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Research Article

Optimal Harvest of a Stochastic Predator-Prey Model

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We firstly show the permanence of hybrid prey-predator system. Then, when both white and color noises are taken into account, we examine the asymptotic properties of stochastic prey-predator model with Markovian switching. Finally, the optimal harvest policy of stochastic prey-predator model perturbed by white noise is considered.

1. Introduction

Population systems have long been an important theme in mathematical biology due to their universal existence and importance. As a result, interest in mathematical models for populations with interaction between species has been on the increase. Generally, many models in theoretical ecology take the classical Lotka-Volterra model of interacting species as a starting point as follows:

$$\frac{dx(t)}{dt} = \operatorname{diag}(x_1(t), \dots, x_n(t))[b + Ax(t)], \tag{1.1}$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$, $b = (b_i)_{1\times n}$ and $A = (a_{ij})_{n\times n}$. The Lotka-Volterra model (1.1) has been studied extensively by many authors. Specifically, the dynamics relationship between predators and their preys also is an important topic in both ecology and mathematical ecology. For two-species predator-prey model, the population model has the form

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left[a_1 - b_1 x(t) - c_1 y(t)\right], \\ \frac{dy(t)}{dt} &= y(t)\left[a_2 - b_2 y(t) + c_2 x(t)\right], \end{aligned} \tag{1.2}$$

where $x(t)$, $y(t)$ represent the prey and the predator populations at time t, respectively, and $a_i, b_i, c_i,\quad i = 1,2$ are all positive constants.

Up to now, few work has been done with the following hybrid predator-prey model:

$$\begin{aligned}\frac{dx(t)}{dt} &= x\left[a_1(\alpha(t)) - b_1(\alpha(t))x - c_1(\alpha(t))y\right],\\ \frac{dy(t)}{dt} &= y\left[a_2(\alpha(t)) - b_2(\alpha(t))y + c_2(\alpha(t))x\right].\end{aligned} \tag{1.3}$$

when a population model is discussed, one of most important and interesting themes is its permanence, which means that the population system will survive forever. In this paper, we show that the hybrid model (1.3) is permanent.

As a matter of fact, due to environmental fluctuations, parameters involved in population models are not absolute constants. Thus, it is important to reveal how environmental noises affect the population systems. There are various types of environmental noises (e.g., white noise and color noise) affect population system significantly. Recently, many authors have considered population systems perturbed by white noise (see e.g., [1–7]). The color noise can be illustrated as a switching between two or more regimes of environmental, which differ by factors such as nutrition or as rain falls [8, 9]. When both white noise and color noise are taken into account, there are many results on corresponding population systems [10–14]. Especially, [10] investigated a Lotka-Volterra system under regime switching, the existence of global positive solutions, stochastic permanence, and extinction were discussed, and the limit of the average in time of the sample path was estimated. Reference [12] considered competitive Lotka-Volterra model in random environments and obtained nice results. Here, we consider the stochastic predator-prey system under regime switching which reads

$$\begin{aligned}dx(t) &= x(t)\left(a_1(\alpha(t)) - b_1(\alpha(t))x(t) - c_1(\alpha(t))y(t)\right)dt + x(t)\sigma_1(\alpha(t))dB_1(t),\\ dy(t) &= y(t)\left(a_2(\alpha(t)) - b_2(\alpha(t))y(t) + c_2(\alpha(t))x(t)\right)dt + y(t)\sigma_2(\alpha(t))dB_2(t).\end{aligned} \tag{1.4}$$

where $\alpha(t)$ is a Markov chain. Therefore, we aim to obtain its dynamical properties in more detail.

As we know, the optimal management of renewable resources, which has a direct relationship to sustainable development, is always a significant problem and focus. Many authors have studied the optimal harvest of its corresponding population model [15–20]. To the best of our knowledge, there is a very little amount of work has been done on the optimal harvest of stochastic predator-prey system. When the predator-prey model (1.2) is perturbed by white noise, we have the stochastic system as follows:

$$\begin{aligned}dx(t) &= x(t)\left[a_1 - b_1x(t) - c_1y(t)\right]dt + \sigma_1x(t)dB_1(t),\\ dy(t) &= y(t)\left[a_2 - b_2y(t) + c_2x(t)\right]dt + \sigma_2y(t)dB_2(t).\end{aligned} \tag{1.5}$$

Suppose that the resource population described by the stochastic system (1.5) is subject to exploitation, under the harvesting effort E_1, E_2 of $x(t)$, $y(t)$, respectively, the model of the harvested population has the form

$$dx(t) = x(t)\left[a_1 - E_1 - b_1 x(t) - c_1 y(t)\right]dt + \sigma_1 x(t) dB_1(t),$$
$$dy(t) = y(t)\left[a_2 - E_2 - b_2 y(t) + c_2 x(t)\right]dt + \sigma_2 y(t) dB_2(t). \tag{1.6}$$

In this paper, based on the arguments on model (1.4), we will obtain the the optimal harvest policy of stochastic predator-prey system (1.6).

The organization of the paper is as follows: we recall the fundamental theory about stochastic differential equation with Markovian switching in Section 2. We show that the hybrid system (1.3) is permanent in Section 3. Since stochastic predator-prey system (1.4) describes population dynamics, it is necessary for the solution of the system to be positive and not to explode to infinity in a finite time. Section 4 is devoted to the existence, uniqueness of global solution by comparison theorem, and its asymptotic properties. Based on the arguments of Section 4, in Section 5 predator-prey model perturbed by white noise (1.5) is considered, and the limit of the average in time of the sample path of the solution is obtain, moreover, optimal harvest policy of population model is derived. Finally, we close the paper with conclusions in Section 6. The important contributions of this paper are therefore clear.

