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Research Article

Determining Consecutive Periods of the Lorenz Maps

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Based on symbolic dynamics, the paper provides a satisfactory and necessary condition of existence for consecutive periodic orbits of the Lorenz maps. In addition, a new algorithm with computer assistance based on symbolic dynamics is proposed to find all periodic orbits up to a certain number with little computer time. Examples for consecutive periods of orbits are raised for the Lorenz maps. With a little variation, the theorems and algorithm can be applied to some other dynamic systems.

1. Introduction

The Lorenz system of (1.1) introduced by Lorenz in [1] is one of the chaotic dynamic systems discussed early. It is a deterministic chaos:

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = xy - bz. \tag{1.1}$$

On the Poincaré section, some geometrical structure of the Lorenz flow may be reduced to a one-dimensional Lorenz map (1.2) [2, 3]:

$$f(x, \mu_L, \mu_R) = \begin{cases} f_L(x) = 1 - \mu_L |x|^{\xi} + h.o.t., & x < 0, \\ f_R(x) = -1 + \mu_R(x) |x|^{\xi} + h.o.t., & x > 0, \end{cases} \tag{1.2}$$

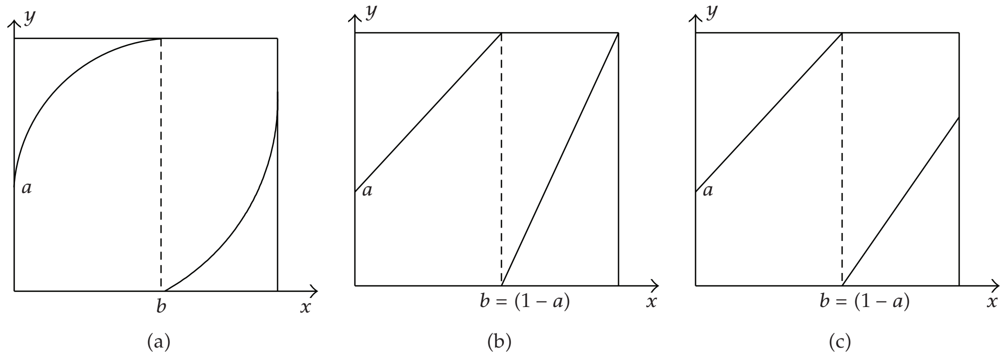

Figure 1: (a) Lorenz map (1.3); (b) Lorenz map (1.4); (c) Lorenz map (1.5).

where ξ is a constant greater than 1. Generally, a Lorenz map with a discontinuity point is as follows (1.3):

$$f(x,b) = \begin{cases} f_L(x), & x < b, \\ f_R(x), & x > b, \end{cases} \tag{1.3}$$

where f is piecewise increasing but undefined at $x = b$, the point $(b, \lim_{x\to b^+} f(x))$ is a discontinuity point and denoted by C, $x \in I = [c,b) \bigcup (b,d]$, and f is a map from $[c,d]$ into $[c,d]$. Furthermore, $\lim_{x\to b^+} f(x)$ and $\lim_{x\to b^-} f(x)$ are denoted by C_+ and C_-, respectively. To simplify this, we suppose that $C_+ = 0$, $C_- = 1$. Thus, $I = [0,b) \bigcup (b,1]$. In this paper, our main discussion is focused on the Lorenz map (1.3). The next two equations (1.4) and (1.5) are among the examples discussed in our paper. Equations (1.4) and (1.5) are two particular cases of (1.3). Figures of (1.3)~(1.5) are shown in Figure 1:

$$S : [0,1] \longrightarrow [0,1] \quad (0 < a < 1), \qquad S(x) = \begin{cases} x + a, & x \in [0, 1-a), \\ \dfrac{(x + a - 1)}{a}, & x \in (1-a, 1], \end{cases} \tag{1.4}$$

$$S : [0,1] \longrightarrow [0,1] \quad (0 < a < 1), \qquad S(x) = \begin{cases} x + a, & x \in [0, 1-a), \\ h(x + a - 1), & x \in (1-a, 1], \end{cases} \tag{1.5}$$

where $1 < h \le 1/a$. The main goal of symbolic dynamics is to determine all of the possible motions of a system under study. In practice, all of the allowed short periodic sequences up to a certain period are very important [3].

