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Research Article

Nonlinear Integral Inequalities in Two Independent Variables on Time Scales

Wei Nian Li

Department of Mathematics, Binzhou University, Shandong 256603, China

Correspondence should be addressed to Wei Nian Li, wnli@263.net

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We investigate some nonlinear integral inequalities in two independent variables on time scales. Our results unify and extend some integral inequalities and their corresponding discrete analogues which established by Pachpatte. The inequalities given here can be used as handy tools to study the properties of certain partial dynamic equations on time scales.

1. Introduction

The theory of dynamic equations on time scales unifies existing results in differential and finite difference equations and provides powerful new tools for exploring connections between the traditionally separated fields. During the last few years, more and more scholars have studied this theory. For example, we refer the reader to [1, 2] and the references cited therein. At the same time, some integral inequalities used in dynamic equations on time scales have been extended by many authors [3–11].

On the other hand, a few authors have focused on the theory of partial dynamic equations on time scales [12–17]. However, only [10, 11] have studied integral inequalities useful in the theory of partial dynamic equations on time scales, as far as we know. In this paper, we investigate some nonlinear integral inequalities in two independent variables on time scales, which can be used as handy tools to study the properties of certain partial dynamic equations on time scales.

Throughout this paper, a knowledge and understanding of time scales and time scale notation is assumed. For an excellent introduction to the calculus on time scales, we refer the reader to [1, 2].

2. Main Results

In what follows, $\mathbb{T}$ is an arbitrary time scale, C_{rd} denotes the set of rd-continuous functions, $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, $\mathcal{R}^+ = \{p \in \mathcal{R} : 1+\mu(t)p(t) > 0$ for all $t \in \mathbb{T}\}$, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0,\infty)$, and $\mathbb{N}_0 = \{0,1,2,\ldots\}$ denotes the set of nonnegative integers. We use the usual conventions that empty sums and products are taken to be 0 and 1, respectively. Throughout this paper, we always assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales, $t_0 \in \mathbb{T}_1$, $s_0 \in \mathbb{T}_2$, $t \geq t_0$, $s \geq s_0$, $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$, and we write $x^{\Delta_t}(t,s)$ for the partial delta derivatives of $x(t,s)$ with respect to t, and $x^{\Delta_t\Delta_s}(t,s)$ for the partial delta derivatives of $x^{\Delta_t}(t,s)$ with respect to s.

The following two lemmas are useful in our main results.

Lemma 2.1 (see [18]). *If $x, y \in \mathbb{R}_+$, and $1/p + 1/q = 1$ with $p > 1$, then*

$$x^{1/p}y^{1/q} \leq \frac{x}{p} + \frac{y}{q}, \tag{2.1}$$

with equality holding if and only if $x = y$.

Lemma 2.2 (Comparison Theorem [1]). *Suppose $u, b \in C_{rd}$, $a \in \mathcal{R}^+$. Then,*

$$u^{\Delta}(t) \leq a(t)u(t) + b(t), \quad t \in \mathbb{T} \tag{2.2}$$

implies

$$u(t) \leq u(t_0)e_a(t,t_0) + \int_{t_0}^{t} e_a(t,\sigma(\tau))b(\tau)\Delta\tau, \quad t \in \mathbb{T}. \tag{2.3}$$

Next, we establish our main results.

Theorem 2.3. *Assume that $u(t,s)$, $a(t,s)$, $b(t,s)$, $g(t,s)$, and $h(t,s)$ are nonnegative functions defined for $(t,s) \in \Omega$ that are right-dense continuous for $(t,s) \in \Omega$, and $p > 1$ is a real constant. Then,*

$$u^p(t,s) \leq a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)u^p(\tau,\eta) + h(\tau,\eta)u(\tau,\eta)\right]\Delta\eta\Delta\tau, \quad (t,s) \in \Omega \tag{2.4}$$

implies

$$u(t,s) \leq \left\{a(t,s) + b(t,s)m(t,s)e_{y(\cdot,s)}(t,t_0)\right\}^{1/p}, \quad (t,s) \in \Omega, \tag{2.5}$$

where

$$m(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}\left[a(\tau,\eta)g(\tau,\eta)+\left(\frac{p-1}{p}+\frac{a(\tau,\eta)}{p}\right)h(\tau,\eta)\right]\Delta\eta\Delta\tau, \tag{2.6}$$

$$y(t,s)=\int_{s_0}^{s}\left[g(t,\eta)+\frac{h(t,\eta)}{p}\right]b(t,\eta)\Delta\eta, \quad (t,s)\in\Omega. \tag{2.7}$$

