+1}\int_0^\infty e^{-\lambda t}\widehat{S}(t)x\,dt &= \lambda^n(\mu - \lambda)\int_0^\infty e^{-\lambda t}S_0(t)(\mu - A)^{-1}x\,dt + (\mu - A)^{-1}Cx \\ &= (\mu - \lambda)(\lambda - A)^{-1}C(\mu - A)^{-1}x + (\mu - A)^{-1}Cx \\ &= (\lambda - A)^{-1}Cx. \end{aligned} \tag{2.19}$$

It follows from the uniqueness of Laplace transforms that $S(t)x = \widehat{S}(t)x$, $x \in X$. So we get (2.16). By the hypothesis $\limsup_{\lambda\to\infty}\|\lambda(\lambda - A)^{-1}\| \le M$, we see

$$\begin{aligned} S(t)x &= \lim_{\mu\to\infty}\left(\mu\int_0^t S_0(s)(\mu-A)^{-1}x\,ds - S_0(t)(\mu-A)^{-1}x + \frac{t^n}{n!}(\mu-A)^{-1}Cx\right) \\ &= \lim_{\mu\to\infty}\mu\int_0^t S_0(s)(\mu-A)^{-1}Cx\,ds, \end{aligned} \tag{2.20}$$

and the proof is completed. □

Now, we study the relation between differentiable $(n+1)$-times integrated C-semigroups and singular n-times integrated C-semigroups.

Theorem 2.4. *Let $\omega \geq 0$, and let A be a closed operator satisfying $(\omega,\infty) \subset \rho_C(A)$. Assume that $\varphi(t)$ is the nonnegative measurable function on $[0,\infty)$. The following two assertions are equivalent:*

1. *A is the subgenerator of a singular n-times integrated C-semigroup $\{U(t)\}_{t\geq 0}$ satisfying $\|U(t)\| \leq \varphi(t)e^{\omega t}$.*
2. *A is the subgenerator of an exponentially bounded $(n+1)$-times integrated C-semigroup $\{S(t)\}_{t\geq 0}$ satisfying*

$$\begin{aligned} \|S(t)-S(s)\| &\leq \int_t^s \varphi(\tau)e^{\omega\tau}d\tau, \quad 0 \leq t \leq s, \\ S(t)x &\in C^1((0,+\infty),X), \quad \text{for } x \in X. \end{aligned} \tag{2.21}$$

Proof. (1)⇒(2): we set

$$S(t)x := \int_0^t U(s)x\,ds, \quad t \geq 0. \tag{2.22}$$

Since $U(t)x$ is locally integrable on $[0,+\infty)$, $S(t)x$ is well-defined for any $x \in X$. It is easy to check that $S(t)x$ belongs to $C^1((0,+\infty),X)$.

For every $\lambda > \omega$, since

$$\|S(t)x\| = \left\|\int_0^t e^{-\lambda s}e^{\lambda s}U(s)x\,ds\right\| \leq e^{\lambda t}\int_0^t e^{-\lambda s}\|U(s)x\|ds \leq Me^{\lambda t}\|x\|, \tag{2.23}$$

we deduce that $S(t)$ is exponentially bounded.

Moreover, for $\lambda > \omega$, we have

$$\begin{aligned} (\lambda-A)^{-1}Cx &= \lambda^n\int_0^\infty e^{-\lambda t}U(t)x\,dt = \lambda^{n+1}\int_0^\infty e^{-\lambda t}S(t)x\,dt, \\ \|S(t)-S(s)\| &= \left\|\int_t^s U(\tau)d\tau\right\| \leq \int_t^s \varphi(\tau)e^{\omega\tau}d\tau, \quad 0 \leq t \leq s. \end{aligned} \tag{2.24}$$

Thus $\{S(t)\}_{t\geq 0}$ is the desired semigroup in (2).

(2)⇒(1): for any $x \in X$, we set

$$U(t)x := \frac{d}{dt}S(t)x, \quad \text{for } t > 0,$$
$$U(0)x := 0, \quad \text{for } t = 0. \tag{2.25}$$

Then $U(t)x \in C((0,+\infty), X)$ and $U(0) = 0$.

