Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2011, Article ID 237219, 14 pages
doi:10.1155/2011/237219

Research Article

Asymptotic Behavior of Solutions of Higher-Order Dynamic Equations on Time Scales

Taixiang Sun,[1] Hongjian Xi,[2] and Xiaofeng Peng[1]

[1] *College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, China*
[2] *Department of Mathematics, Guangxi College of Finance and Economics, Nanning, Guangxi 530003, China*

Correspondence should be addressed to Taixiang Sun, stx1963@163.com

Received 18 November 2010; Accepted 23 February 2011

Academic Editor: Abdelkader Boucherif

We investigate the asymptotic behavior of solutions of the following higher-order dynamic equation $x^{\Delta^n}(t) + f(t, x(t), x^{\Delta}(t), \ldots, x^{\Delta^{n-1}}(t)) = 0$, on an arbitrary time scale $\mathbf{T}$, where the function f is defined on $\mathbf{T} \times \mathbf{R}^n$. We give sufficient conditions under which every solution x of this equation satisfies one of the following conditions: (1) $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = 0$; (2) there exist constants a_i $(0 \le i \le n-1)$ with $a_0 \neq 0$, such that $\lim_{t\to\infty} x(t)/\sum_{i=0}^{n-1} a_i h_{n-i-1}(t, t_0) = 1$, where $h_i(t, t_0)$ $(0 \le i \le n-1)$ are as in Main Results.

1. Introduction

In this paper, we investigate the asymptotic behavior of solutions of the following higher-order dynamic equation

$$x^{\Delta^n}(t) + f\left(t, x(t), x^{\Delta}(t), \ldots, x^{\Delta^{n-1}}(t)\right) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbf{T}$, where the function f is defined on $\mathbf{T} \times \mathbf{R}^n$.

Since we are interested in the asymptotic and oscillatory behavior of solutions near infinity, we assume that $\sup \mathbf{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbf{T}} = \{t \in \mathbf{T} : t \ge t_0\}$, where $t_0 \in \mathbf{T}$. By a solution of (1.1), we mean a nontrivial real-valued function $x \in C_{\mathrm{rd}}([T_x, \infty)_{\mathbf{T}}, \mathbf{R}), T_x \ge t_0$, which has the property that $x^{\Delta^n}(t) \in C_{\mathrm{rd}}([T_x, \infty)_{\mathbf{T}}, \mathbf{R})$ and satisfies (1.1) on $[T_x, \infty)_{\mathbf{T}}$, where C_{rd} is the space of rd-continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger's landmark paper [1] in order to create a theory that can unify continuous and discrete analysis. The cases when a time scale is equal to the real numbers or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [2]). Not only the new theory of the so-called "dynamic equations" unifies the theories of differential equations and difference equations but also extends these classical cases to cases "in between," for example, to the so-called q-difference equations when $\mathbf{T} = q^{\mathbf{N}_0}$, which has important applications in quantum theory (see [3]).

On a time scale $\mathbf{T}$, the forward jump operator, the backward jump operator, and the graininess function are defined as

$$\sigma(t) = \inf\{s \in \mathbf{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbf{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t, \tag{1.2}$$

respectively. We refer the reader to [2, 4] for further results on time scale calculus. Let $p \in C_{\mathrm{rd}}(\mathbf{T}, \mathbf{R})$ with $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbf{T}$, then the delta exponential function $e_p(t, t_0)$ is defined as the unique solution of the initial value problem

$$\begin{gathered} y^{\Delta} = p(t)y, \\ y(t_0) = 1. \end{gathered} \tag{1.3}$$

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to [5–18].

Recently, Erbe et al. [19–21] considered the asymptotic behavior of solutions of the third-order dynamic equations

$$\begin{gathered} \left(a(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\Delta} + p(t)f(x(t)) = 0, \\ x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0, \\ \left(a(t)\left\{\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\gamma}\right)^{\Delta} + f(t, x(t)) = 0, \end{gathered} \tag{1.4}$$

respectively, and established some sufficient conditions for oscillation.

Karpuz [22] studied the asymptotic nature of all bounded solutions of the following higher-order nonlinear forced neutral dynamic equation

$$[x(t) + A(t)x(\alpha(t))]^{\Delta^n} + f\left(t, x(\beta(t)), x(\gamma(t))\right) = \varphi(t). \tag{1.5}$$

Chen [23] derived some sufficient conditions for the oscillation and asymptotic behavior of the nth-order nonlinear neutral delay dynamic equations

$$\left\{a(t)\Psi(x(t))\left[\left|\left(x(t) + p(t)x(\tau(t))\right)^{\Delta^{n-1}}\right|^{\alpha-1}\left(x(t) + p(t)x(\tau(t))\right)^{\Delta^{n-1}}\right]^{\gamma}\right\}^{\Delta} + \lambda F(t, x(\delta(t))) = 0, \tag{1.6}$$

on an arbitrary time scale $\mathbf{T}$. Motivated by the above studies, in this paper, we study (1.1) and give sufficient conditions under which every solution x of (1.1) satisfies one of the following conditions: (1) $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = 0$; (2) there exist constants a_i $(0 \le i \le n-1)$ with $a_0 \ne 0$, such that $\lim_{t\to\infty} x(t)/\sum_{i=0}^{n-1} a_i h_{n-i-1}(t,t_0) = 1$, where $h_i(t,t_0)$ $(0 \le i \le n-1)$ are as in Section 2.