Proof. Define a function $z(t,s)$ by

$$z(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)u^p(\tau,\eta)+h(\tau,\eta)u(\tau,\eta)\right]\Delta\eta\Delta\tau, \quad (t,s)\in\Omega. \tag{2.8}$$

Then, (2.4) can be written as

$$u^p(t,s)\le a(t,s)+b(t,s)z(t,s), \quad (t,s)\in\Omega. \tag{2.9}$$

From (2.9), by Lemma 2.1, we have

$$\begin{aligned}u(t,s)&\le (a(t,s)+b(t,s)z(t,s))^{1/p}(1)^{(p-1)/p}\\&\le \frac{a(t,s)}{p}+\frac{b(t,s)z(t,s)}{p}+\frac{p-1}{p}, \quad (t,s)\in\Omega.\end{aligned} \tag{2.10}$$

It follows from (2.8)–(2.10) that

$$\begin{aligned}z(t,s)\le{}&\int_{t_0}^{t}\int_{s_0}^{s}\Bigg\{g(\tau,\eta)\left[a(\tau,\eta)+b(\tau,\eta)z(\tau,\eta)\right]\\&\qquad+h(\tau,\eta)\left[\frac{p-1+a(\tau,\eta)}{p}+\frac{b(\tau,\eta)z(\tau,\eta)}{p}\right]\Bigg\}\Delta\eta\Delta\tau\\={}&m(t,s)+\int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)+\frac{h(\tau,\eta)}{p}\right]b(\tau,\eta)z(\tau,\eta)\Delta\eta\Delta\tau, \quad (t,s)\in\Omega,\end{aligned} \tag{2.11}$$

where $m(t,s)$ is defined by (2.6). It is easy to see that $m(t,s)$ is nonnegative, right-dense continuous, and nondecreasing for $(t,s)\in\Omega$. Let $\varepsilon>0$ be given, and from (2.11), we obtain

$$\frac{z(t,s)}{m(t,s)+\varepsilon}\le 1+\int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)+\frac{h(\tau,\eta)}{p}\right]b(\tau,\eta)\frac{z(\tau,\eta)}{m(\tau,\eta)+\varepsilon}\Delta\eta\Delta\tau, \quad (t,s)\in\Omega. \tag{2.12}$$

Define a function $v(t,s)$ by

$$v(t,s)=1+\int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)+\frac{h(\tau,\eta)}{p}\right]b(\tau,\eta)\frac{z(\tau,\eta)}{m(\tau,\eta)+\varepsilon}\Delta\eta\Delta\tau, \quad (t,s)\in\Omega. \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$z(t,s) \le (m(t,s)+\varepsilon)v(t,s), \quad (t,s)\in\Omega. \tag{2.14}$$

From (2.13), a delta derivative with respect to t yields

$$\begin{aligned} v^{\Delta_t}(t,s) &= \int_{s_0}^{s}\left[g(t,\eta)+\frac{h(t,\eta)}{p}\right]b(t,\eta)\frac{z(t,\eta)}{m(t,\eta)+\varepsilon}\Delta\eta \\ &\le \int_{s_0}^{s}\left[g(t,\eta)+\frac{h(t,\eta)}{p}\right]b(t,\eta)v(t,\eta)\Delta\eta \\ &\le \left(\int_{s_0}^{s}\left[g(t,\eta)+\frac{h(t,\eta)}{p}\right]b(t,\eta)\Delta\eta\right)v(t,s) \\ &= y(t,s)v(t,s), \quad (t,s)\in\Omega, \end{aligned} \tag{2.15}$$

where $y(t,s)$ is defined by (2.7). Noting that $v(t_0,s)=1$, $y(t,s)\ge 0$, and using Lemma 2.2, from (2.15), we obtain

$$v(t,s) \le e_{y(\cdot,s)}(t,t_0), \quad (t,s)\in\Omega. \tag{2.16}$$

It follows from (2.9), (2.14), and (2.16) that

$$u(t,s) \le \left\{a(t,s)+b(t,s)(m(t,s)+\varepsilon)e_{y(\cdot,s)}(t,t_0)\right\}^{1/p}, \quad (t,s)\ \in\Omega. \tag{2.17}$$