2. Stochastic Differential Equation with Markovian Switching

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets.) Let $\alpha(t)$, $t \geq 0$, be a right-continuous Markov chain in the probability space tasking values in a finite state space $S = \{1, 2, \ldots, m\}$ with generator $\Gamma = (\gamma_{ij})_{m\times m}$ given by

$$P\{\alpha(t+\Delta t) = j \mid \alpha(t) = i\} = \gamma_{ij}\Delta + o(\Delta) \quad i \neq j,$$
$$P\{\alpha(t+\Delta t) = j \mid \alpha(t) = i\} = 1 + \gamma_{ii}\Delta + o(\Delta) \quad i = j, \tag{2.1}$$

where $\Delta > 0$. Here, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assume that the Markov chain $\alpha(t)$ is independent of the Brownian motion. And almost every sample path of $\alpha(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+.

We assume, as a standing hypothesis in following of the paper, that the Markov chain is irreducible. The algebraic interpretation of irreducibility is $\operatorname{rank}(\Gamma) = m - 1$. Under this condition, the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_1, \ldots, \pi_m) \in R^{1\times m}$ which can be determined by solving the following linear equation:

$$\pi\Gamma = 0, \tag{2.2}$$

subject to

$$\sum_{j=1}^{m}\pi_j = 1, \quad \pi_j > 0, \ \forall j \in S. \tag{2.3}$$

Consider a stochastic differential equation with Markovian switching

$$dx(t) = f(x(t), t, \alpha(t))dt + g(x(t), t, \alpha(t))dB(t), \tag{2.4}$$

on $t \geq 0$ with initial value $x(0) = x_0 \in R^n$, where

$$f : R^n \times R_+ \times S \longrightarrow R^n, \qquad g : R^n \times R_+ \times S \longrightarrow R^{n\times m}. \tag{2.5}$$

For the existence and uniqueness of the solution, we should suppose that the coefficients of the above equation satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \ldots$, there is an $h_k > 0$ such that

$$\left|f(x,t,\alpha) - f(y,t,\alpha)\right| \bigvee \left|g(x,t,\alpha) - g(y,t,\alpha)\right| \leq h_k\left|x - y\right|, \tag{2.6}$$

for all $t \geq 0$, $\alpha \in S$ and those $x, y \in R^n$ with $|x| \bigvee |y| \leq k$, and there is an $h > 0$ such that

$$\left|f(x,t,\alpha)\right| \bigvee \left|g(x,t,\alpha)\right| \leq h(1 + |x|), \tag{2.7}$$

for all $(x,t,\alpha) \in R^n \times R_+ \times S$.

Let $C^{2,1}(R^n \times R_+ \times S, R_+)$ denote the family of all nonnegative functions $V(x,t,\alpha)$ on $R^n \times R_+ \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{2,1}(R^n \times R_+ \times S, R_+)$, define an operator LV from $R^n \times R_+ \times S$ to R by

$$\begin{aligned} \mathrm{LV}(x,t,\alpha) = {} & V_t(x,t,\alpha) + V_x(x,t,\alpha)f(x,t,\alpha) + \sum_{j=1}^{m}\gamma_{ij}V(x,t,j) \\ & + \frac{1}{2}\mathrm{trace}\left[g^T(x,t,\alpha)V_{xx}(x,t,\alpha)g(x,t,\alpha)\right]. \end{aligned} \tag{2.8}$$

In particular, if V is independent of α, that is, $V(x,t,\alpha) = V(x,t)$, then

$$\mathrm{LV}(x,t,\alpha) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\mathrm{trace}\left[g^T(x,t)V_{xx}(x,t)g(x,t)\right]. \tag{2.9}$$

For convenience and simplicity in the following discussion, for any sequence $c(i)$, $i \in S$, we define

$$\widehat{c} = \min_{i\in S} c(i), \qquad \check{c} = \max_{i\in S} c(i). \tag{2.10}$$

For any sequence c_i, $\quad i = 1, 2$, we define

$$\widehat{c} = \min_{i=1,2} c_i, \qquad \check{c} = \max_{i=1,2} c_i. \tag{2.11}$$

And throughout the paper, we use K to denote a positive constant whose exact value may be different in different appearances.

3. Hybrid Predator-Prey Model

In this section, we mainly consider the permanence of the hybrid prey-predator system (1.3).

Lemma 3.1. *The solution* $X(t) = (x(t), y(t))$ *of* (1.3) *obeys*

$$\limsup_{t\to\infty} x(t) \le \frac{\check{a}}{\widehat{b}} =: \beta, \qquad \limsup_{t\to+\infty} y(t) \le \frac{\check{a}\left(\widehat{b}+\check{c}\right)}{\widehat{b}^2} =: \gamma. \tag{3.1}$$

Proof. It follows from the first equation that

$$x'(t) \le x(t)\left(\check{a} - \widehat{b}x(t)\right). \tag{3.2}$$

Using the comparison theorem, we can derive the first inequality. Then, for any $\epsilon > 0$ arbitrarily, there exists $T > 0$, such that for any $t > T$

$$x(t) < \beta + \epsilon. \tag{3.3}$$

So, for any $t > T$, we obtain

$$y'(t) \le y(t)\left[\check{a} - \widehat{b} + \check{c}(\beta+\epsilon)\right]. \tag{3.4}$$

By comparison theorem, then

$$\limsup_{t\to+\infty} y(t) \le \frac{\check{a} + \check{c}(\beta+\epsilon)}{\widehat{b}}. \tag{3.5}$$

For $\epsilon > 0$ is arbitrary, we therefore conclude

$$\limsup_{t\to+\infty} y(t) \le \frac{\check{a}\left(\widehat{b}+\check{c}\right)}{\widehat{b}^2} =: \gamma. \tag{3.6}$$

□

Lemma 3.2. *Assume that* $\widehat{a} - \check{c}\gamma > 0$ *holds. Then, the solution* $X(t) = (x(t), y(t))$ *to* (1.3) *satisfies*

$$\liminf_{t\to\infty} x(t) \ge \frac{\widehat{a} - \check{c}\gamma}{\widehat{b}}, \qquad \liminf_{t\to+\infty} y(t) \ge \frac{\widehat{a}}{\check{b}}. \tag{3.7}$$