2. Main Results

Let k be a nonnegative integer and $s,t \in \mathbf{T}$, then we define a sequence of functions $h_k(t,s)$ as follows:

$$h_k(t,s) = \begin{cases} 1 & \text{if } k = 0, \\ \displaystyle\int_s^t h_{k-1}(\tau,s)\Delta\tau & \text{if } k \ge 1. \end{cases} \tag{2.1}$$

To obtain our main results, we need the following lemmas.

Lemma 2.1. *Let n be a positive integer, then there exists $T_n > t_0$, such that*

$$h_{k+1}(t,t_0) - h_k(t,t_0) \ge 1 \quad \textit{for } t \ge T_n,\ 0 \le k \le n-1. \tag{2.2}$$

Proof. We will prove the above by induction. First, if $k = 0$, then we take $T_1 \ge t_0 + 2$. Thus,

$$h_1(t,t_0) - h_0(t,t_0) = t - t_0 - 1 \ge 1 \quad \text{for } t \ge T_1. \tag{2.3}$$

Next, we assume that there exists $T_m > t_0$, such that $h_{k+1}(t,t_0) - h_k(t,t_0) \ge 1$ for $t \ge T_m$ and $0 \le k \le m$ with $0 \le m < n-1$, then

$$\begin{aligned} h_{m+1}(t,t_0) - h_m(t,t_0) &= \int_{t_0}^{t} (h_m(\tau,t_0) - h_{m-1}(\tau,t_0))\Delta\tau \\ &= \int_{t_0}^{T_m} (h_m(\tau,t_0) - h_{m-1}(\tau,t_0))\Delta\tau + \int_{T_m}^{t} (h_m(\tau,t_0) - h_{m-1}(\tau,t_0))\Delta\tau \\ &\ge \int_{t_0}^{T_m} (h_m(\tau,t_0) - h_{m-1}(\tau,t_0))\Delta\tau + \int_{T_m}^{t} \Delta\tau \\ &= \int_{t_0}^{T_m} (h_m(\tau,t_0) - h_{m-1}(\tau,t_0))\Delta\tau + t - T_m, \end{aligned} \tag{2.4}$$

from which it follows that there exists $T_{m+1} > T_m$, such that $h_{k+1}(t,t_0) - h_k(t,t_0) \ge 1$ for $t \ge T_{m+1}$ and $0 \le k \le m+1$. The proof is completed. □

Lemma 2.2 (see [24]). *Let $p \in C_{\mathrm{rd}}(\mathbf{T}, [0,\infty))$, then*

$$1 + \int_{t_0}^{t} p(s)\Delta s \le e_p(t, t_0) \le e^{\int_{t_0}^{t} p(s)\Delta s}. \tag{2.5}$$

Lemma 2.3 (see [2]). *Let $y, p \in C_{\mathrm{rd}}(\mathbf{T}, [0,\infty))$ and $A \in [0,\infty)$, then*

$$y(t) \le A + \int_{t_0}^{t} y(\tau)p(\tau)\Delta\tau, \quad \forall t \in \mathbf{T} \tag{2.6}$$

implies

$$y(t) \le A e_p(t, t_0), \quad \forall t \in \mathbf{T}. \tag{2.7}$$

Lemma 2.4 (see [2]). *Let n be a positive integer. Suppose that x is n times differentiable on $\boldsymbol{T}$. Let $\alpha \in T^{\kappa^{n-1}}$ and $t \in \mathbf{T}$, then*

$$x(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) x^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\tau)) x^{\Delta^n}(\tau)\Delta\tau. \tag{2.8}$$

Lemma 2.5 (see [2]). *Assume that f and g are differentiable on $\boldsymbol{T}$ with $\lim_{t\to\infty} g(t) = \infty$. If there exists $T > t_0$, such that*

$$g(t) > 0, \quad g^{\Delta}(t) > 0, \quad \forall t \ge \mathbf{T}, \tag{2.9}$$

then

$$\lim_{t\to\infty} \frac{f^{\Delta}(t)}{g^{\Delta}(t)} = r \ (or\ \infty) \ implies \ \lim_{t\to\infty} \frac{f(t)}{g(t)} = r \ (or\ \infty). \tag{2.10}$$

Lemma 2.6 (see [23]). *Let x be defined on $[t_0,\infty)_{\mathbf{T}}$, and $x(t) > 0$ with $x^{\Delta^n}(t) \le 0$ for $t \ge t_0$ and not eventually zero. If x is bounded, then*

(1) $\lim_{t\to\infty} x^{\Delta^i}(t) = 0$ *for* $1 \le i \le n-1$,

(2) $(-1)^{i+1} x^{\Delta^{n-i}}(t) > 0$ *for all* $t \ge t_0$ *and* $1 \le i \le n-1$.