Letting $\varepsilon\to 0$ in (2.17), we immediately obtain the required (2.5). The proof of Theorem 2.3 is complete. □

Remark 2.4. Letting $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}_+$ and $\mathbb{T}_1=\mathbb{T}_2=\mathbb{N}_0$, respectively, we easily see that Theorem 2.3 reduces to Theorem 2.3.3(c_1) and Theorem 5.2.4(d_1) in [19].

Theorem 2.5. *Assume that all assumptions of Theorem 2.3 hold. If $a(t,s)>0$ and $a(t,s)$ is nondecreasing for $(t,s)\in\Omega$, then*

$$u^p(t,s)\le a^p(t,s)+b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)u^p(\tau,\eta)+h(\tau,\eta)u(\tau,\eta)\right]\Delta\eta\Delta\tau, \quad (t,s)\in\Omega \tag{2.18}$$

implies

$$u(t,s)\le a(t,s)\left\{1+b(t,s)n(t,s)e_{w(\cdot,s)}(t,t_0)\right\}^{1/p}, \quad (t,s)\in\Omega, \tag{2.19}$$

where

$$n(t,s) = \int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta) + h(\tau,\eta)a^{1-p}(\tau,\eta)\right]\Delta\eta\Delta\tau,$$
$$w(t,s) = \int_{s_0}^{s}\left[g(t,\eta) + \frac{h(t,\eta)a^{1-p}(\tau,\eta)}{p}\right]b(t,\eta)\Delta\eta, \quad (t,s)\in\Omega. \tag{2.20}$$

Proof. Noting that $a(t,s) > 0$ and $a(t,s)$ is nondecreasing for $(t,s)\in\Omega$, from (2.18), we have

$$\left(\frac{u(t,s)}{a(t,s)}\right)^p \le 1 + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)\left(\frac{u(\tau,\eta)}{a(\tau,\eta)}\right)^p + h(\tau,\eta)a^{1-p}(\tau,\eta)\frac{u(\tau,\eta)}{a(\tau,\eta)}\right]\Delta\eta\Delta\tau,$$
$$(t,s)\in\Omega. \tag{2.21}$$

By Theorem 2.3, from (2.21), we easily obtain the desired (2.19). This completes the proof of Theorem 2.5. □

Remark 2.6. If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}_+$ in Theorem 2.5, then we easily obtain Theorem 2.3.3(c_2) in [19].

Theorem 2.7. *Assume that $u(t,s)$, $a(t,s)$, and $b(t,s)$ are nonnegative functions defined for $(t,s)\in\Omega$ that are right-dense continuous for $(t,s)\in\Omega$, and $p>1$ is a real constant. If $f:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is right-dense continuous on Ω and continuous on $\mathbb{R}_+$ such that*

$$0 \le f(t,s,x) - f(t,s,y) \le \phi(t,s,y)(x-y), \tag{2.22}$$

for $(t,s)\in\Omega$, $x\ge y\ge 0$, where $\phi:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is right-dense continuous on Ω and continuous on $\mathbb{R}_+$, then

$$u^p(t,s) \le a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s} f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau, \quad (t,s)\in\Omega \tag{2.23}$$

implies

$$u(t,s) \le \left\{a(t,s) + b(t,s)\widetilde{m}(t,s)e_{\widetilde{w}(\cdot,s)}(t,t_0)\right\}^{1/p}, \quad (t,s)\in\Omega, \tag{2.24}$$

where

$$\widetilde{m}(t,s) = \int_{t_0}^{t}\int_{s_0}^{s} f\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p}\right)\Delta\eta\Delta\tau, \tag{2.25}$$

$$\widetilde{w}(t,s) = \int_{s_0}^{s}\phi\left(t,\eta,\frac{p-1+a(t,\eta)}{p}\right)\frac{b(t,\eta)}{p}\Delta\eta, \quad (t,s)\in\Omega. \tag{2.26}$$