In this paper a periodic sequence means its nonrepeating sequence.

In principle, one can enumerate all possible sequences and then check their admissibility. But it is too time consuming and sometimes impossible. In a study on the Lorenz system (1.3), Procaccia et al. in [3] tried to derive some propositions which were intended to make the work easier. By some propositions and yet with much work, he finally generated admissible periodic sequences up to period 6. In practice, by his method, to find out all admissible periodic sequences up to a greater period will be more time consuming and the method is not easy to be applied to other systems.

Symbolic dynamics is a powerful tool in studying the Lorenz maps and sometimes computer-assisted proof is used [4–12]. In [5–7], with computer assistance the authors used symbolic dynamics and obtained some dynamic properties of the Lorenz maps but existence of periodic points was not proved. With computer assistance, Galias and Zgliczyński [8] were able to present that the Lorenz system with "classical" (most popular) parameter values $(\sigma, b, r) = (10, 28, 8/3)$ has infinitely many qualitatively distinct periodic trajectories [8]. But the procedure is still very time consuming and consecutive periods cannot be proved by a computer program itself. And because no symbolic dynamics is used, the method cannot be extended to other systems. The methods used in [9, 10] were complex and very time-consuming because of no computer assistance.

To study chaos of a system we care not only the lengths of periodic orbits but also all the possible periods.

In 1964, Sharkovsky [13] and Štefan [14] proposed a theorem about periods for continuous maps. And the conclusion that period 3 implies chaos [15] is just a particular case in Sarkovskii's theorem. But Sarkovskii's conclusion holds on condition that the map is continuous and cannot easily be applied to discontinuous maps such the Lorenz maps (1.3).

In this paper, new concepts are put forward to reduce the complexity in finding out periodic orbits. By number theorems and symbolic dynamics the Lorenz map (1.3) is discussed and some necessary and satisfactory conditions for the existence of consecutive periods are given. Based on a new algorithm, a program is designed and the time to find out periodic orbits is shortened remarkably.

2. Symbolic Dynamics for the Lorenz Map and Consecutive Periods

2.1. Description for the Lorenz Map with Symbolic Dynamics

In symbolic dynamics, a one-dimensional point is always expressed by a symbolic sequence. Contrary to unimodal continuous map such as the Logistic map, there exists a discontinuity point in the Lorenz map (1.3) which makes dynamic behaviours more complex than those of the unimodal continuous map. In our paper we study the Lorenz systems of (1.3)–(1.5), where the two piecewise functions are increasing. To apply symbolic dynamics, we divide the interval I in (1.3) into two subintervals $I_0 = [0, b)$ and $I_1 = (b, 1]$ and symbols "0" and "1" represent the points in I_0 and I_1, respectively. Starting from any point $x_0 \in I$, by finite iterations we obtain a sequence of 0,1 and C and denote the sequence by $S(x_0) = s_0 s_1 \cdots C$; or by infinite iterations we obtain a sequence of 0 and 1 and denote the sequence by $S(x_0) = s_0 s_1 \cdots$. We denote the sequence beginning with m 0's and then followed by n 1's by $0^m 1^n$.

A kneading pair (K_+, K_-) is the pair of symbolic sequences starting from initial points $(f(C_+), f(C_-))$.

A superstable kneading pair is the kneading pair with C contained.

Furthermore, σ is the shift operator; for example, $\sigma(s_1 s_2 \cdots) = s_2 s_3 \cdots$. If $S(x)$ is a periodic sequence, then $\sigma(S(x))$ is also a periodic sequence.