Now, one states and proves the main results.

Theorem 2.7. *Assume that there exists $t_1 > t_0$, such that the function $f(t, u_0, \dots, u_{n-1})$ satisfies*

$$\left| f(t, u_0, \dots, u_{n-1}) \right| \le \sum_{i=0}^{n-1} p_i(t)|u_i|, \quad \forall (t, u_0, \dots, u_{n-1}) \in [t_1, \infty)_{\mathbf{T}} \times \mathbf{R}^n, \tag{2.11}$$

where $p_i(t)$ $(0 \le i \le n-1)$ *are nonnegative functions on* $[t_1, \infty)_{\mathbf{T}}$ *and*

$$\lim_{t\to\infty} e_q(t, t_1) < \infty, \tag{2.12}$$

with $q(t) = \sum_{i=0}^{n-1} p_i(t) h_{n-i-1}(t, t_0)$ $(t \ge t_1)$, *then every solution* x *of* (1.1) *satisfies one of the following conditions:*

(1) $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = 0$,

(2) *there exist constants* a_i $(0 \le i \le n-1)$ *with* $a_0 \ne 0$, *such that*

$$\lim_{t\to\infty} \frac{x(t)}{\sum_{i=0}^{n-1} a_i h_{n-i-1}(t, t_0)} = 1. \tag{2.13}$$

Proof. Let x be a solution of (1.1), then it follows from Lemma 2.4 that for $0 \le m \le n-1$,

$$x^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} h_k(t, t_1) x^{\Delta^{k+m}}(t_1) + \int_{t_1}^{\rho^{n-m-1}(t)} h_{n-m-1}(t, \sigma(\tau)) x^{\Delta^n}(\tau) \Delta\tau \quad \text{for } t \ge t_1. \tag{2.14}$$

By (2.11) and Lemma 2.1, we see that there exists $T > t_1$, such that for $t \ge T$ and $0 \le m \le n-1$,

$$\left| x^{\Delta^m}(t) \right| \le h_{n-m-1}(t, t_0) \left[\sum_{k=0}^{n-m-1} \left| x^{\Delta^{k+m}}(t_1) \right| + \int_{t_1}^{t} \sum_{i=0}^{n-1} p_i(\tau) \left| x^{\Delta^i}(\tau) \right| \Delta\tau \right]. \tag{2.15}$$

Then we obtain

$$\left| x^{\Delta^m}(t) \right| \le h_{n-m-1}(t, t_0) F(t) \quad \text{for } t \ge T,\ 0 \le m \le n-1, \tag{2.16}$$

where

$$F(t) = A + \int_T^t \sum_{i=0}^{n-1} p_i(\tau) \left| x^{\Delta^i}(\tau) \right| \Delta\tau, \tag{2.17}$$

with

$$A = \max_{0 \le m \le n-1} \left\{ \sum_{k=0}^{n-m-1} \left| x^{\Delta^{k+m}}(t_1) \right| \right\} + \int_{t_1}^{T} \sum_{i=0}^{n-1} p_i(\tau) \left| x^{\Delta^i}(\tau) \right| \Delta\tau. \tag{2.18}$$

Using (2.16) and (2.17), it follows that

$$F(t) \le A + \int_T^t \sum_{i=0}^{n-1} p_i(\tau) h_{n-i-1}(\tau, t_0) F(\tau) \Delta\tau \quad \text{for } t \ge T. \tag{2.19}$$

By Lemma 2.3, we have

$$F(t) \leq A e_q(t,T) \quad \forall t \geq T, \tag{2.20}$$

with $q(t) = \sum_{i=0}^{n-1} p_i(t) h_{n-i-1}(t,t_0)$. Hence from (2.12), there exists a finite constant $c > 0$, such that $F(t) \leq c$ for $t \geq T$. Thus, inequality (2.20) implies that

$$\left| x^{\Delta^m}(t) \right| \leq h_{n-m-1}(t,t_0) c \quad \text{for } t \geq T,\ 0 \leq m \leq n-1. \tag{2.21}$$

By (1.1), we see that if $t \geq T$, then

$$x^{\Delta^{n-1}}(t) = x^{\Delta^{n-1}}(T) - \int_T^t f\left(\tau, x(\tau), x^{\Delta}(\tau), \ldots, x^{\Delta^{n-1}}(\tau)\right) \Delta\tau. \tag{2.22}$$