Proof. Define a function $z(t,s)$ by

$$z(t,s) = \int_{t_0}^{t}\int_{s_0}^{s} f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau, \quad (t,s) \in \Omega. \tag{2.27}$$

As in the proof of Theorem 2.3, from (2.23), we easily see that (2.9) and (2.10) hold. Combining (2.10), (2.27) and noting the assumptions on f, we have

$$\begin{aligned} z(t,s) \le \int_{t_0}^{t}\int_{s_0}^{s}\Bigg[& f\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p}+\frac{b(\tau,\eta)z(\tau,\eta)}{p}\right) - f\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p}\right) \\ & +f\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p}\right)\Bigg]\Delta\eta\Delta\tau \\ \le \tilde{m}(t,s) + & \int_{t_0}^{t}\int_{s_0}^{s}\phi\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p}\right)\frac{b(\tau,\eta)}{p}z(\tau,\eta)\Delta\eta\Delta\tau, \end{aligned} \tag{2.28}$$

where $\tilde{m}(t,s)$ is defined by (2.25). It is easy to see that $\tilde{m}(t,s)$ is nonnegative, right-dense continuous, and nondecreasing for $(t,s) \in \Omega$. The remainder of the proof is similar to that of Theorem 2.3 and we omit it. □

Remark 2.8. Letting $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}_+$ and $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{N}_0$ in Theorem 2.7, respectively, we can obtain Theorem 2.3.4(d_1) and Theorem 5.2.4(d_2) in [19].

Theorem 2.9. *Assume that $u(t,s)$, $a(t,s)$, and $b(t,s)$ are nonnegative functions defined for $(t,s) \in \Omega$ that are right-dense continuous for $(t,s) \in \Omega$, and $p > 1$ is a real constant. If $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is right-dense continuous on Ω and continuous on $\mathbb{R}_+$, and $\Psi \in C(\mathbb{R}_+,\mathbb{R}_+)$ such that*

$$0 \le f(t,s,x) - f(t,s,y) \le \phi(t,s,y)\Psi^{-1}(x-y), \tag{2.29}$$

for $(t,s) \in \Omega$, $x \ge y \ge 0$, where $\phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is right-dense continuous on Ω and continuous on $\mathbb{R}_+$, Ψ^{-1} is the inverse function of Ψ, and

$$\Psi^{-1}(xy) \le \Psi^{-1}(x)\Psi^{-1}(y), \quad x,y \in \mathbb{R}_+, \tag{2.30}$$

then

$$u^p(t,s) \le a(t,s) + b(t,s)\Psi\left(\int_{t_0}^{t}\int_{s_0}^{s} f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau\right), \quad (t,s) \in \Omega \tag{2.31}$$

implies

$$u(t,s) \le \left\{a(t,s) + b(t,s)\Psi\left(\tilde{m}(t,s)e_{\overline{w}(\cdot,s)}(t,t_0)\right)\right\}^{1/p}, \quad (t,s) \in \Omega, \tag{2.32}$$

where $\widetilde{m}(t,s)$ *is defined by* (2.25), *and*

$$\overline{w}(t,s) = \int_{s_0}^{s} \phi\left(t, \eta, \frac{p-1+a(t,\eta)}{p}\right)\Psi^{-1}\left(\frac{b(t,\eta)}{p}\right)\Delta\eta, \quad (t,s) \in \Omega. \tag{2.33}$$

Proof. Define a function $z(t,s)$ by (2.27). Similar to the proof of Theorem 2.3, we have

$$u^p(t,s) \leq a(t,s) + b(t,s)\Phi(z(t,s)), \tag{2.34}$$

$$u(t,s) \leq \frac{p-1+a(t,s)}{p} + \frac{b(t,s)}{p}\Phi(z(t,s)), \quad (t,s) \in \Omega. \tag{2.35}$$