Proof. By virtue of the first equation of (1.3), we can get

$$\begin{aligned}\left(\frac{1}{x(t)}\right)' &= -a_1(\alpha(t))\frac{1}{x(t)} + b_1(\alpha(t)) + c_1(\alpha(t))\frac{y(t)}{x(t)} \\ &= b_1(\alpha(t)) - \left[a_1(\alpha(t)) - c_1(\alpha(t))y(t)\right]\frac{1}{x(t)}.\end{aligned} \tag{3.8}$$

Then, it follows from the assumption that there exists sufficiently small $\epsilon > 0$, such that $\hat{a} - \check{c}(\gamma + \epsilon) > 0$. From Lemma 3.1, for above $\epsilon > 0$, there exists $T > 0$, such that $y(t) < \gamma + \epsilon$ for any $t > T$. Thus,

$$\left(\frac{1}{x(t)}\right)' \leq \check{b} - \left[\hat{a} - \check{c}(\gamma + \epsilon)\right]\frac{1}{x(t)}. \tag{3.9}$$

By the comparison theorem, we have

$$\limsup_{t \to +\infty} \frac{1}{x(t)} \leq \frac{\check{b}}{\hat{a} - \check{c}(\gamma + \epsilon)}. \tag{3.10}$$

Then,

$$\liminf_{t \to \infty} x(t) \geq \frac{\hat{a} - \check{c}(\gamma + \epsilon)}{\check{b}}. \tag{3.11}$$

Consequently,

$$\liminf_{t \to \infty} x(t) \geq \frac{\hat{a} - \check{c}\gamma}{\check{b}}. \tag{3.12}$$

On the other hand, the second equation of (1.3) implies that

$$y'(t) \geq a_2(\alpha(t))y(t) - b_2(\alpha(t))y^2(t) > \hat{a}y(t) - \check{b}y^2(t). \tag{3.13}$$

Using the comparison theorem, similar to the proof of the first assertion, we directly obtain

$$\liminf_{t \to +\infty} y(t) \geq \frac{\hat{a}}{\check{b}}. \tag{3.14}$$

□

Theorem 3.3. *Suppose that $a_i(\alpha) > 0$, $b_i(\alpha) > 0$, $c_i(\alpha) > 0$, $i = 1, 2$, $\alpha = 1, 2, \ldots, m$ and $\hat{a} - \check{c}\gamma > 0$ hold. Then, the solution $X(t) = (x(t), y(t))$ to* (1.3) *is permanent.*

Proof. From Lemma 3.1 and Lemma 3.2, we immediately get

$$\frac{\hat{a}-\check{c}\gamma}{\check{b}} \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq \frac{\check{a}}{\hat{b}},$$
$$\frac{\hat{a}}{\check{b}} \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq \frac{\check{a}\left(\hat{b}+\check{c}\right)}{\hat{b}^2}. \tag{3.15}$$

□

4. Stochastic Predator-Prey Model With Markovian Switching

In this section, we consider the stochastic differential equation with regime switching (1.4). If stochastic differential equation has a unique global (i.e., no explosion in a finite time) solution for any initial value, the coefficients of the equation are required to obey the linear growth condition and local Lipschitz condition. It is easy to see that the coefficients of (1.4) satisfy the local Lipschitz condition; therefore, there is a unique local solution $X(t) = (x(t), y(t))$ on $t \in [0, \tau_e)$ with initial value $(x_0, y_0) > 0$, $\alpha \in S$, where τ_e is the explosion time.

And since our purpose is to reveal the effect of environmental noises, we impose the following hypothesis on intensities of environmental noises.

Assumption 4.1 ($\hat{a} - (1/2)\check{\sigma}^2 > 0$). By virtue of comparison theorem, we will demonstrate that the local solution to (1.4) is global, which is motivated by [12]

$$\Phi(t) = \frac{\exp\int_0^t \left(a_1(\alpha(s)) - (1/2)\sigma_1^2(\alpha(s))\right)ds + \sigma_1(\alpha(s))dB_1(s)}{(1/x_0) + \int_0^t b_1(\alpha(s)) \exp\int_0^s \left(a_1(\alpha(\tau)) - (1/2)\sigma_1^2(\alpha(\tau))\right)d\tau + \sigma_1(\alpha(\tau))dB_1(\tau)ds}. \tag{4.1}$$

Thus, $\Phi(t)$ is the unique solution of

$$d\Phi(t) = \Phi(t)[a_1(\alpha(t)) - b_1(\alpha(t))\Phi(t)]dt + \Phi(t)\sigma_1(\alpha(t))dB_1(t), \tag{4.2}$$

with $\Phi(0) = x_0$. By the comparison theorem, we get $x(t) \leq \Phi(t)$ for $t \in [0, \tau_e)$ a.s. It is easy to see that

$$d\varphi(t) = \varphi(t)\left(a_2(\alpha(t)) - b_2\varphi(t)\right)dt + \varphi(t)\sigma_2(\alpha(t))dB_2(t), \tag{4.3}$$

with $\varphi(0) = y_0$, has a unique solution

$$\varphi(t) = \frac{\exp\int_0^t \left(a_2(\alpha(s)) - (1/2)\sigma_2^2(\alpha(s))\right)ds + \sigma_2(\alpha(s))dB_2(s)}{(1/y_0) + \int_0^t b_2(\alpha(s)) \exp\int_0^s \left(a_2(\alpha(\tau)) - (1/2)\sigma_2^2(\alpha(\tau))\right)d\tau + \sigma_2(\alpha(\tau))dB_2(\tau)ds}. \tag{4.4}$$

Obviously, $\varphi(t) \leq y(t)$, $t \in [0, \tau_e)$ a.s.