Since condition (2.12) and Lemma 2.2 implies that

$$\lim_{t \to \infty} \int_T^t \sum_{i=0}^{n-1} p_i(\tau) h_{n-i-1}(\tau,t_0) \Delta\tau < \infty, \tag{2.23}$$

we find from (2.11) and (2.21) that the sum in (2.22) converges as $t \to \infty$. Therefore, $\lim_{t\to\infty} x^{\Delta^{n-1}}(t)$ exists and is a finite number. Let $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = a_0$. If $a_0 \neq 0$, then it follows from Lemma 2.5 that

$$\lim_{t \to \infty} \frac{x(t)}{h_{n-1}(t,t_0)} = \lim_{t \to \infty} x^{\Delta^{n-1}}(t) = a_0, \tag{2.24}$$

and x has the desired asymptotic property. The proof is completed. □

Theorem 2.8. *Assume that there exist functions $p_i : [t_0,\infty)_{\mathbf{T}} \to (0,\infty)$ $(0 \leq i \leq n)$, and nondecreasing continuous functions $g_i : (0,\infty) \to (0,\infty)$ $(0 \leq i \leq n-1)$, and $t_1 > t_0$ such that*

$$\left| f(t, u_0, \ldots, u_{n-1}) \right| \leq \sum_{i=0}^{n-1} p_i(t) g_i\left(\frac{|u_i|}{h_{n-i-1}(t,t_0)} \right) + p_n(t) \quad \textit{for } t \geq t_1, \tag{2.25}$$

with

$$\begin{gathered} \int_{t_1}^{\infty} p_i(t) \Delta t = P_i < \infty \quad \textit{for } 0 \leq i \leq n, \\ \int_{\varepsilon}^{\infty} \frac{ds}{\sum_{i=0}^{n-1} g_i(s)} = \infty \quad \textit{for any } \varepsilon > 0, \end{gathered} \tag{2.26}$$

then every solution x *of* (1.1) *satisfies one of the following conditions:*

(1) $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = 0$,

(2) *there exist constants* a_i $(0 \le i \le n-1)$ *with* $a_0 \ne 0$ *such that*

$$\lim_{t\to\infty} \frac{x(t)}{\sum_{i=0}^{n-1} a_i h_{n-i-1}(t,t_0)} = 1. \tag{2.27}$$

Proof. Let x be a solution of (1.1), then it follows from Lemma 2.4 that for $0 \le m \le n-1$,

$$x^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} h_k(t,t_1) x^{\Delta^{k+m}}(t_1) + \int_{t_1}^{\rho^{n-m-1}(t)} h_{n-m-1}(t,\sigma(\tau)) x^{\Delta^n}(\tau)\Delta\tau \quad \text{for } t \ge t_1. \tag{2.28}$$

By Lemma 2.1 and (2.25), we see that there exists $T > t_1$, such that for $t \ge T$ and $0 \le m \le n-1$,

$$\left|x^{\Delta^m}(t)\right| \le h_{n-m-1}(t,t_0)\left[\sum_{k=0}^{n-m-1}\left|x^{\Delta^{k+m}}(t_1)\right| + \int_{t_1}^{t}\left[\sum_{i=0}^{n-1} p_i(\tau) g_i\left(\frac{\left|x^{\Delta^i}(\tau)\right|}{h_{n-i-1}(\tau,t_0)}\right) + p_n(\tau)\right]\Delta\tau\right]. \tag{2.29}$$

Then, we obtain

$$\left|x^{\Delta^m}(t)\right| \le h_{n-m-1}(t,t_0)F(t), \quad \text{for } t \ge T,\ 0 \le m \le n-1, \tag{2.30}$$

where

$$F(t) = A + \int_T^t \sum_{i=0}^{n-1} p_i(\tau) g_i\left(\frac{\left|x^{\Delta^i}(\tau)\right|}{h_{n-i-1}(\tau,t_0)}\right)\Delta\tau, \tag{2.31}$$

with

$$A = \max_{0\le m\le n-1}\left\{\sum_{k=0}^{n-m-1}\left|x^{\Delta^{k+m}}(t_1)\right|\right\} + \int_{t_1}^{T}\sum_{i=0}^{n-1} p_i(\tau) g_i\left(\frac{\left|x^{\Delta^i}(\tau)\right|}{h_{n-i-1}(\tau,t_0)}\right)\Delta\tau + P_n. \tag{2.32}$$

Using (2.30) and (2.31), it follows that

$$F(t) \le A + \int_T^t \sum_{i=0}^{n-1} p_i(\tau) g_i(F(\tau))\Delta\tau \quad \text{for } t \ge T. \tag{2.33}$$

Write

$$u(t) = A + \int_T^t \sum_{i=0}^{n-1} p_i(\tau) g_i(F(\tau)) \Delta\tau \quad \text{for } t \geq T, \tag{2.34}$$

$$G(y) = \int_A^y \frac{ds}{\sum_{i=0}^{n-1} g_i(s)}, \tag{2.35}$$

then

$$\begin{aligned}
[G(u(t))]^{\Delta} &= u^{\Delta}(t) \int_0^1 G'(hu(t) + (1-h)u^{\sigma}(t)) dh \\
&= \left(\sum_{i=0}^{n-1} p_i(t) g_i(F(t)) \right) \int_0^1 \frac{dh}{\sum_{i=0}^{n-1} g_i(hu(t) + (1-h)u^{\sigma}(t))} \\
&\leq \frac{\sum_{i=0}^{n-1} p_i(t) g_i(u(t))}{\sum_{i=0}^{n-1} g_i(u(t))} \\
&\leq \sum_{i=0}^{n-1} p_i(t),
\end{aligned} \tag{2.36}$$

from which it follows that

$$G(u(t)) \leq G(u(T)) + \int_T^t \sum_{i=0}^{n-1} p_i(\tau) \Delta\tau \leq G(u(T)) + \sum_{i=0}^{n-1} P_i. \tag{2.37}$$