In symbolic dynamics, an allowed word, or simply word, is a sequence can be obtained by iterations; otherwise, the sequences will be called forbidden words. If a sequence $S(x_1) = s_1 s_2 \cdots$ is an allowed word, then $\sigma(s_1 s_2 \cdots)$ is also an allowed one.

For the Lorenz map (1.3), the ordering rule for any allowed word is very simple. The ordering rule is [3, 4]

$$\Sigma 0 \cdots < \Sigma C < \Sigma 1 \cdots, \tag{2.1}$$

where Σ is the common beginning sequence. Inequality (2.1) is called the ordering rule of the Lorenz map (1.3).

Any sequence $S(x)$ must satisfy the following condition:

$$A(x) \le K_-, \qquad B(x) \ge K_+, \tag{2.2}$$

where $A(x)$ mean the subsequences following 0 in the sequence $S(x)$ while $B(x)$ mean the subsequences following 1 in the sequence $S(x)$. Any kneading sequence (K_+, K_-) itself must also satisfy condition (2.2), too. But if K_+ (or K_-) is superstable, then the inequality sign in inequality (2.2) will change from "$\ge$" to "$>$" and "$\le$" (or "$<$") because a superstable sequence corresponds to only one point but not an interval. For example, if K_- is superstable but K_+ is not, any sequence $S(x)$ satisfies $A(x) < K_-$ and $B(x) \ge K_+$ [3, 4].

Consider the case that $S(x)$ is periodic. Though $x = b$ is not defined, it will do when we define $S(b)$ as 01^∞ (or 10^∞), which is something like $1 = 1.00\cdots = 0.99\cdots$. For a given kneading pair (K_+, K_-), whether it is superstable or not, we will determine all admissible periodic sequences according to ordering rules (2.3) and admissibility conditions (2.4):

$$\Sigma 0 \cdots < \Sigma 1 \cdots, \tag{2.3}$$

$$A(x) < K_-, \qquad B(x) > K_+. \tag{2.4}$$

In this paper we denote the greatest common divisor of two integers a and b by $[a, b]$, while the least common multiple is denoted by (a, b) if not confused with intervals. For simplicity of notation, when we say a periodic sequence we mean its nonrepeating symbols. The length of a word W is denoted by $|W|$.

Theorem 2.1. *Given the kneading pair as*

$$(K_+, K_-) = \left(0^{m_1}1^{n_1}0^{m_2}1^{n_2}\cdots 0^{m_i}1^{n_i}\cdots, 1^{l_1}0^{r_1}1^{l_2}0^{r_2}\cdots 0^{l_j}1^{r_j}\cdots\right), \tag{2.5}$$

then it follows that $\max(l_1, l_2, \dots, n_1, n_2, \dots) = l_1$ *and* $\max(m_1, m_2, \dots, r_1, r_2, \dots) = m_1$.

Proof. By the condition that $A(x) \le K_-$ we hold that $\max(l_1, l_2, \dots, n_1, n_2, \dots) = l_1$, and by the condition that $B(x) \ge K_+$ we hold that $\max(m_1, m_2, \dots, r_1, r_2, \dots) = m_1$; thus, Theorem 2.1 follows. □

2.2. Some Preparations on Number Theory

At first we present a lemma about number theory. The proof is trivial and thus omitted.

Lemma 2.2. *Let a and b be any two positive coprime integers. There exist two nonnegative integers m and n such that $a \cdot m + b \cdot n = c$ holds, where c is any integer not less than $a \cdot b$. In this paper the expression $a \cdot m + b \cdot n$ is called nonnegative linear combination of a and b.*

Remark 2.3. By Lemma 2.2 a set such as $A = \{c, c+1, \ldots\}$ exists, where A is generated by nonnegative linear combinations of a and b. It does not necessarily follow that $a \cdot b = c$ and there is sometimes the case that $a \cdot b > c$. For example, let $a = 2$ and $b = 3$; thus, we can get $A = \{2, 3, \ldots\}$ though $2 \cdot 3 = 6$. If we want to find the least integer c, we have to do a further analysis but it is easy work and does not affect our discussion in this paper.

