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Research Article

Value Distributions and Uniqueness of Difference Polynomials

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We investigate the zeros distributions of difference polynomials of meromorphic functions, which can be viewed as the Hayman conjecture as introduced by (Hayman 1967) for difference. And we also study the uniqueness of difference polynomials of meromorphic functions sharing a common value, and obtain uniqueness theorems for difference.

1. Introduction

A meromorphic function means meromorphic in the whole complex plane. Given a meromorphic function f, recall that $\alpha \not\equiv 0, \infty$ is a small function with respect to f, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. We use notations $\rho(f)$, $\lambda(1/f)$ to denote the order of growth of f and the exponent of convergence of the poles of f, respectively. We say that meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). We assume that the reader is familiar with standard notations and fundamental results of Nevanlinna Theory [1–3].

As we all know that a finite value a is called the Picard exception value of f, if $f-a$ has no zeros. The Picard theorem shows that a transcendental entire function has at most one Picard exception value, a transcendental meromorphic function has at most two Picard exception values. The Hayman conjecture [4], is that if f is a transcendental meromorphic function and $n \in \mathbb{N}$, then $f^n f'$ takes every finite nonzero value infinitely often. This conjecture has been solved by Hayman [5] for $n \geq 3$, by Mues [6] for $n = 2$, by Bergweiler and Eremenko [7] for $n = 1$. From above, it is showed that the Picard exception value of $f^n f'$ may only be zero. Recently, for an analog of Hayman conjecture for difference, Laine and Yang [8, Theorem 2] proved the following.

Theorem A. *Let f be a transcendental entire function with finite order and c be a nonzero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Remark 1.1. Theorem A implies that the Picard exception value of $f(z)^n f(z+c)$ cannot be nonzero constant. However, Theorem A does not remain valid for meromorphic functions. For example, $f(z) = (e^z - 1)/(e^z + 1)$, $n = 2, 3$, $c = i\pi$. Thus, we get that $f(z)^2 f(z+c) = (e^z - 1)/(e^z + 1)$ never takes the value -1, and $f(z)^3 f(z+c) = ((e^z - 1)/(e^z + 1))^2$ never takes the value 1.

As the improvement of Theorem A to the case of meromorphic functions, we first obtain the following theorem. In the following, we assume that $\alpha(z)$ and $\beta(z)$ are small functions with respect of f, unless otherwise specified.

Theorem 1.2. *Let f be a transcendental meromorphic function with finite order and c be a nonzero complex constant. If $n \geq 6$, then the difference polynomial $f(z)^n f(z+c) - \alpha(z)$ has infinitely many zeros.*

Remark 1.3. The restriction of finite order in Theorem 1.2 cannot be deleted. This can be seen by taking $f(z) = 1/P(z)e^{e^z}$, $e^c = -n$ $(n \geq 6)$, $P(z)$ is a nonconstant polynomial, and $R(z)$ is a nonzero rational function. Then $f(z)$ is of infinite order and has finitely many poles, while

$$f(z)^n f(z+c) - R(z) = \frac{1 - P(z)^n P(z+c) R(z)}{P(z)^n P(z+c)} \tag{1.1}$$

has finitely many zeros. We have given the example when $n = 2, 3$ in Remark 1.1 to show that $f(z)^n f(z+c) - \alpha(z)$ may have finitely many zeros. But we have not succeed in reducing the condition $n \geq 6$ to $n \geq 4$ in Theorem 1.2.

In the following, we will consider the zeros of other difference polynomials. Using the similar method of the proof of Theorem 1.2 below, we also can obtain the following results.