Since $\lim_{y\to\infty} G(y) = \infty$ and $G(y)$ is strictly increasing, there exists a constant $c > 0$, such that $u(t) \leq c$ for $t \geq T$. By (2.30), (2.33), and (2.34), we have

$$\left| x^{\Delta^m}(t) \right| \leq h_{n-m-1}(t, t_0) c \quad \text{for } t \geq T,\ 0 \leq m \leq n-1. \tag{2.38}$$

It follows from (1.1) that if $t \geq T$, then

$$x^{\Delta^{n-1}}(t) = x^{\Delta^{n-1}}(T) - \int_T^t f\left(\tau, x(\tau), x^{\Delta}(\tau), \dots, x^{\Delta^{n-1}}(\tau)\right) \Delta\tau. \tag{2.39}$$

Since (2.38) and condition (2.25) implies that

$$\begin{aligned}
&\int_T^t \left| f\left(\tau, x(\tau), x^{\Delta}(\tau), \ldots, x^{\Delta^{n-1}}(\tau)\right) \right| \Delta\tau \\
&\quad \le \int_T^t \left[\sum_{i=0}^{n-1} p_i(\tau) g_i\left(\frac{\left| x^{\Delta^i}(\tau) \right|}{h_{n-i-1}(\tau, t_0)} \right) + p_n(\tau) \right] \Delta\tau \\
&\quad \le \sum_{i=0}^{n-1} P_i g_i(c) + P_n \\
&\quad = M < \infty,
\end{aligned} \tag{2.40}$$

we see that the sum in (2.39) converges as $t \to \infty$. Therefore, $\lim_{t\to\infty} x^{\Delta^{n-1}}(t)$ exists and is a finite number. Let $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = a_0$. If $a_0 \ne 0$, then it follows from Lemma 2.5 that

$$\lim_{t\to\infty} \frac{x(t)}{h_{n-1}(t, t_0)} = \lim_{t\to\infty} x^{\Delta^{n-1}}(t) = a_0, \tag{2.41}$$

and x has the desired asymptotic property. The proof is completed. □

Theorem 2.9. *Assume that there exist positive functions $p : [t_0, \infty)_{\mathbf{T}} \to (0, \infty)$, and nondecreasing continuous functions $g_i : (0, \infty) \to (0, \infty)$ $(0 \le i \le n-1)$, and $t_1 > t_0$, such that*

$$\left| f(t, u_0, \ldots, u_{n-1}) \right| \le p(t) \prod_{i=0}^{n-1} g_i\left(\frac{|u_i|}{h_{n-i-1}(t, t_0)} \right) \quad \textit{for } t \ge t_1, \tag{2.42}$$

with

$$\begin{gathered}
\int_{t_1}^{\infty} p(t)\Delta t = P < \infty, \\
\int_{\varepsilon}^{\infty} \frac{ds}{\prod_{i=0}^{n-1} g_i(s)} = \infty, \quad \textit{for any } \varepsilon > 0,
\end{gathered} \tag{2.43}$$

then every solution x of (1.1) *satisfies one of the following conditions:*

(1) $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) = 0$,

(2) *there exist constants a_i $(0 \le i \le n-1)$ with $a_0 \ne 0$, such that*

$$\lim_{t\to\infty} \frac{x(t)}{\sum_{i=0}^{n-1} a_i h_{n-i-1}(t, t_0)} = 1. \tag{2.44}$$

Proof. Arguing as in the proof of Theorem 2.8, we see that there exists $T > t_1$, such that for $t \geq T$ and $0 \leq m \leq n-1$,

$$\left|x^{\Delta^m}(t)\right| \leq h_{n-m-1}(t,t_0)\left[\sum_{k=0}^{n-m-1}\left|x^{\Delta^{k+m}}(t_1)\right| + \int_{t_1}^{t}\prod_{i=0}^{n-1}p(\tau)g_i\left(\frac{\left|x^{\Delta^i}(\tau)\right|}{h_{n-i-1}(\tau,t_0)}\right)\Delta\tau\right], \tag{2.45}$$

from which we obtain

$$\left|x^{\Delta^m}(t)\right| \leq h_{n-m-1}(t,t_0)F(t) \quad \text{for } t \geq T,\ 0 \leq m \leq n-1, \tag{2.46}$$

where

$$F(t) = A + \int_{T}^{t}\prod_{i=0}^{n-1}p(\tau)g_i\left(\frac{\left|x^{\Delta^i}(\tau)\right|}{h_{n-i-1}(\tau,t_0)}\right), \tag{2.47}$$

$$A = \max_{0\leq m\leq n-1}\left\{\sum_{k=0}^{n-m-1}\left|x^{\Delta^{k+m}}(t_0)\right|\right\} + \int_{t_1}^{T}\prod_{i=0}^{n-1}p(\tau)g_i\left(\frac{\left|x^{\Delta^i}(\tau)\right|}{h_{n-i-1}(\tau,t_0)}\right). \tag{2.48}$$