Remark 2.4. Suppose that a and b are two positive integers such that $[a, b] = d > 1$, then a set $A = \{ab, ab + d, ab + 2d, \ldots\}$ exists, where A is a set with elements generated by nonnegative linear combinations of a and b.

2.3. The Lorenz Maps with Simple Kneading Pairs

A Superstable Kneading pair always means quick and easy conclusions about existence of consecutive periods. If a kneading pair (K_+, K_-) for the Lorenz system (1.3) is superstable, by Theorem 2.1 and Lemma 2.2 we will soon have the following results.

Corollary 2.5. *Given the kneading pair (K_+, K_-) for the Lorenz system* (1.3)*, by the allowed condition* (2.4) *and Lemma 2.2, one has the following.*

(1) *If $(K_+, K_-) = (0^{m_1}1^{m_2}0^{m_3}C, 1^\infty)$, where $m_1 \geq m_3 + 1$, $m_2, m_3 \geq 0$, then $W = 01^k$ ($k = 1, 2, \ldots$) are allowed periodic orbits if $m_1 \geq 2$ and $W = (01^{k+m_2})^\infty$ ($k = 1, 2, \ldots$) are allowed periodic orbits if $m_1 = 1$, which means that consecutive periods exist.*

(2) *If $(K_+, K_-) = (0C, 1C)$, then the system only has periodic orbits such as $[(01)^k]^\infty$ ($k = 1, 2, \ldots$) and $[(10)^k]^\infty$ ($k = 1, 2, \ldots$) and thus no consecutive periods exist.*

(3) *If $(K_+, K_-) = (0C, 1^kC)$, where $k \geq 2$, then $W_1 = (01)^\infty$ and $W_2 = (011)^\infty$ are two periodic sequences, which by Theorem 2.1 means that there exist consecutive periods.*

(4) *If $(K_+, K_-) = (0^mC, 1^nC)$ ($m \geq 2$, $n \geq 2$), or $(K_+, K_-) = (0^mC, 1^\infty)$ ($m \geq 1$), then $W_1 = (01)^\infty$ and $W_2 = (001)^\infty$ are two periodic sequences, which by Theorem 2.1 means that there exist periods with lengths no less than 2.*

Corollary 2.6. *There exist consecutive periods in the Lorenz map* (1.4)*. The set of periods is $A = \{1, m+1, m+2, \ldots\}$, where m is the minimal value of positive integers satisfying $a^m < 1 - a$ or $(m+1)a \geq 1 - a$, which means consecutive periods exist for the system.*

Proof. We discuss the problem in 4 cases as follows.

(1) If $a > 1 - a$, then $a^m > 1 - a$ and $a^{m+1} < 1 - a$ ($m \geq 1$) imply that $a > 1 - a, a^2 > 1 - a, \ldots, a^m > 1 - a$, and $a^{m+1} < 1 - a$ ($m \geq 1$). If $x_0 = 0$, then, by iteration, we have $(x_0, x_1, \ldots, x_m, x_{m+1}, \ldots) = (0, a, (a + a - 1)/a, \ldots, (a^m + a - 1)/a^m, (a^m + a - 1)/a^m + a, \ldots)$. $(K_+, K_-) = (01^m0\cdots, 1^\infty)$ ($m \geq 1$).

(2) If $a > 1 - a$, then $a^m = 1 - a$ ($m \geq 2$) implies that $a > 1 - a$, $a^2 > 1 - a, \ldots, a^{m-1} > 1 - a$, $a^m = 1 - a$, and $a^{m+1} < 1 - a$ ($m \geq 2$). If $x_0 = 0$, then, by iteration, we have $(x_0, x_1, \ldots, x_m, x_{m+1}, \ldots) = (0, a, (a + a - 1)/a, \ldots, (a^{m-2} + a - 1)/a^{m-2}, C)$. $(K_+, K_-) = (01^{m-1}C, 1^\infty)$ ($m \geq 2$).