Moreover,

$$dy(t) \leq y(t)\left(a_2(\alpha(t)) - b_2(\alpha(t))y + c_2\Phi(t)\right)dt + y(t)\sigma_2(\alpha(t))dB_2(t). \tag{4.5}$$

So, we get $y(t) \le \Psi(t)$, $t \in [0, \tau_e)$ a.s., where

$$\Psi(t) = \frac{\exp \int_0^t \left(a_2(\alpha(s)) + c_2\Phi(s) - (1/2)\sigma_2^2(\alpha(s))\right)ds + \sigma_2(\alpha(s))dB_2(s)}{(1/y_0) + \int_0^t b_2(\alpha(s))F_1(s)ds}, \tag{4.6}$$

$$F_1(s) = \exp \int_0^s \left(a_2(\alpha(\tau)) + c_2\Phi(\tau) - (1/2)\sigma_2^2(\alpha(\tau))\right)d\tau + \sigma_2(\alpha(\tau))dB_2(\tau). \tag{4.7}$$

Besides,

$$dx(t) \ge x(t)(a_1(\alpha(t)) - b_1(\alpha(s))x(t) - c_1\Psi(t))dt + x(t)\sigma_1(\alpha(t))dB_1(t). \tag{4.8}$$

Then,

$$x(t) \ge \phi(t), \tag{4.9}$$

where

$$\begin{gathered} \phi(t) = \frac{\exp \int_0^t \left(a_1(\alpha(s)) - c_1(\alpha(s))\Psi(s) - (1/2)\sigma_1^2(\alpha(s))\right)ds + \sigma_1(\alpha(s))dB_1(s)}{(1/x_0) + \int_0^t b_1(\alpha(s))F_2(s)ds}, \\ F_2(s) = \exp \int_0^s \left(a_1(\alpha(\tau)) - c_1(\alpha(\tau))\Psi(\tau) - (1/2)\sigma_1^2(\alpha(\tau))\right)d\tau + \sigma_1(\alpha(\tau))dB_1(\tau). \end{gathered} \tag{4.10}$$

To sum up, we obtain

$$\phi(t) \le x(t) \le \Phi(t), \quad \varphi(t) \le y(t) \le \Psi(t), \quad t \in [0, \tau_e) \text{ a.s.} \tag{4.11}$$

It can be easily verified that $\phi(t)$, $\Phi(t)$, $\varphi(t)$, $\Psi(t)$ all exist on $t \ge 0$, hence

Theorem 4.2. *There is a unique positive solution $X(t) = (x(t), y(t))$ of (1.4) for any initial value $(x_0, y_0) \in R_+^2$, $\alpha \in S$, and the solution has the properties*

$$\phi(t) \le x(t) \le \Phi(t), \quad \varphi(t) \le y(t) \le \Psi(t), \quad t > 0 \text{ a.s.} \tag{4.12}$$

where $\phi(t)$, $\Phi(t)$, $\varphi(t)$, $\Psi(t)$ are defined as (4.1),(4.4),(4.6), *and* (4.9).

Theorem 4.2 tells us that (1.4) has a unique global solution, which makes us to further discuss its properties.

Now, we will investigate certain asymptotic limits of the population model (1.4). Referred to [12], it is not difficult to imply that

$$\lim_{t\to\infty}\frac{\ln\Phi(t)}{t}=0,\quad \lim_{t\to\infty}\frac{\ln\varphi(t)}{t}=0\quad \text{a.s.} \tag{4.13}$$

Then, we give the following essential theorems which will be used.

Theorem 4.3. *Let Assumption 4.1 hold. Then, the solution $y(t)$ has the property*

$$\lim_{t\to\infty}\frac{\ln y(t)}{t}=0\quad \textit{a.s.} \tag{4.14}$$

Proof. The proof is motivated by [12]. By (4.12) and (4.13), then

$$0=\liminf_{t\to\infty}\frac{\ln\varphi(t)}{t}\le\liminf_{t\to\infty}\frac{\ln y(t)}{t}\quad \text{a.s.} \tag{4.15}$$

Thus, it remains to show that

$$\limsup_{t\to\infty}\frac{\ln y(t)}{t}\le 0\quad \text{a.s.} \tag{4.16}$$

Note the quadratic variation of $\int_0^t\sigma_i(\alpha(s))dB_i(s)$ is $\int_0^t\sigma_i^2(\alpha(s))ds\le Kt$, thus the strong law of large numbers for local martingales yields that

$$\frac{\int_0^t\sigma_i(\alpha(s))dB_i(s)}{t}\longrightarrow 0\quad \text{a.s. as } t\longrightarrow\infty. \tag{4.17}$$

Therefore, for any $\epsilon>0$, there exists some positive constant $T<\infty$ such that for any $t\ge T$

$$\left|\int_0^t\sigma_i(\alpha(s))dB_i(s)\right|<\epsilon t\quad \text{a.s.} \tag{4.18}$$

Then, for any $t>s\ge T$, we have

$$\left|\int_s^t\sigma_i(\alpha(\tau))dB_i(\tau)\right|\le\left|\int_0^t\sigma_i(\alpha(\tau))dB_i(\tau)\right|+\left|\int_0^s\sigma_i(\alpha(\tau))dB_i(\tau)\right|\le\epsilon(t+s)\quad \text{a.s.} \tag{4.19}$$

Moreover, it follows from (4.13) that for the above ϵ and T, we get

$$|\ln\Phi(t)|\le\epsilon t\quad \text{a.s.} \tag{4.20}$$

By virtue of (4.6), we can derive for $t > s \geq T$

$$\frac{1}{\Psi(t)} \geq \int_T^t b_2(\alpha(s)) \exp\left[\int_s^t -\left(a_2(\alpha(\tau)) - c_2(\alpha(s))\Phi(\tau) + \frac{1}{2}\sigma_2^2(\alpha(\tau))\right)d\tau - \sigma_2(\alpha(\tau))dB_2(\tau)\right]ds. \tag{4.21}$$