Theorem 1.4. *Let f be a transcendental meromorphic function with finite order and c be a nonzero complex constant. If $n \geq 7$, then the difference polynomial $f(z)^n [f(z+c) - f(z)] - \alpha(z)$ has infinitely many zeros.*

Theorem 1.5. *Let f be a transcendental meromorphic function with finite order and c be a nonzero complex constant. If $n \geq 6$, $m, n \in \mathbb{N}$, then the difference polynomial $f(z)^n (f(z)^m - a) f(z+c) - \alpha(z)$ has infinitely many zeros.*

Remark 1.6. The above two theorems also are not true when f is of infinite order, which can be seen by function $f(z) = e^{e^z}/z$, $e^c = -n$, where $\alpha(z) = 1/z^n(z+c)$ in Theorem 1.4 and $\alpha(z) = -a/z^n(z+c)$ in Theorem 1.5.

Theorem 1.7. *Let f be a transcendental meromorphic function with finite order and c be a nonzero complex constant. If $n \geq 4m + 4$, $m, n \in \mathbb{N}$, then the difference polynomial $f(z)^n + \beta(z)[f(z+c) - f(z)]^m - \alpha(z)$ has infinitely many zeros.*

Corollary 1.8. *There is no transcendental finite order meromorphic solution of the nonlinear difference equation*

$$f(z)^n + H(z)\left[f(z+c) - f(z)\right]^m = R(z), \tag{1.2}$$

where $n \geq 4m + 4$ and $H(z)$, $R(z)$ are rational functions.

Remark 1.9. Some results about the zeros distributions of difference polynomials of entire functions or meromorphic functions with the condition $\lambda(1/f) < \rho(f)$ can be found in [9–12]. Theorem 1.7 is a partial improvement of [11, Theorem 1.1] for f is an entire function and is also an improvement of [13, Theorem 1.1] for the case of $m = 1$.

The uniqueness problem of differential polynomials of meromorphic functions has been considered by many authors, such as Fang and Hua [14], Qiu and Fang [15], Xu and Yi [16], Yang and Hua [17], and Lahiri and Rupa [18]. The uniqueness results for difference polynomials of entire functions was considered in a recent paper [15], which can be stated as follows.

Theorem B (see [19, Theorem 1.1]). *Let f and g be transcendental entire functions with finite order, and c be a nonzero complex constant. If $n \geq 6$, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share z CM, then $f = t_1 g$ for a constant t_1 that satisfies $t_1^{n+1} = 1$.*

Theorem C (see [19, Theorem 1.2]). *Let f and g be transcendental entire functions with finite order, and c be a nonzero complex constant. If $n \geq 6$, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share 1 CM, then $fg = t_2$ or $f = t_3 g$ for some constants t_2 and t_3 that satisfy $t_2^{n+1} = 1$ and $t_3^{n+1} = 1$.*

In this paper, we improve Theorems B and C to meromorphic functions and obtain the following results.

Theorem 1.10. *Let f and g be transcendental meromorphic functions with finite order. Suppose that c is a nonzero constant and $n \in \mathbb{N}$. If $n \geq 14$, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share 1 CM, then $f = tg$ or $fg = t$, where $t^{n+1} = 1$.*

Theorem 1.11. *Under the conditions of Theorem 1.10, if $n \geq 26$, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share 1 IM, then $f = tg$ or $fg = t$, where $t^{n+1} = 1$.*

Remark 1.12. Let $f(z) = (e^z - 1)/(e^z + 1)$ and $g(z) = (e^z + 1)/(e^z - 1)$, $c = i\pi$. Thus, $f(z)^n f(z+c) = ((e^z - 1)/(e^z + 1))^{n-1}$ and $g(z)^n g(z+c) = ((e^z + 1)/(e^z - 1))^{n-1}$ share the value 1 CM. From above, the case $fg = t$, where $t^{n+1} = 1$ may occur in Theorems 1.10 and 1.11.

From the proof of Theorem 1.11 and (2.7) below, we obtain easily the next result.