(3) If $a < 1-a$, then $ma < 1-a$ and $(m+1)a > 1-a$ $(m \geq 1)$ imply that $a < 1-a$, $2a < 1-a, \ldots, ma < 1-a$, $(m+1)a > 1-a$ $(m \geq 1)$. If $x_0 = 0$, then, by iteration, we have $(x_0, x_1, \ldots, x_m, x_{m+1}, \ldots) = (0, a, 2a, \ldots, ma, (ma+a-1)/a, \ldots)$ $(m \geq 1)$. $(K_+, K_-) = (0^m 1 \cdots, 1^\infty)$ $(m \geq 2)$.

(4) If $a < 1-a$, then $ma = 1-a$ implies that $a < 1-a, 2a < 1-a, \ldots, (m-1)a < 1-a$, and $ma = 1-a$ $(m = 2, 3, \ldots)$. If $x_0 = 0$, then, by iteration, we have $(x_0, x_1, \ldots, x_m, x_{m+1}, \ldots) = (0, a, 2a, \ldots, (m-1)a, C)$ $(m \geq 2)$. $(K_+, K_-) = (0^m C, 1^\infty)$ $(m \geq 2)$.

In cases of (1) and (2), by (2.1) and (2.2), $(01^{m+k})^\infty$ $(k = 1, 2, \ldots)$ are admissible periodic orbits; in cases of (3) and (4), by Corollary 2.5 and (2.3) and (2.4), $(01^{m-1}1^k)^\infty$ $(k = 2, 3, \ldots)$ are admissible periodic orbits. In addition, 1^∞ is an admissible periodic orbit in all of the four cases. Thus, Corollary 2.6 is complete. □

Remark 2.7. The conclusions above also hold if the Lorenz system (1.3) is not piecewise linear but just possesses the same kneading pair as that in Corollary 2.6. So the results can be extended to other systems.

2.4. The Lorenz Systems with Complex Kneading Pairs [7–10]

To consider the periods for the Lorenz map (1.3) with kneading pairs more complex than those in Corollaries 2.5 and 2.6, we make the following definitions.

Definition 2.8. Suppose that $(K_+, K_-) = (1^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots, 0^{r_1}1^{l_1}0^{r_2}1^{l_2}\cdots)$. A string such as 1^m0^r $(m_1 \geq m \geq 1$, $r_1 \geq r \geq 1)$ is called a *basic 1-string* and all of the basic 1-strings form a set denoted by Φ, while $0^r(m_1 1^m$ is called a *basic 0-string*. All of the basic 0-strings form a set denoted by Ψ. Both basic 1-strings and basic 0-strings are called *basic strings*. If two basic strings $W_1, W_2 \in \Phi$ (or $W_1, W_2 \in \Psi$) such that $W_1 \leq W_2$, then the combined string W_1W_2 is called an *increasing string* and is otherwise called a *decreasing string*.

An increasing string or a decreasing string can be extended to the sequences composed of more basic strings.

By conditions (2.2) and (2.4) for any kneading pair $((K_+, K_-))$, K_- is composed of basic 1-strings and K_+ is composed of basic 0-strings.

Definition 2.9. If the Lorenz system (1.3) contains a periodic sequence W and $|W| = P$, by shift map σ we get another periodic sequence of the same period P. Denote the P periodic sequences generated by the shift map on W by $S_\sigma(W)$. Denote the subset of $S_\sigma(W)$ beginning with 1 by $S_\sigma^1(W)$ and the subset of $S_\sigma(W)$ beginning with 0 by $S_\sigma^0(W)$. If a periodic sequence W begins with 1^m0 $(m \geq 1)$, we shift 1^m to the end of W and get another period which is denoted by $\sigma_1(W)$. If a period W begins with 0^m1 $(m \geq 1)$, we shift 0^m to the end of W and get another period which is denoted by $\sigma_0(W)$.