Using the generalized Itô Lemma, we can conclude that

$$\ln\Phi(t) - \ln\Phi(s) = \int_s^t \left[a_1(\alpha(\tau)) - b_1(\alpha(\tau))\Phi(\tau) - \frac{1}{2}\sigma_1^2(\alpha(\tau))\right]d\tau + \sigma_1(\alpha(\tau))dB_1(\tau). \tag{4.22}$$

Consequently,

$$\int_s^t b_1(\alpha(\tau))\Phi(\tau)d\tau = \int_s^t \left[a_1(\alpha(\tau)) - \frac{1}{2}\sigma_1^2(\alpha(\tau))\right]d\tau + \sigma_1(\alpha(\tau))dB_1(\tau) - \ln\Phi(t) + \ln\Phi(s). \tag{4.23}$$

Then, for $t > s \geq T$

$$\begin{aligned}
\frac{1}{\Psi(t)} &\geq \int_T^t b_2(\alpha(s)) \exp\left[\int_s^t -\left(a_2(\alpha(\tau)) - c_2(\alpha(s))\Phi(\tau) + \frac{1}{2}\sigma_2^2(\alpha(\tau))\right)d\tau - \sigma_2(\alpha(\tau))dB_2(\tau)\right]ds \\
&\geq \hat{b}\int_T^t \exp\left[-\int_s^t \left(a_2(\alpha(\tau)) - \frac{1}{2}\sigma_2^2(\alpha(\tau))\right)d\tau - \frac{\check{c}}{\hat{b}}\int_s^t \left(a_1(\alpha(\tau)) - \frac{1}{2}\sigma_1^2(\alpha(\tau))\right)d\tau \right. \\
&\qquad\qquad \left. - \frac{2\check{c}\epsilon}{\hat{b}}(t+s) - \epsilon(t+s)\right]ds \\
&\geq \hat{b}\int_T^t \exp\left[-\left(1 + \frac{\check{c}}{\hat{b}}\right)\left(\check{a} - \frac{1}{2}\hat{\sigma}^2\right)(t-s) - \frac{2\check{c}\epsilon}{\hat{b}}(t+s) - \epsilon(t+s)\right]ds.
\end{aligned} \tag{4.24}$$

Denote $F = (1+(\check{c}/\hat{b}))(\check{a}-(1/2)\hat{\sigma}^2)+(2\check{c}\epsilon/\hat{b})+\epsilon$ and $f = (1+(\check{c}/\hat{b}))(\check{a}-(1/2)\hat{\sigma}^2)-(2\check{c}\epsilon/\hat{b})-\epsilon$. It is obvious that $F > f$. Hence,

$$\frac{1}{\Psi(t)} \geq \hat{b}e^{-Ft}\int_T^t e^{fs}ds = \frac{\hat{b}}{f}e^{-Ft}\left(e^{ft} - e^{fT}\right). \tag{4.25}$$

So, we obtain

$$Ft - \ln\Psi(t) \geq \ln\frac{b_2}{f} + \ln\left(e^{ft} - e^{fT}\right). \tag{4.26}$$

That is,

$$\ln \Psi(t) \leq Ft - \ln \frac{\widehat{b}}{f} - \ln\left(e^{ft} - e^{fT}\right). \tag{4.27}$$

Therefore,

$$\limsup_{t\to\infty} \frac{\ln \Psi(t)}{t} \leq \frac{4\check{c}\epsilon}{\widehat{b}} + 2\epsilon \quad \text{a.s.} \tag{4.28}$$

Note the fact that $\epsilon > 0$ is arbitrary, we obtain that

$$\limsup_{t\to\infty} \frac{\ln \Psi(t)}{t} \leq 0 \quad \text{a.s.} \tag{4.29}$$

By (4.12), we have $\limsup_{t\to\infty}(\ln y(t)/t) \leq 0$ a.s. Consequently,

$$0 = \liminf_{t\to\infty} \frac{\ln \varphi(t)}{t} \leq \liminf_{t\to\infty} \frac{\ln y(t)}{t} \leq \limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq \limsup_{t\to\infty} \frac{\ln \Psi(t)}{t} \leq 0 \quad \text{a.s.} \tag{4.30}$$

So, we complete the proof. □

Theorem 4.4. *Let Assumption 4.1 and* $(\widehat{a} - (1/2)\check{\sigma}^2) - (\check{c}/\widehat{b})(1 + (\check{c}/\widehat{b}))(\check{a} - (1/2)\widehat{\sigma}^2) > 0$ *hold. Then,* $x(t)$ *has the property*

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0 \quad \textit{a.s.} \tag{4.31}$$

Proof. From (4.12) and (4.13), we have

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leq \limsup_{t\to\infty} \frac{\ln \Phi(t)}{t} = 0 \quad \text{a.s.}, \tag{4.32}$$

Now, we only show that $\liminf_{t\to\infty}(\ln x(t)/t) \geq 0$. By virtue of (4.12), it remains to demonstrate that $\liminf_{t\to\infty}(\ln \phi(t)/t) \geq 0$. From the proof of Theorem 4.3, we know that for any $\epsilon > 0$, there exists some positive constant $T < \infty$ such that for any $t > s \geq T$

$$\left| \int_s^t \sigma_i(\alpha(\tau))dB_i(\tau) \right| \leq \epsilon(t+s), \quad |\ln \Phi(t)| \leq \epsilon t \quad \text{a.s.} \tag{4.33}$$

$$0 = \lim_{t\to\infty} \frac{\ln \varphi(t)}{t} \leq \liminf_{t\to\infty} \frac{\ln \Psi(t)}{t} \leq \limsup_{t\to\infty} \frac{\ln \Psi(t)}{t} \leq 0 \quad \text{a.s.} \tag{4.34}$$