Corollary 1.13. *Let f and g be transcendental entire functions with finite order, and c be a nonzero complex constant. If $n \geq 12$, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share 1 IM, then $f = tg$ or $fg = t$, where $t^{n+1} = 1$.*

2. Some Lemmas

The difference logarithmic derivative lemma of functions with finite order, given by Chiang and Feng [20, Corollary 2.5], Halburd and Korhonen [21, Theorem 2.1], plays an important part in considering the difference Nevanlinna theory. Here, we state the following version.

Lemma 2.1 (see [22, Theorem 5.6]). *Let f be a transcendental meromorphic function of finite order, and let $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f), \tag{2.1}$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.2 (see [20, Theorem 2.1]). *Let $f(z)$ be a transcendental meromorphic function of finite order. Then,*

$$T(r, f(z+c)) = T(r,f) + S(r,f). \tag{2.2}$$

For the proof of Theorem 1.4, we need the following lemma.

Lemma 2.3. *Let $f(z)$ be a transcendental meromorphic function of finite order. Then,*

$$T(r, f(z)^n[f(z+c) - f(z)]) \geq (n-1)T(r,f) + S(r,f). \tag{2.3}$$

Proof. Assume that $G(z) = f(z)^n[f(z+c) - f(z)]$, then

$$\frac{1}{f(z)^{n+1}} = \frac{1}{G}\frac{f(z+c) - f(z)}{f(z)}. \tag{2.4}$$

Using the first and second main theorems of Nevanlinna theory and Lemma 2.1, we get

$$\begin{aligned}(n+1)T(r,f) &\leq T(r, G(z)) + T\left(r, \frac{f(z+c) - f(z)}{f(z)}\right) + O(1)\\ &\leq T(r, G(z)) + m\left(r, \frac{f(z+c) - f(z)}{f(z)}\right) + N\left(r, \frac{f(z+c) - f(z)}{f(z)}\right) + O(1)\\ &\leq T(r, G(z)) + N\left(r, \frac{f(z+c)}{f(z)}\right) + S(r,f)\\ &\leq T(r, G(z)) + 2T(r,f) + S(r,f),\end{aligned} \tag{2.5}$$

thus, we get the (2.3). □

In order to prove Theorem 1.5 and Corollary 1.13, we also need the next result.

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic function with finite order, $F = f(z)^n(f(z)^m - a)f(z+c)$. Then*

$$T(r, F) \geq (n+m-1)T(r,f) + S(r,f). \tag{2.6}$$

If f is a transcendental entire function with finite order, and $m = 0$, $a \neq 1$, then

$$T(r, f(z)^n f(z+c)) = (n+1)T(r, f) + S(r, f). \tag{2.7}$$

Proof. We deduce from Lemma 2.1 and the standard Valiron-Mohon'ko [23] theorem,

$$\begin{aligned}
(n+m+1)T(r, f) &= T\left(r, f^{n+1}(f^m - a)\right) \\
&\leq m\left(r, f^{n+1}(f^m - a)\right) + N\left(r, f^{n+1}(f^m - a)\right) \\
&\leq m\left(r, F(z)\frac{f(z)}{f(z+c)}\right) + N\left(r, F(z)\frac{f(z)}{f(z+c)}\right) \\
&\leq T(r, F) + m\left(r, \frac{f(z)}{f(z+c)}\right) + N\left(r, \frac{f(z)}{f(z+c)}\right) + S(r, f) \\
&\leq T(r, F) + 2T(r, f) + S(r, f).
\end{aligned} \tag{2.8}$$

Thus, (2.6) follows from (2.8). If f is entire and $m = 0$, $a \neq 1$, then from above, we get

$$T(r, f(z)^n f(z+c)) \geq (n+1)T(r, f) + S(r, f). \tag{2.9}$$

Moreover, $T(r, f(z)^n f(z+c)) \leq (n+1)T(r, f) + S(r, f)$ follows by Lemma 2.2. Thus (2.7) is proved. □

Lemma 2.5 (see [17, Lemma 3]). *Let F and G be two nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

(i) $\max\{T(r, F), T(r, G)\} \leq N_2(r, 1/F) + N_2(r, F) + N_2(r, 1/G) + N_2(r, G) + S(r, F) + S(r, G)$,

(ii) $F = G$,

(iii) $F \cdot G = 1$,

where $N_2(r, 1/F)$ denotes the counting function of zeros of F such that simple zeros are counted once and multiple zeros are counted twice.