Using (2.46) and (2.47), it follows that

$$F(t) \leq A + \int_{T}^{t}\prod_{i=0}^{n-1}p(\tau)g_i(F(\tau))\Delta\tau \quad \text{for } t \geq T. \tag{2.49}$$

Write

$$u(t) = A + \int_{T}^{t}\prod_{i=0}^{n-1}p(\tau)g_i(F(\tau))\Delta\tau \quad \text{for } t \geq T, \tag{2.50}$$

$$G(y) = \int_{A}^{y}\frac{ds}{\prod_{i=0}^{n-1}g_i(s)}, \tag{2.51}$$

then

$$\begin{aligned}
[G(u(t))]^{\Delta} &= u^{\Delta}(t)\int_0^1 G'(hu(t) + (1-h)u^{\sigma}(t))dh \\
&= \left(\prod_{i=0}^{n-1}p(t)g_i(F(t))\right)\int_0^1\frac{dh}{\prod_{i=0}^{n-1}g_i(hu(t) + (1-h)u^{\sigma}(t))} \\
&\leq \frac{\prod_{i=0}^{n-1}p(t)g_i(u(t))}{\prod_{i=0}^{n-1}g_i(u(t))} \\
&= p(t),
\end{aligned} \tag{2.52}$$

from which it follows that

$$G(u(t)) \le G(u(T)) + \int_T^t p(\tau)\Delta\tau \le G(u(T)) + P. \tag{2.53}$$

The rest of the proof is similar to that of Theorem 2.8, and the details are omitted. The proof is completed. □

Theorem 2.10. *Assume that the function $f(t, u_0, \ldots, u_{n-1})$ satisfies*

(1) $f(t, u_0, \ldots, u_{n-1}) = p(t)F(u_0, \ldots, u_{n-1})$ *for all* $(t, u_0, \ldots, u_{n-1}) \in [t_0, \infty)_T \times \mathbf{R}^n$,

(2) $p(t) \ge 0$ *for* $t \ge t_0$ *and* $\int_{t_0}^{\infty} h_{n-1}(\tau, t_0)p(\tau)\Delta\tau = \infty$,

(3) $u_0 F(u_0, \ldots, u_{n-1}) > 0$ *for* $u_0 \ne 0$ *and* $F(u_0, \ldots, u_{n-1})$ *is continuous at* $(u_0, 0, \ldots, 0)$ *with* $u_0 \ne 0$,

then (1) if n is even, then every bounded solution of (1.1) *is oscillatory; (2) if n is odd, then every bounded solution $x(t)$ of* (1.1) *is either oscillatory or tends monotonically to zero together with* $x^{\Delta^i}(t)$ $(1 \le i \le n-1)$.

Proof. Assume that (1.1) has a nonoscillatory solution x on $[t_0, \infty)$, then, without loss of generality, there is a $t_1 \ge t_0$, sufficiently large, such that $x(t) > 0$ for $t \ge t_1$. It follows from (1.1) that $x^{\Delta^n}(t) \le 0$ for $t \ge t_1$ and not eventually zero. By Lemma 2.6, we have

$$\begin{gathered} \lim_{t\to\infty} x^{\Delta^i}(t) = 0, \quad \text{for } 1 \le i \le n-1, \\ (-1)^{i+1} x^{\Delta^{n-i}}(t) > 0 \quad \forall t \ge t_1,\ 1 \le i \le n-1, \end{gathered} \tag{2.54}$$

and $x(t)$ is eventually monotone. Also $x^{\Delta}(t) > 0$ for $t \ge t_1$ if n is even and $x^{\Delta}(t) < 0$ for $t \ge t_1$ if n is odd. Since $x(t)$ is bounded, we find $\lim_{t\to\infty} x(t) = c \ge 0$. Furthermore, if n is even, then $c > 0$.

We claim that $c = 0$. If not, then there exists $t_2 > t_1$, such that

$$F\Big(x(t), x^{\Delta}(t), \ldots, x^{\Delta^{n-1}}(t)\Big) > \frac{F(c, 0, \ldots, 0)}{2} > 0 \quad \text{for } t \ge t_2, \tag{2.55}$$

since F is continuous at $(c, 0, \ldots, 0)$ by the condition (3). From (1.1) and (2.55), we have

$$x^{\Delta^n}(t) + p(t)\frac{F(c, 0, \ldots, 0)}{2} \le 0, \quad \text{for } t \ge t_2. \tag{2.56}$$

Multiplying the above inequality by $h_{n-1}(t, t_0)$, and integrating from t_2 to t, we obtain

$$\int_{t_2}^{t} h_{n-1}(\tau, t_0)x^{\Delta^n}(\tau)\Delta\tau + \int_{t_2}^{t} h_{n-1}(\tau, t_0)p(\tau)\frac{F(c, 0, \ldots, 0)}{2}\Delta\tau \le 0, \quad \text{for } t \ge t_2. \tag{2.57}$$