It follows from (4.34) that

$$\lim_{t\to\infty}\frac{\ln\Psi(t)}{t}=0\quad \text{a.s.} \tag{4.35}$$

For above ϵ and T, we get for $t\geq T|\ln\Psi(t)|\leq \epsilon t$ a.s. By the generalized Itô Lemma, then

$$\begin{aligned}\int_s^t b_1(\alpha(\tau))\Phi(\tau)d\tau &= \int_s^t\left[a_1(\alpha(\tau))-\frac{1}{2}\sigma_1^2(\alpha(\tau))\right]d\tau+\sigma_1(\alpha(\tau))dB_1(\tau)\\ &\quad -\ln\Phi(t)+\ln\Phi(s),\\ \int_s^t b_2(\alpha(\tau))\Psi(\tau)d\tau &= \int_s^t\left[a_2(\alpha(\tau))+c_2\Phi(\tau)-\frac{1}{2}\sigma_2^2(\alpha(\tau))\right]d\tau+\sigma_2(\alpha(\tau))dB_2(\tau)\\ &\quad -\ln\Psi(t)+\ln\Psi(s).\end{aligned} \tag{4.36}$$

Therefore,

$$\begin{aligned}\int_s^t\Psi(\tau)d\tau &\leq \frac{1}{\widehat{b}}\left[\int_T^t\left(a_2(\alpha(\tau))-\frac{1}{2}\sigma_2^2(\alpha(\tau))\right)d\tau+2\epsilon\left(1+\frac{\check{c}}{\widehat{b}}\right)(t+s)\right.\\ &\qquad\left.+\frac{\check{c}}{\widehat{b}}\int_T^t\left(a_1(\alpha(\tau))-\frac{1}{2}\sigma_1^2(\alpha(\tau))\right)d\tau\right]\\ &\leq \frac{1}{\widehat{b}}\left(1+\frac{\check{c}}{\widehat{b}}\right)\left[2\epsilon(t+s)+\left(\check{a}-\frac{1}{2}\widehat{\sigma}^2\right)(t-s)\right].\end{aligned} \tag{4.37}$$

Thus, it is easy to imply that

$$-\int_T^t\left(a_1(\alpha(\tau))-c_1\Psi(\tau)-\frac{1}{2}\sigma_1^2(\alpha(\tau))\right)d\tau+\sigma_1(\alpha(\tau))dB_1(\tau)\leq -H(t-T)+h\epsilon(t+T), \tag{4.38}$$

where $H=(\widehat{a}-(1/2)\check{\sigma}^2)-(\check{c}/\widehat{b})(1+(\check{c}/\widehat{b}))(\check{a}-(1/2)\widehat{\sigma}^2)$ and $h=(2\check{c}/\widehat{b})(1+(\check{c}/\widehat{b}))+1$. By (4.9), we imply for $t>T$

$$\begin{aligned}\frac{1}{\phi(t)} &= e^{-\int_T^t(a_1(\alpha(\tau))-c_1(\alpha(\tau))\Psi(\tau)-(1/2)\sigma_1^2(\alpha(\tau)))d\tau+\sigma_1(\alpha(\tau))dB_1(\tau)}\\ &\quad\times\left[\frac{1}{x(T)}+\int_T^t b_1(\alpha(\tau))e^{\int_T^s(a_1(\alpha(\tau))-c_1(\alpha(\tau))\Psi(\tau)-(1/2)\sigma_1^2(\alpha(\tau)))d\tau+\sigma_1(\alpha(\tau))dB_1(\tau)}ds\right]\\ &\leq \frac{1}{x(T)}e^{-H(t-T)+h\epsilon(t+T)}+\check{b}\int_T^t e^{-H(t-s)+h\epsilon(t+s)}ds\\ &\leq \frac{1}{x(T)}e^{h\epsilon(t+T)}+\frac{\check{b}}{H+h\epsilon}e^{2h\epsilon t}.\end{aligned} \tag{4.39}$$

Consequently,

$$\left[e^{-2h\epsilon(t+T)}\right]\frac{1}{\phi(t)} \leq \frac{1}{x(T)}e^{-h\epsilon(t+T)} + \frac{\check{b}}{H+h\epsilon}e^{-2h\epsilon T} \leq K \quad \text{a.s.} \tag{4.40}$$

It follows from (4.40) that

$$\limsup_{t\to\infty}\frac{\ln(1/\phi(t))}{t} \leq \limsup_{t\to\infty}\left(\frac{\ln K}{t} + \frac{2h\epsilon(t+T)}{t}\right) = 2h\epsilon \quad \text{a.s.} \tag{4.41}$$

Then,

$$\liminf_{t\to\infty}\frac{\ln\phi(t)}{t} \geq -2h\epsilon \quad \text{a.s.} \tag{4.42}$$

For $\epsilon > 0$ is arbitrary, we imply

$$\liminf_{t\to\infty}\frac{\ln\phi(t)}{t} \geq 0 \quad \text{a.s.} \tag{4.43}$$

Finally, we obtain

$$0 \leq \liminf_{t\to\infty}\frac{\ln\phi(t)}{t} \leq \liminf_{t\to\infty}\frac{\ln x(t)}{t} \leq \limsup_{t\to\infty}\frac{\ln x(t)}{t} \leq \limsup_{t\to\infty}\frac{\ln \Phi(t)}{t} = 0 \quad \text{a.s.} \tag{4.44}$$

□

Theorem 4.5. *Let Assumption 4.1 and* $(\widehat{a} - (1/2)\check{\sigma}^2) - (\check{c}/\widehat{b})(1 + (\check{c}/\widehat{b}))(\check{a} - (1/2)\widehat{\sigma}^2) > 0$ *hold. Then, the solution* $X(t) = (x(t), y(t))$ *to* (1.4) *obeys*