For the proof of Theorem 1.11, we need the following lemma.

Lemma 2.6 (see [16, Lemma 2.3]). *Let F and G be two nonconstant meromorphic functions, and F and G share 1 IM. Let*

$$H = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}. \tag{2.10}$$

If $H \not\equiv 0$, then

$$T(r,F)+T(r,G)\le 2\left(N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right)$$
$$+3\left(\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right)+S(r,F)+S(r,G). \tag{2.11}$$

3. Proof of the Theorems

Proof of Theorem 1.2. Since f is a transcendental meromorphic function, assume that $G(z) = f(z)^n f(z+c) - \alpha(z)$, then we can get

$$\begin{aligned} T(r,G(z)) &\ge T\left(r,f(z)^n f(z+c)\right)+S(r,f) \\ &\ge T\left(r,f(z)^n\right)-T\left(r,f(z+c)\right)+S(r,f) \\ &\ge (n-1)T\left(r,f(z)\right)+S(r,f). \end{aligned} \tag{3.1}$$

Using the second main theorem, we have

$$\begin{aligned} (n-1)T(r,f) &\le T(r,G)+S(r,f) \\ &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G+\alpha(z)}\right)+S(r,G) \\ &\le \overline{N}(r,f)+\overline{N}(r,f(z+c))+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f(z+c)}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f) \\ &\le 4T(r,f)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f). \end{aligned} \tag{3.2}$$

So the condition $n \ge 6$ implies that G must have infinitely many zeros. □

Proof of Theorem 1.7. Let

$$\varphi := \frac{\beta(z)\left[f(z+c)-f(z)\right]^m-\alpha(z)}{f(z)^n}. \tag{3.3}$$

We proceed to prove that $\varphi + 1$ has infinitely many zeros, which implies that $f(z)^n + \beta(z)[f(z+c)-f(z)]^m - \alpha(z)$ has infinitely many zeros. We first prove that

$$T(r,\varphi) \ge (n-2m)T(r,f)+S(r,f). \tag{3.4}$$

Applying the first main theorem and Lemma 2.2, we observe that

$$\begin{aligned} T(r, f(z)^n) &= T\left(r, \psi \cdot \frac{1}{\beta(z)\left[f(z+c) - f(z)\right]^m - R(z)}\right) + O(1) \\ &\le T(r, \psi) + T\left(r, \beta(z)\left[f(z+c) - f(z)\right]^m - \alpha(z)\right) + O(1) \\ &\le T(r, \psi) + 2mT(r, f) + S(r, f). \end{aligned} \tag{3.5}$$

From (3.5), we easily obtain the inequality (3.4). Concerning the zeros and poles of ψ, we have

$$\begin{aligned} \overline{N}(r, \psi) &\le \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f) \\ &\le 2T(r, f) + S(r, f), \end{aligned} \tag{3.6}$$

$$\begin{aligned} \overline{N}\left(r, \frac{1}{\psi}\right) &\le \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{\left[f(z+c) - f(z)\right]^m - \alpha(z)/\beta(z)}\right) + S(r, f) \\ &\le T(r, f) + 2mT(r, f) + S(r, f). \end{aligned} \tag{3.7}$$

Using the second main theorem, Lemma 2.2, (3.6) and (3.7), we get

$$\begin{aligned} (n - 2m)T(r, f) &\le T(r, \psi) + S(r, f) \\ &\le \overline{N}(r, \psi) + \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi + 1}\right) + S(r, f) \\ &\le (3 + 2m)T(r, f) + \overline{N}\left(r, \frac{1}{\psi + 1}\right) + S(r, f). \end{aligned} \tag{3.8}$$

Since $n \ge 4m + 4$, then (3.8) implies that $\psi + 1$ has infinitely many zeros, completing the proof. □

Remark 3.1. It is easy to know that if $\alpha(z) \equiv 0$, then (3.7) can be replaced by

$$\overline{N}\left(r, \frac{1}{\psi}\right) \le 3T(r, f) + S(r, f), \tag{3.9}$$

which implies that $n \ge 2m + 6$ in Theorem 1.7.