Noting that

$$\|S(t+h) - S(t)\| \leq \int_t^{t+h} \varphi(s)e^{\omega s} ds, \tag{2.26}$$

we find

$$\left\| \frac{S(t+h) - S(t)}{h} \right\| \leq \frac{1}{h} \int_t^{t+h} \varphi(s)e^{\omega s} ds. \tag{2.27}$$

Since $S(t)x$ is continuously differentiable for $t > 0$, we get

$$\|U(t)\| \leq \varphi(t)e^{\omega t} \quad \text{(a.e.)}. \tag{2.28}$$

Moreover, for $\lambda > \omega$, we have

$$\int_0^\infty e^{-\lambda t}\|U(t)\| dt \leq \int_0^\infty e^{-(\lambda-\omega)t}\varphi(t) dt < \infty,$$
$$(\lambda - A)^{-1}Cx = \lambda^{n+1}\int_0^\infty e^{-\lambda t}S(t)x\, dt = \lambda^n \int_0^\infty e^{-\lambda t}U(t)x\, dt. \tag{2.29}$$

Thus, $\{U(t)\}_{t\geq 0}$ is a singular n-times integrated C-semigroup with subgenerator A. □

Theorem 2.5. *Let $M > 0$, $\omega \geq 0$ be constants, and let A be a closed operator satisfying $(\omega, \infty) \subset \rho(A)$. Let $\varphi(t)$ be the function in Theorem 2.4. If A is the subgenerator of a singular n-times integrated C-semigroup $\{U(t)\}_{t\geq 0}$, satisfying $\|U(t)\| \leq \varphi(t)e^{\omega t}$, and satisfies*

$$\limsup_{\lambda\to\infty} \left\| \lambda(\lambda - A)^{-1} \right\| \leq M \quad (\lambda > \omega), \tag{2.30}$$

then

(1) *for $\lambda > \omega$, $x \in X$, $U(t)x = (\lambda - A_0)S_0(t)(\lambda - A)^{-1}x$,*

(2) *for $x \in \overline{D(A)}$, $\lim_{t\to 0^+} U(t)x = 0$,*

(3) *for $\lambda > \omega$, $x \in X$, $U(t)x = \lim_{\lambda\to\infty} \lambda S_0(t)(\lambda - A)^{-1}x$,*

(4) *for $\lambda > \omega$, $x \in \overline{D(A)}$ if and only if $\lim_{\lambda\to\infty} \lambda^{n+1} \int_0^\infty e^{-\lambda t}U(t)x\, dt = Cx$,*

where A_0 and $S_0(t)$ are the symbols mentioned in Theorem 2.3.

Proof. It follows from Theorems 2.3 and 2.4 that A subgenerates an $(n+1)$-times integrated C-semigroup $\{S(t)\}_{t\geq 0}$, which is continuously differentiable for $t > 0$ and satisfies (2.16) and (2.17).

Differentiating (2.16) with respect to t, we obtain

$$U(t)x = \frac{d}{dt}S(t)x = (\lambda - A_0)S_0(t)(\lambda - A)^{-1}x, \quad x \in X, \ \lambda > \omega. \tag{2.31}$$

This completes the proof of (1).

To show (2), for $x \in \overline{D(A)}$, we have

$$U(t)x = (\lambda - A_0)S_0(t)(\lambda - A)^{-1}x = S_0(t)x. \tag{2.32}$$

Letting $t \to 0^+$, we get

$$\lim_{t\to 0^+} U(t)x = 0, \quad x \in \overline{D(A)}. \tag{2.33}$$

To show (3), for $x \in X$, since $S(t)x \in C^1((0,+\infty), X)$, it follows from (2.17) that $\lim_{\lambda\to\infty}\lambda S_0(t)(\lambda - A)^{-1}x$ is continuous for $t > 0$, thus, we have

$$U(t)x = \frac{d}{dt}S(t)x = \lim_{\lambda\to\infty}\lambda S_0(t)(\lambda - A)^{-1}x, \quad t > 0. \tag{2.34}$$

Obviously, the equality above is true for $t = 0$.

Noting that

$$\limsup_{\lambda\to\infty}\left\|\lambda(\lambda - A)^{-1}\right\| \leq M \quad (\lambda > \omega), \tag{2.35}$$

we can deduce that $x \in \overline{D(A)}$ implies $\lim_{\lambda\to\infty}\lambda(\lambda - A)^{-1}Cx = Cx$, and from

$$(\lambda - A)^{-1}Cx = \lambda^n \int_0^\infty e^{-\lambda t}U(t)x\,dt, \tag{2.36}$$

assertion (4) is immediate if we note that $\lim_{\lambda\to\infty}\lambda(\lambda - A)^{-1}Cx = Cx$ implies $x \in \overline{D(A)}$. □

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