From (2.27), (2.35) and the assumptions on f and Ψ, we obtain

$$\begin{aligned} z(t,s) \leq & \int_{t_0}^{t}\int_{s_0}^{s}\left[f\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p} + \frac{b(\tau,\eta)\Psi(z(\tau,\eta))}{p}\right) - f\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p}\right)\right. \\ & \left. + f\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p}\right)\right]\Delta\eta\Delta\tau \\ \leq & \ \widetilde{m}(t,s) + \int_{t_0}^{t}\int_{s_0}^{s}\phi\left(\tau,\eta,\frac{p-1+a(\tau,\eta)}{p}\right)\Psi^{-1}\left(\frac{b(\tau,\eta)}{p}\right)z(\tau,\eta)\Delta\eta\Delta\tau, \end{aligned} \tag{2.36}$$

where $\widetilde{m}(t,s)$ is defined by (2.25). Obviously, $\widetilde{m}(t,s)$ is nonnegative, right-dense continuous, and nondecreasing for $(t,s) \in \Omega$. The remainder of the proof is similar to that of Theorem 2.3, and we omit it here. This completes the proof of Theorem 2.9. □

Remark 2.10. We note that when $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}_+$, Theorem 2.9 reduces to Theorem 2.3.4(d_2) in [19].

Remark 2.11. Using our main results, we can obtain many integral inequalities for some peculiar time scales. For example, letting $\mathbb{T}_1 = \mathbb{R}_+$, $\mathbb{T}_2 = \mathbb{N}_0$, from Theorem 2.3, we easily obtain the following result.

Corollary 2.12. *Assume that* $u(t,s)$, $a(t,s)$, $b(t,s)$, $g(t,s)$ *and* $h(t,s)$ *are nonnegative functions defined for* $t \in \mathbb{R}_+$, $s \in \mathbb{N}_0$ *that are continuous for* $t \in \mathbb{R}_+$, *and* $p > 1$ *is a real constant. Then,*

$$u^p(t,s) \leq a(t,s) + b(t,s)\int_0^t\left\{\sum_{\eta=0}^{s-1}\left[g(\tau,\eta)u^p(\tau,\eta) + h(\tau,\eta)u(\tau,\eta)\right]\right\}d\tau, \quad t \in \mathbb{R}_+, \ s \in \mathbb{N}_0 \tag{2.37}$$

implies

$$u(t,s) \le \left\{ a(t,s) + b(t,s)m^*(t,s) \times \exp\left(\int_0^t \left[\sum_{\eta=0}^{s-1} \left(g(\tau,\eta) + \frac{h(\tau,\eta)}{p} \right) b(\tau,\eta) \right] d\tau \right) \right\}^{1/p}, \quad t \in \mathbb{R}_+, \; s \in \mathbb{N}_0, \tag{2.38}$$

where

$$m^*(t,s) = \int_0^t \left\{ \sum_{\eta=0}^{s-1} \left[a(\tau,\eta)g(\tau,\eta) + \left(\frac{p-1}{p} + \frac{a(\tau,\eta)}{p} \right) h(\tau,\eta) \right] \right\} d\tau. \tag{2.39}$$

3. Some Applications

In this section, we present two applications of our main results.

Example 3.1. Consider the following partial dynamic equation on time scales

$$(u^p(t,s))^{\Delta_t \Delta_s} = F(t,s,u(t,s)) + r(t,s), \quad (t,s) \in \Omega, \tag{3.1}$$

with the initial boundary conditions

$$u(t,s_0) = \alpha(t), \qquad u(t_0,s) = \beta(s), \qquad u(t_0,s_0) = \gamma, \tag{3.2}$$

where $p > 1$ is a constant, $F : \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R} \to \mathbb{R}$ is right-dense continuous on Ω and continuous on $\mathbb{R}$, $r : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is right-dense continuous on Ω, $\alpha : \mathbb{T}_1 \to \mathbb{R}$ and $\beta : \mathbb{T}_2 \to \mathbb{R}$ are right-dense continuous, and $\gamma \in \mathbb{R}$ is a constant.