Proof of Theorem 1.10. Let $F(z) = f(z)^n f(z+c)$ and $G(z) = g(z)^n g(z+c)$. Thus, F and G share the value 1 CM. Suppose first that $F \not\equiv G$ and $F \cdot G \not\equiv 1$. From the beginning of the proof of Theorem 1.2, we obtain

$$\begin{aligned} T(r, F) &\ge (n - 1)T(r, f) + S(r, f), \\ T(r, G) &\ge (n - 1)T(r, g) + S(r, g). \end{aligned} \tag{3.10}$$

Moreover, from Lemma 2.2, it is easy to get

$$\begin{aligned} T(r,G) &\le (n+1)T(r,g)+S(r,g),\\ T(r,F) &\le (n+1)T(r,f)+S(r,f). \end{aligned} \tag{3.11}$$

Using the second main theorem, we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+S(r,F)\\ &\le \overline{N}(r,f)+\overline{N}(r,f(z+c))+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f(z+c)}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\ &\le 4T(r,f)+T(r,G)+S(r,f)\\ &\le 4T(r,f)+(n+1)T(r,g)+S(r,g)+S(r,f). \end{aligned} \tag{3.12}$$

Thus,

$$(n-5)T(r,f) \le (n+1)T(r,g)+S(r,g)+S(r,f). \tag{3.13}$$

Similarly, we obtain

$$(n-5)T(r,g) \le (n+1)T(r,f)+S(r,g)+S(r,f). \tag{3.14}$$

Therefore, from (3.13) and (3.14), $S(r,f)=S(r,g)$ follows. From the definition of F, we get

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &\le 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f(z+c)}\right)+S(r,f)\\ &\le 3T(r,f)+S(r,f). \end{aligned} \tag{3.15}$$

Similarly, we can get

$$N_2\left(r,\frac{1}{G}\right) \le 3T(r,g)+S(r,f), \tag{3.16}$$

$$N_2(r,F) \le 3T(r,f)+S(r,f), \tag{3.17}$$

$$N_2(r,G) \le 3T(r,g)+S(r,g). \tag{3.18}$$

Thus,

$$\begin{aligned} T(r,F)+T(r,G) &\le 2N_2\left(r,\frac{1}{F}\right)+2N_2(r,F)+2N_2\left(r,\frac{1}{G}\right)+2N_2(r,G)+S(r,f)\\ &\le 12(T(r,f)+T(r,g))+S(r,f). \end{aligned} \tag{3.19}$$

Then, from (3.10), and (3.19), we have

$$(n-1)(T(r,f)+T(r,g)) \leq 12(T(r,f)+T(r,g))+S(r,f), \tag{3.20}$$

which is in contradiction with $n \geq 14$.Therefore, applying Lemma 2.5, we must have either $F = G$ or $F \cdot G = 1$. If $F = G$, thus, $f^n f(z+c) = g^n g(z+c)$. Let $H(z) = f(z)/g(z)$. Assume that $H(z)$ is not a constant. Then we get

$$H(z)^n = \frac{1}{H(z+c)}. \tag{3.21}$$

Thus, from Lemma 2.2, we get

$$nT(r,H) = T(r,H(z+c)) + O(1) = T(r,H) + S(r,H), \tag{3.22}$$

which is a contradiction with $n \geq 14$. Hence H must be a constant, which implies that $H^{n+1} = 1$, thus, $f = tg$ and $t^{n+1} = 1$.