Definition 2.10. Let $A_1, A_2, \ldots, A_{m+1}$ be the beginning $m+1$ $(m \geq 1)$ basic 1-strings for a sequence of the Lorenz system. $A_1A_2\cdots A_m$ is called the *first decreasing string* if $A_1 \geq A_2 \geq \cdots \geq A_m < A_{m+1}$ and is denoted by D_1. Let $B_1, B_2, \ldots, B_{n+1}$ be the beginning $n+1$ $(n \geq 1)$ basic 0-strings for a sequence of the Lorenz map (1.3). $B_1B_2\cdots B_n$ is called the *first increasing string* if $B_1 \leq B_2 \leq \cdots \leq B_n > B_{n+1}$ and is denoted by C_1. Similarly, we can get $D_2, C_2, D_3, C_3, \ldots$.

Example 2.11. To generate a kneading pair for analysis, we let $a = 0.3$ and $h = 3.2$ in the Lorenz map (1.5) and get the kneading pair as follows:

$$\begin{gathered}
K_- = (11101000110101001001010010001101011010110010100\cdots), \\
K_+ = (00010110010011010101101000111001001011001011001\cdots), \\
\Phi = \{(10), (100), (1000), (110), (1100), (11000), (1110), (11100), (111000)\}, \\
\Psi = \{(01), (001), (0001), (011), (0011), (00011), (0111), (00111), (000111)\}, \\
D_1 = \{(11101000)\}
\end{gathered} \tag{2.6}$$

because $(1110) > (1000) < A_3 = (110)$. $C_1 = \{(0001011)\}$ because $(0001) < (011) > B_3 = (001)$. Similarly, we have $D_2 = \{(11010100100)\}$, $D_3 = \{(101001000)\}$, $C_2 = \{(0010011)\}$, and $C_3 = \{(0101011)\}$.

Both the first decreasing strings and the first increasing strings of the kneading pair are very important because by Theorem 2.1 and conditions (2.2) and (2.4) all basic strings are subjected to limitation of the First Decreasing Strings and the First Increasing Strings.

Obviously, by inequalities (2.1)~(2.4) we hold that $D_1 \geq D_2 \geq D_3 \geq \cdots$ and $C_1 \leq C_2 \leq C_3 \leq \cdots$. Thus, we have Theorem 2.12 as follows.

Theorem 2.12. *For a Lorenz map, suppose that W_1 and W_2 are the two sequences composed of basic 1-strings and V_1 and V_2 are the two sequences composed of basic 0-strings, where $W_2 \leq W_1 < D_1$ and $C_1 < V_1 \leq V_2$, and $\{\sigma_1(W_1), \sigma_1(W_2)\} = \{V_1, V_2\}$ or $\{\sigma_0(V_1), \sigma_0(V_2)\} = \{W_1, W_2\}$. Then the sequences composed of V_1's and V_2's are periodic sequences of the Lorenz map* (1.3) *and there exist consecutive periods if* $[|V_1|, |V_2|] = 1$.

Corollary 2.13. *A satisfactory and necessary condition for the existence of consecutive periods for the Lorenz map* (1.3) *is that two coprime periods W_1 and W_2 satisfying the conditions in Theorem 2.12 exist for the Lorenz map* (1.3).

3. Finding Out Periodic Sequences Quickly with Computer Assistance

3.1. Designing an Algorithm and Steps

Theorem 2.12 and Corollary 2.13 provide not only a satisfactory and necessary condition for the existence of consecutive periods for the Lorenz map (1.3) but also an algorithm to find consecutive periods. Yet there may be some short periods less than the periods of orbits generated by Theorem 2.12. In practice without an efficient method, to find all of the periodic sequences up to certain period may be very time consuming [3–6, 10, 13]. In this section we provide a method used to design a program to solve the problem quickly.

To avoid accounting the same period more than once, we consider as only one periodic sequence the set of periodic sequences in which the other is just the shift map of another one; that is, we think of $S_\sigma(W)$ as only one periodic sequence.

Table 1: Consecutive periodic sequences for the Lorenz map (1.3).