Since

$$\int_{t_2}^{t} h_{n-1}(\tau,t_0)x^{\Delta^n}(\tau)\Delta\tau \geq \sum_{i=1}^{n}(-1)^{i+1}h_{n-i}(\tau,t_0)x^{\Delta^{n-i}}(\tau)\Big|_{t_2}^{t} \geq \sum_{i=1}^{n}(-1)^{i}h_{n-i}(t_2,t_0)x^{\Delta^{n-i}}(t_2)+(-1)^{n+1}x(t), \tag{2.58}$$

we get

$$A+(-1)^{n+1}x(t)+\int_{t_2}^{t} h_{n-1}(\tau,t_0)p(\tau)\frac{F(c,0,\dots,0)}{2}\Delta\tau \leq 0, \quad \text{for } t \geq t_2, \tag{2.59}$$

where $A=\sum_{i=1}^{n}(-1)^{i}h_{n-i}(t_2,t_0)x^{\Delta^{n-i}}(t_2)$. Thus, $\int_{t_2}^{\infty} h_{n-1}(\tau,t_0)p(\tau)\Delta\tau<\infty$ since $x(t)$ is bounded, which gives a contradiction to the condition (2). The proof is completed. □

3. Examples

Example 3.1. Consider the following higher-order dynamic equation:

$$x^{\Delta^n}(t)+\sum_{i=0}^{n-1}\frac{1}{t^{\beta_i}}\frac{x^{\Delta^i}(t)}{h_{n-i-1}(t,t_0)}=0, \tag{3.1}$$

where $t \geq t_1 > t_0 > 0$ and $\beta_i > 1$ $(0 \leq i \leq n-1)$. Let $p_i(t)=1/[t^{\beta_i}h_{n-i-1}(t,t_0)]$ $(0 \leq i \leq n-1)$ and

$$f(t,u_0,\dots,u_{n-1})=\sum_{i=0}^{n-1}\frac{1}{t^{\beta_i}}\frac{u_i}{h_{n-i-1}(t,t_0)}, \tag{3.2}$$

then we have

$$\left|f(t,u_0,\dots,u_{n-1})\right| \leq \sum_{i=0}^{n-1}p_i(t)|u_i|, \quad \forall (t,u_0,\dots,u_{n-1}) \in [t_1,\infty)_{\mathbf{T}} \times \mathbf{R}^n,$$
$$e_{\sum_{i=0}^{n-1}p_i(t)h_{n-i-1}}(t,t_1)=e_{\sum_{i=0}^{n-1}1/t^{\beta_i}}(t,t_1) \leq e^{\int_{t_1}^{t}\sum_{i=0}^{n-1}1/\tau^{\beta_i}\Delta\tau}<\infty, \tag{3.3}$$

by Example 5.60 in [4]. Thus, it follows from Theorem 2.7 that if x is a solution of (3.1) with $\lim_{t\to\infty}x^{\Delta^{n-1}}(t)\neq 0$, then there exist constants a_i $(0 \leq i \leq n-1)$ with $a_0 \neq 0$, such that $\lim_{t\to\infty}x(t)/\sum_{i=0}^{n-1}a_ih_{n-i-1}(t,t_0)=1$.

Example 3.2. Consider the following higher-order dynamic equation:

$$x^{\Delta^n}(t)+\sum_{i=0}^{n-1}\frac{1}{t^{\beta_i}}\left(\frac{x^{\Delta^i}(t)}{h_{n-i-1}(t,t_0)}\right)^{\alpha_i}+\frac{1}{t^{\beta_n}}=0, \tag{3.4}$$

where $t > t_0 > 0$, $\alpha_i \in (0,1)$ $(0 \le i \le n-1)$, and $\beta_i > 1$ $(0 \le i \le n)$. Let $g_i(u) = u^{\alpha_i}$ $(0 \le i \le n-1)$, $p_i(t) = 1/t^{\beta_i}$ $(0 \le i \le n)$, and

$$f(t, u_0, \ldots, u_{n-1}) = \sum_{i=0}^{n-1} \frac{1}{t^{\beta_i}} \left(\frac{u_i}{h_{n-i-1}(t, t_0)} \right)^{\alpha_i} + \frac{1}{t^{\beta_n}}. \tag{3.5}$$

It is easy to verify that $f(t, u_0, \ldots, u_{n-1})$ satisfies the conditions of Theorem 2.8. Thus, it follows that if x is a solution of (3.4) with $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) \neq 0$, then there exist constants a_i $(0 \le i \le n-1)$ with $a_0 \neq 0$, such that $\lim_{t\to\infty} x(t) / \sum_{i=0}^{n-1} a_i h_{n-i-1}(t, t_0) = 1$.