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\int_0^t b_1(\alpha(s))x(s) + c_1(\alpha(s))y(s)ds = \sum_{\alpha=1}^{m}\pi_\alpha\left(a_1(\alpha) - \frac{\sigma_1^2(\alpha)}{2}\right) \quad \text{a.s.,}\\
&\lim_{t\to\infty}\frac{1}{t}\int_0^t c_2(\alpha(s))x(s) - b_2(\alpha(s))y(s)ds = \sum_{\alpha=1}^{m}\pi_\alpha\left(a_2(\alpha) - \frac{\sigma_2^2(\alpha)}{2}\right) \quad \text{a.s.}
\end{aligned} \tag{4.45}$$

Proof. Using the generalized Itô Lemma, we have

$$\begin{aligned}
\ln x(t) &= \ln x_0 + \int_0^t \left(a_1(\alpha(s)) - \frac{\sigma_1^2(\alpha(s))}{2} \right) ds \\
&\quad - \left[\int_0^t b_1(\alpha(s)) x(s) + c_1(\alpha(s)) y(s) ds \right] + \int_0^t \sigma_1(\alpha(s)) dB_1(s), \\
\ln y(t) &= \ln y_0 + \int_0^t \left(a_2(\alpha(s)) - \frac{\sigma_2^2(\alpha(s))}{2} \right) ds \\
&\quad + \left[\int_0^t c_2(\alpha(s)) x(s) - b_2(\alpha(s)) y(s) ds \right] + \int_0^t \sigma_2(\alpha(s)) dB_2(s).
\end{aligned} \tag{4.46}$$

Therefore,

$$\begin{aligned}
\frac{1}{t} \ln x(t) &= \frac{1}{t} \ln x_0 + \frac{1}{t} \int_0^t \left(a_1(\alpha(s)) - \frac{\sigma_1^2(\alpha(s))}{2} \right) ds + \frac{1}{t} \int_0^t \sigma_1(\alpha(s)) dB_1(s) \\
&\quad - \frac{1}{t} \int_0^t b_1(\alpha(s)) x(s) + c_1(\alpha(s)) y(s) ds, \\
\frac{1}{t} \ln y(t) &= \frac{1}{t} \ln y_0 + \frac{1}{t} \int_0^t \left(a_2(\alpha(s)) - \frac{\sigma_2^2(\alpha(s))}{2} \right) ds + \frac{1}{t} \int_0^t \sigma_2(\alpha(s)) dB_2(s) \\
&\quad + \frac{1}{t} \int_0^t c_2(\alpha(s)) x(s) - b_2(\alpha(s)) y(s) ds.
\end{aligned} \tag{4.47}$$

Thus, let $t \to \infty$, by the ergodic properties of Markov chains, we have

$$\begin{aligned}
\lim_{t \to \infty} \frac{\int_0^t \left(a_1(\alpha(s)) - \left(\sigma_1^2(\alpha)/2 \right) \right) ds}{t} &= \sum_{\alpha=1}^{m} \pi_\alpha \left(a_1(\alpha) - \frac{\sigma_1^2(\alpha)}{2} \right) \quad \text{a.s.}, \\
\lim_{t \to \infty} \frac{\int_0^t \left(a_2(\alpha(s)) - \left(\sigma_2^2(\alpha)/2 \right) \right) ds}{t} &= \sum_{\alpha=1}^{m} \pi_\alpha \left(a_2(\alpha) - \frac{\sigma_2^2(\alpha)}{2} \right) \quad \text{a.s.}
\end{aligned} \tag{4.48}$$

Hence, by virtue of the strong law of large numbers of local martingales, we get

$$\begin{aligned}
\lim_{t \to \infty} \frac{1}{t} \int_0^t b_1(\alpha(s)) x(s) + c_1(\alpha(s)) y(s) ds &= \sum_{\alpha=1}^{m} \pi_\alpha \left(a_1(\alpha) - \frac{\sigma_1^2(\alpha)}{2} \right) \quad \text{a.s.}, \\
\lim_{t \to \infty} \frac{1}{t} \int_0^t c_2(\alpha(s)) x(s) - b_2(\alpha(s)) y(s) ds &= \sum_{\alpha=1}^{m} \pi_\alpha \left(a_2(\alpha) - \frac{\sigma_2^2(\alpha)}{2} \right) \quad \text{a.s.}
\end{aligned} \tag{4.49}$$

The proof is complete. □

5. Optimal Harvest Policy

When the harvesting problems of population resources is discussed, we aim to obtain the optimal harvesting effort and the corresponding maximum sustainable yield.

In the same way of Theorems 4.2–4.5, we can conclude the following results. Here, we do not list the corresponding proofs in detail, only show the main results.

Theorem 5.1. *Assume* $\widehat{a}-\check{E}-(1/2)\check{\sigma}^2 > 0$ *and* $(\widehat{a}-\check{E}-(1/2)\check{\sigma}^2)-(\check{c}/\widehat{b})(1+(\check{c}/\widehat{b}))(\check{a}-\check{E}-(1/2)\widehat{\sigma}^2) > 0$ *hold. Then,* (1.6) *has a unique global solution* $X(t) = (x(t), y(t))$ *for any initial value* $(x_0, y_0) \in R_+^2$*. In addition, the solution has the properties*

$$\lim_{t\to\infty}\frac{\ln x(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{\ln y(t)}{t} = 0 \quad a.s. \tag{5.1}$$

When the harvesting problems are considered, the corresponding average population level is derived below.

Theorem 5.2. *Suppose that the conditions of Theorem 5.1 hold. If*

$$\lim_{t\to\infty}\frac{\int_0^t x(s)ds}{t}, \quad \lim_{t\to\infty}\frac{\int_0^t y(s)ds}{t} \quad \textit{exist a.s.}, \tag{5.2}$$

then the solution $X(t) = (x(t), y(t))$ *to* (1.6) *obeys*

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_0^t x(s)ds}{t} &= \frac{b_2\left(a_1 - E_1 - (\sigma_1^2/2)\right) - c_1\left(a_2 - E_2 - (\sigma_2^2/2)\right)}{b_1b_2 + c_1c_2} \quad a.s.,\\ \lim_{t\to\infty}\frac{\int_0^t y(s)ds}{t} &= \frac{c_2\left(a_1 - E_1 - (\sigma_1^2/2)\right) + b_1\left(a_2 - E_2 - (\sigma_2^2/2)\right)}{b_1b_2 + c_1c_2} \quad a.s.\end{aligned} \tag{5.3}$$

When the two species are both subjected to exploitation, it is important and necessary to discuss the corresponding maximum sustainable revenue.