(K_+, K_-)	Periodic sequences with coprime periods	The least number of consecutive periods
$(0001010011\cdots, 11001010010\cdots)$	$01, 001$	2
$(0^{m_1}10^{m_2}1^{n_2}\cdots, 1^{l_1}0^{r_1}1^{l_2}\cdots)$, $m_1 \geq 3$, $l_1 \geq 2$	$01, 001$	2
$(01^501^601^701^6\cdots, 1^701^701^701^50\cdots)$	$01^601^7, 01^6$	105
$(((01^5)^6((01^6)^201^5\cdots, (1^60)^3(1^50)^21^60\cdots)$	$01^501^6, 01^501^601^6$	247
$((001)^n01001\cdots, 10(100)^m10\cdots)$, $n-m \geq 3$	$10(100)^{m+1}, 10(100)^{m+2}$	$(3m+5)(3m+8)$
$((001)^n01001\cdots, 1010(100)^m10\cdots)$, $n-m \geq 2$	$10(100)^m, 10(100)^{m+1}$	$(3m+2)(3m+5)$

Basic steps for the program are as follows.

Step 1. Let P be the period considered. Generate the kneading pair (K_+, K_-) with length long enough (generally about 3 times of P) for a given Lorenz map. If (K_+, K_-) is a superstable kneading pair, then we substitute C with 01^∞ or 10^∞. Find all of the basic 1-strings.

Step 2. Find out all of the possible periodic sequences with period P composed of the basic 1-strings.

Step 3. Check against the ordering rule (inequality (2.3)) and condition (2.4) and find out all of the true periodic sequences with period P.

Step 4. Find out the set of periodic sequences in which no one is the shift map of any other one.

Step 5. Change P and turn to Step 2 to get periodic sequences with different periods.

Let us call the above program *Program 1*.

To compare with the enumeration algorithm mentioned in the papers in [3–6], we now give the program used in the papers in [3–6] which can be obtained just by replacing Step 2 in Program 1 with Step 2′.

Step 2′. Generate P-dimension data arrays with every element being 0 or 1 and we get 2^P arrays in which some are the shift maps for other ones. Give the order to the arrays. Let us call the program *Program 2*.

By the steps we have a computer program in Matlab 7.0 (see the appendix).

3.2. Results

3.2.1. Examples of Coprime Periods for the Lorenz Map (1.3)

For most Lorenz maps by Theorem 2.12 and Corollary 2.13 we can find the consecutive periods if the Lorenz maps have ones (see Table 1).

The middle column in Table 1 can be easily obtained by the ordering rule (inequality (2.1)) and the concepts of basic strings. By the method of Successive Division we can determine whether two numbers are coprime or not. For the kneading pair $(K_+, K_-) = ((001)^{m+k}01001\cdots, 101010(100)^m10\cdots)$, $m \geq 1$, $k \geq 0$, since the lengths of basic strings of

K_- are $3n_1 + 2n_2$ ($m \le n_1 \le m + k$, $1 \le n \le 3$) and $[3n_1 + 2n_2, 3n_1' + 2n_2'] \ge 2$ ($m \le n_1$, $n_1' \le m + k$, $1 \le n_2$, $n_2' \le 3$), we hold that there exist no consecutive periods for the corresponding Lorenz map but there exist doubled consecutive periods no less than $(3m + 2)(3m + 4)$ by Remark 2.4. In the Lorenz map (1.5), if we set the coefficients as $h = 2.7$, $a = 0.3$ and $h = 1.12$, $a = 0.802$ we get the kneading pairs as $(01^5 01^6 01^7 01^6 \cdots, 1^7 01^7 01^7 01^5 0 \cdots)$ and $((01^5)^6((01)^6)^2 01^5 \cdots, (1^6 0)^3 (1^5 0)^2 1^6 0 \cdots)$, respectively, and the Lyapunov exponents for them are 0.3991 and 0.095, respectively.