If $F \cdot G = 1$, implies that

$$f(z)^n f(z+c) g(z)^n g(z+c) = 1. \tag{3.23}$$

Let $M(z) = f(z)g(z)$, similar as above, $M(z)$ must be a constant. Thus $fg = t$, $t^{n+1} = 1$ follows; we have completed the proof. □

Proof of Theorem 1.11. Let $F(z) = f(z)^n f(z+c)$ and $G(z) = g(z)^n g(z+c)$, let H be defined in Lemma 2.6. Using the similar proof as the proof of Theorem 1.10 up to (3.18), combining with Lemma 2.6 and

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F}\right) &\leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f(z+c)}\right) + S(r,f) \\ &\leq 2T(r,f) + S(r,f), \end{aligned} \tag{3.24}$$

we can get

$$(n-1)(T(r,f)+T(r,g)) \leq 24(T(r,f)+T(r,g))+S(r,f), \tag{3.25}$$

which is in contradiction with $n \geq 26$. Thus, we get $H \equiv 0$. The following is standard. For the convenience of reader, we give a complete proof here. By integratiing (2.10) twice, we have

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \tag{3.26}$$

which implies $T(r,F) = T(r,G) + O(1)$. From (3.10)-(3.11), thus,

$$(n-1)T(r,f) \le (n+1)T(r,g) + S(r,f) + S(r,g), \tag{3.27}$$

$$(n-1)T(r,g) \le (n+1)T(r,f) + S(r,f) + S(r,g). \tag{3.28}$$

In the following, we will prove that $F = G$ or $F \cdot G = 1$.

Case 1 $(b \neq 0, -1)$. If $a - b - 1 \neq 0$, then by (3.26), we get

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{G - (a-b-1)/(b+1)}\right). \tag{3.29}$$

Combining the Nevanlinna second main theorem with Lemma 2.4 and (3.27), we have

$$\begin{aligned}
(n-1)T(r,g) + S(r,g) &\le T(r,G) \\
&\le \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G - (a-b-1)/(b+1)}\right) + S(r,G) \\
&\le \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,G) \\
&\le \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g(z+c)}\right) + \overline{N}(r,g) + \overline{N}(r,g(z+c)) \\
&\quad + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + S(r,g) \\
&\le 4T(r,g) + 2T(r,f) + S(r,g) \\
&\le \left(4 + 2\frac{n-1}{n+1}\right)T(r,g) + S(r,g).
\end{aligned} \tag{3.30}$$

This implies $n^2 - 6n - 3 \le 0$, which is in contradiction with $n \ge 26$. Thus, $a - b - 1 = 0$, hence

$$F = \frac{(b+1)G}{bG+1}. \tag{3.31}$$

Using the same method as above,

$$\begin{aligned}(n-1)T(r,g)+S(r,g) &\leq T(r,G)\\ &\leq \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G+1/b}\right)+S(r,G)\\ &\leq \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+\overline{N}(r,F)+S(r,G)\\ &\leq \left(4+2\frac{n-1}{n+1}\right)T(r,g)+S(r,g),\end{aligned} \tag{3.32}$$

which is also a contradiction.

Case 2 ($b=0$, $a\neq 1$). From (3.26), we have

$$F=\frac{G+a-1}{a}. \tag{3.33}$$

Similarly, we also can get a contradiction. Thus, $a=1$ follows, implies that $F=G$. Thus, we get $f=tg$ and $t^{n+1}=1$.

Case 3 ($b=-1$, $a\neq -1$). From (3.26), we obtain

$$F=\frac{a}{a+1-G}. \tag{3.34}$$

Similarly, we get a contradiction, $a=-1$ follows. Thus, we get $F\cdot G=1$ also implies $fg=t$, $t^{n+1}=1$. Thus, we have completed the proof. □

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