Example 3.3. Consider the following higher-order dynamic equation:

$$x^{\Delta^n}(t) + \frac{1}{t^{\beta}} \prod_{i=0}^{n-1} \left(\frac{x^{\Delta^i}(t)}{h_{n-i-1}(t, t_0)} \right)^{\alpha_i} = 0, \tag{3.6}$$

where $t > t_0 > 0$, $\alpha_i \in (0,1)$ $(0 \le i \le n-1)$ with $0 < \sum_{i=0}^{n-1} \alpha_i < 1$ and $\beta > 1$. Let $g_i(u) = u^{\alpha_i}$ $(0 \le i \le n-1)$, $p(t) = 1/t^{\beta}$, and

$$f(t, u_0, \ldots, u_{n-1}) = \prod_{i=0}^{n-1} \frac{1}{t^{\beta}} \left(\frac{u_i}{h_{n-i-1}(t, t_0)} \right)^{\alpha_i}. \tag{3.7}$$

It is easy to verify that $f(t, u_0, \ldots, u_{n-1})$ satisfies the conditions of Theorem 2.9. Thus, it follows that if x is a solution of (3.6) with $\lim_{t\to\infty} x^{\Delta^{n-1}}(t) \neq 0$, then there exist constants a_i $(0 \le i \le n-1)$ with $a_0 \neq 0$, such that $\lim_{t\to\infty} x(t) / \sum_{i=0}^{n-1} a_i h_{n-i-1}(t, t_0) = 1$.

Acknowledgment

This paper was supported by NSFC (no. 10861002) and NSFG (no. 2010GXNSFA013106, no. 2011GXNSFA018135) and IPGGE (no. 105931003060).

References

[1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[3] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York, NY, USA, 2002.

[4] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[5] M. Bohner and S. H. Saker, "Oscillation of second order nonlinear dynamic equations on time scales," *The Rocky Mountain Journal of Mathematics*, vol. 34, no. 4, pp. 1239–1254, 2004.

[6] L. Erbe, "Oscillation results for second-order linear equations on a time scale," *Journal of Difference Equations and Applications*, vol. 8, no. 11, pp. 1061–1071, 2002.

[7] T. S. Hassan, "Oscillation criteria for half-linear dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 176–185, 2008.

[8] R. P. Agarwal, M. Bohner, and S. H. Saker, "Oscillation of second order delay dynamic equations," *The Canadian Applied Mathematics Quarterly*, vol. 13, no. 1, pp. 1–17, 2005.

[9] M. Bohner, B. Karpuz, and Ö. Öcalan, "Iterated oscillation criteria for delay dynamic equations of first order," *Advances in Difference Equations*, vol. 2008, Article ID 458687, 12 pages, 2008.
[10] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear delay dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 505–522, 2007.
[11] Z. Han, B. Shi, and S. Sun, "Oscillation criteria for second-order delay dynamic equations on time scales," *Advances in Difference Equations*, vol. 2007, Article ID 70730, 16 pages, 2007.
[12] Z. Han, S. Sun, and B. Shi, "Oscillation criteria for a class of second-order Emden-Fowler delay dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 847–858, 2007.
[13] Y. Şahiner, "Oscillation of second-order delay differential equations on time scales," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 63, no. 5–7, pp. e1073–e1080, 2005.
[14] E. Akin-Bohner, M. Bohner, S. Djebali, and T. Moussaoui, "On the asymptotic integration of nonlinear dynamic equations," *Advances in Difference Equations*, vol. 2008, Article ID 739602, 17 pages, 2008.
[15] T. S. Hassan, "Oscillation of third order nonlinear delay dynamic equations on time scales," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1573–1586, 2009.
[16] S. R. Grace, R. P. Agarwal, B. Kaymakçalan, and W. Sae-jie, "On the oscillation of certain second order nonlinear dynamic equations," *Mathematical and Computer Modelling*, vol. 50, no. 1-2, pp. 273–286, 2009.
[17] T. Sun, H. Xi, and W. Yu, "Asymptotic behaviors of higher order nonlinear dynamic equations on time scales," *Journal of Applied Mathematics and Computing*. In press.
[18] T. Sun, H. Xi, X. Peng, and W. Yu, "Nonoscillatory solutions for higher-order neutral dynamic equations on time scales," *Abstract and Applied Analysis*, vol. 2010, Article ID 428963, 16 pages, 2010.
[19] L. Erbe, A. Peterson, and S. H. Saker, "Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales," *Journal of Computational and Applied Mathematics*, vol. 181, no. 1, pp. 92–102, 2005.
[20] L. Erbe, A. Peterson, and S. H. Saker, "Hille and Nehari type criteria for third-order dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 112–131, 2007.
[21] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation and asymptotic behavior of a third-order nonlinear dynamic equation," *The Canadian Applied Mathematics Quarterly*, vol. 14, no. 2, pp. 129–147, 2006.
[22] B. Karpuz, "Asymptotic behaviour of bounded solutions of a class of higher-order neutral dynamic equations," *Applied Mathematics and Computation*, vol. 215, no. 6, pp. 2174–2183, 2009.
[23] D.-X. Chen, "Oscillation and asymptotic behavior for n th-order nonlinear neutral delay dynamic equations on time scales," *Acta Applicandae Mathematicae*, vol. 109, no. 3, pp. 703–719, 2010.
[24] M. Bohner, "Some oscillation criteria for first order delay dynamic equations," *Far East Journal of Applied Mathematics*, vol. 18, no. 3, pp. 289–304, 2005.