Assume that

$$|F(t,s,v)| \le g(t,s)|v|^p + h(t,s)|v|, \tag{3.3}$$

where $g(t,s)$ and $h(t,s)$ are nonnegative right-dense continuous functions for $(t,s) \in \Omega$. If $u(t,s)$ is a solution of (3.1), (3.2), then $u(t,s)$ satisfies

$$|u(t,s)| \le \left\{ a_0(t,s) + M(t,s)e_{\Upsilon(\cdot,s)}(t,t_0) \right\}^{1/p}, \quad (t,s) \in \Omega, \tag{3.4}$$

where

$$a_0(t,s) = \left|\alpha^p(t) + \beta^p(s) - \gamma^p\right| + \int_{t_0}^{t}\int_{s_0}^{s} \left|r(\tau,\eta)\right|\Delta\eta\Delta\tau,$$

$$M(t,s) = \int_{t_0}^{t}\int_{s_0}^{s}\left[a_0(\tau,\eta)g(\tau,\eta) + \left(\frac{p-1}{p} + \frac{a_0(\tau,\eta)}{p}\right)h(\tau,\eta)\right]\Delta\eta\Delta\tau, \tag{3.5}$$

$$Y(t,s) = \int_{s_0}^{s}\left[g(t,\eta) + \frac{h(t,\eta)}{p}\right]\Delta\eta, \quad (t,s)\in\Omega.$$

In fact, the solution $u(t,s)$ of (3.1), (3.2) satisfies

$$u^p(t,s) = \alpha^p(t) + \beta^p(s) - \gamma^p + \int_{t_0}^{t}\int_{s_0}^{s} F(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau + \int_{t_0}^{t}\int_{s_0}^{s} r(\tau,\eta)\Delta\eta\Delta\tau, \quad (t,s)\in\Omega. \tag{3.6}$$

Therefore,

$$|u(t,s)|^p \le a_0(t,s) + \int_{t_0}^{t}\int_{s_0}^{s}\left|F(\tau,\eta,u(\tau,\eta))\right|\Delta\eta\Delta\tau, \quad (t,s)\in\Omega. \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$|u(t,s)|^p \le a_0(t,s) + \int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)\left|u(\tau,\eta)\right|^p + h(\tau,\eta)\left|u(\tau,\eta)\right|\right]\Delta\eta\Delta\tau, \quad (t,s)\in\Omega. \tag{3.8}$$

Using Theorem 2.3, from (3.8), we easily obtain (3.4).

Example 3.2. Consider the following dynamic equation on time scales:

$$u^p(t,s) = K + \int_{t_0}^{t}\int_{s_0}^{s} H(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau, \quad (t,s)\in\Omega, \tag{3.9}$$

where $K > 0$, $p > 1$ are constants, $H : \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R} \to \mathbb{R}$ is right-dense continuous on Ω and continuous on $\mathbb{R}$.

Assume that

$$|H(t,s,v)| \le h(t,s)|v|, \quad (t,s)\in\Omega, \tag{3.10}$$

where $h(t,s)$ is a nonnegative right-dense continuous function for $(t,s)\in\Omega$. If $u(t,s)$ is a solution of (3.9), then

$$|u(t,s)| \le \left\{K\left[1 + \overline{n}(t,s)e_{q(\cdot,s)}(t,t_0)\right]\right\}^{1/p}, \quad (t,s)\in\Omega, \tag{3.11}$$

where

$$\overline{n}(t,s) = K^{(1-p)/p}\int_{t_0}^{t}\int_{s_0}^{s} h(\tau,\eta)\Delta\eta\Delta\tau,$$
$$q(t,s) = \frac{K^{(1-p)/p}}{p}\int_{s_0}^{s} h(t,\eta)\Delta\eta, \quad (t,s)\in\Omega. \tag{3.12}$$

In fact, if $u(t,s)$ is a solution of (3.9), then

$$|u(t,s)|^p \le K + \int_{t_0}^{t}\int_{s_0}^{s} \left|H(\tau,\eta,u(\tau,\eta))\right|\Delta\eta\Delta\tau, \quad (t,s)\in\Omega. \tag{3.13}$$

It follows from (3.10) and (3.13) that

$$|u(t,s)|^p \le K + \int_{t_0}^{t}\int_{s_0}^{s} h(\tau,\eta)\left|u(\tau,\eta)\right|\Delta\eta\Delta\tau, \quad (t,s)\in\Omega. \tag{3.14}$$

Therefore, by Theorem 2.5, from (3.14), we immediately obtain (3.11).

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