Theorem 5.3. *Let the conditions of Theorem 5.1 hold. Then, if* $4pqb_1b_2-(pc_1 - qc_2)^2 \neq 0$*, the optimal harvesting efforts of* $x(t)$ *and* $y(t)$*, respectively, are*

$$E_1^* = \frac{qb_1\left(a_2 - (\sigma_2^2/2)\right)(pc_1 + qc_2) + \left(a_1 - (\sigma_1^2/2)\right)\left[2pqb_1b_2 + qc_2(pc_1 - qc_2)\right]}{(pc_1 - qc_2)^2 - 4pqb_1b_2}, \tag{5.4}$$

$$E_2^* = \frac{\left(a_2 - (\sigma_2^2/2)\right)\left[pc_1(pc_1 - qc_2) - 2pqb_1b_2\right] - pb_2\left(a_1 - (\sigma_1^2/2)\right)(pc_1 + qc_2)}{(pc_1 - qc_2)^2 - 4pqb_1b_2}. \tag{5.5}$$

The optimal sustainable harvesting yield is

$$F(E_1^*, E_2^*) = \frac{(pb_2E_1^* + qc_2E_2^*)\left(a_1 - E_1^* - (\sigma_1^2/2)\right) + (qb_1E_2^* - pc_1E_1^*)\left(a_2 - E_2^* - (\sigma_2^2/2)\right)}{b_1b_2 + c_1c_2}, \tag{5.6}$$

where p, q *are the price of* $x(t)$ *and* $y(t)$.

Proof. Assume that p, q are the price of resources of $x(t)$ and $y(t)$. Then, the maximum sustainable yield reads

$$\begin{aligned} F(E_1, E_2) &= pE_1 \lim_{t\to\infty} \frac{\int_0^t x(s)ds}{t} + qE_2 \lim_{t\to\infty} \frac{\int_0^t y(s)ds}{t} \\ &= p\lim_{t\to\infty} \frac{\int_0^t E_1 x(s)ds}{t} + q\lim_{t\to\infty} \frac{\int_0^t E_2 y(s)ds}{t} \\ &= \frac{(pb_2E_1 + qc_2E_2)(a_1 - (\sigma_1^2/2) - E_1)}{b_1b_2 + c_1c_2} \\ &\quad + \frac{(qb_1E_2 - pc_1E_1)(a_2 - (\sigma_2^2/2) - E_2)}{b_1b_2 + c_1c_2}. \end{aligned} \tag{5.7}$$

Let

$$\frac{\partial F(E_1, E_2)}{\partial E_1} = 0, \qquad \frac{\partial F(E_1, E_2)}{\partial E_2} = 0. \tag{5.8}$$

Then, we can have

$$\begin{aligned} &(pc_1 - qc_2)E_2 - 2pb_2E_1 + pb_2\left(a_1 - \left(\sigma_1^2/2\right)\right) - pc_1\left(a_2 - \left(\sigma_2^2/2\right)\right) = 0, \\ &(pc_1 - qc_2)E_1 - 2qb_1E_2 + qc_2\left(a_1 - \left(\sigma_1^2/2\right)\right) + qb_1\left(a_2 - \left(\sigma_2^2/2\right)\right) = 0. \end{aligned} \tag{5.9}$$

Therefore, there exists a unique extreme value point (E_1^*, E_2^*), where

$$\begin{aligned} E_1^* &= \frac{qb_1(a_2 - (\sigma_2^2/2))(pc_1 + qc_2) + (a_1 - (\sigma_1^2/2))\left[2pqb_1b_2 + qc_2(pc_1 - qc_2)\right]}{(pc_1 - qc_2)^2 - 4pqb_1b_2}, \\ E_2^* &= \frac{(a_2 - (\sigma_2^2/2))\left[pc_1(pc_1 - qc_2) - 2pqb_1b_2\right] - pb_2(a_1 - (\sigma_1^2/2))(pc_1 + qc_2)}{(pc_1 - qc_2)^2 - 4pqb_1b_2}. \end{aligned} \tag{5.10}$$

That is, under the condition $4pqb_1b_2 - (pc_1 - qc_2)^2 \neq 0$, we obtain (5.4) and (5.5). So, we obtain the optimal harvesting efforts of $x(t)$ and $y(t)$.

Substituting (5.4) and (5.5) into the representation of $F(E_1, E_2)$ (5.7), we have the optimal sustainable yield

$$F(E_1^*, E_2^*) = \frac{(pb_2E_1^* + qc_2E_2^*)(a_1 - E_1^* - (\sigma_1^2/2)) + (qb_1E_2^* - pc_1E_1^*)(a_2 - E_2^* - (\sigma_2^2/2))}{b_1b_2 + c_1c_2}, \tag{5.11}$$

as desired. Therefore, we complete the proof. □

6. Conclusions

The optimal management of renewable resources has a direct relationship to sustainable development. When population system is subject to exploitation, it is important and necessary to discuss the optimal harvesting effort and the corresponding maximum sustainable yield. Meanwhile, population systems are often subject to environmental noise. It is also necessary to reveal how the noise affects the population systems. Our work is an attempt to carry out the study of optimal harvest policy of population system in a stochastic setting. When both white noise and color noise are taken into account, we consider the limit of the average in time of the sample path of the stochastic model (1.4). Based on the arguments of (1.4), we discuss the corresponding stochastic system perturbed by white noise (1.5). We obtain the the optimal harvesting effort and the corresponding maximum sustainable yield.

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