3.2.2. *Examples of Periodic Sequences for the Lorenz Systems*

By Program 1 we can find all the periodic sequences up to a certain period without any being lost. Combining Program 1 and Theorems 2.1 and 2.12, we can determine the short periods and whether they have consecutive periods, which is one of the important characteristics of chaos in the sense of Devaney's concept of chaos.

Example 3.1. Still we take (K_+, K_-) mentioned in Section 2.4 as an example:

$$\begin{aligned} K_- &= [11101000110101001001010010001101011010110010100\cdots], \\ K_+ &= [00010110010011010101101000111001001011001011001\cdots]. \end{aligned} \tag{3.1}$$

The set of basic 1-strings is $W = \{10, 100, 1000, 110, 1100, 11000, 1110, 11100, 111000\}$.

When $P = 6$, we get 6 periodic sequences as follows (without considering shift map of the sequences. The same below.):

101010, 100100, 110100, 110110, 110010, 111000.

When $P = 9$, we get 17 periodic sequences as follows:

100101010, 100100100, 110101010, 110100100, 110110100, 110110110, 100010110, 100011010, 110010100, 110010110, 110010010, 110011010, 110001100, 111001010, 111001000, 111001100, 111000110

3.2.3. *Comparison of Different Programs Based on the Two Algorithms*

Example 3.2. Suppose that a kneading pair for the Lorenz map (1.3) is as follows:

$$\begin{aligned} K_- &= [11101000110101001001010010001101011010110010100\cdots], \\ K_+ &= [00010110010011010101101000111001001011001011001\cdots]. \end{aligned} \tag{3.2}$$

By Program 2 we find no periods in no less than 20 within 2 hours of computation time of the computer. Based on Theorem 2.12 and Corollary 2.13, Program 1 can reduce the computation time substantially on the same computer. The results are shown in Table 2.

Table 2: Comparison of different algorithms.

Periods	N	Periods	N	A_1	T_1 (seconds)	A_2	T_2 (seconds)
2	1	9	17	512	5.17	45	0.39
3	2	10	28	1024	9.89	80	0.97
4	2	11	40	2048	24.11	139	2.39
5	4	12	67	4096	75.86	242	7.07
6	6	13	102	8192	274.83	420	18.96
7	6	14	169	16384	979.75	733	59.29
8	12	15	272	32768	3788.41	1273	164.68

```
clear
n1=1000;a=0.3;s=0;k=3.2;b=(a-1)*k;
x(1)=1-a,%to generate F(C)
%x(1)=1,%to generate F(D)
for n=1:n1
        x(n+1)=(x(n)+a).*(x(n)>=0&x(n)<(1-a))+(k.*(x(n))+b).*(x(n)>=(1-a)&x(n)<=1);
        y(n)=log(abs(1.*(x(n)>0&x(n)<1-a)+k.*((x(n)>=1-a)&x(n)<=1)));
    n=n+1;
    hold on;
    plot(n,x(n))
end
LE0=sum([y(3:n1)])/(n1-2)
xx=0:0.001:1;z=(xx+a).*(xx>=0&xx<(1-a))+(k.*xx+b).*(xx>=(1-a)&xx<=1);
plot(z)
w=[x(2:50)]>1-a,S=sum(w);
```

Algorithm 1

In Table 2, the meanings of variables are as follows:

N: number of periodic orbits,

A_i ($i = 1, 2$): number of arrays from which periodic sequences are chosen by enumeration algorithm (Program i(i=2,1)),

T_i ($i = 1, 2$): time spending on finding out periods from 9 to 15 by enumeration algorithm (Program i(i=2,1)) on the same computer.

4. Conclusions

Based on symbolic dynamics and computer assistance, a satisfactory and necessary condition for existence of consecutive periods is studied in the paper. Computer programs and way of designing program are provided to find short periodic sequences. With some variation of the method, the algorithm can be applied to other dynamic systems with different ordering rules or admissibility conditions of symbolic sequences such as the Logistic map and the Metric map.

```
clear
tic
P=11;% P must corresponds to Lines from 34 to 41.
A{1}=[1 0];A{2}=[1 0 0];A{3}=[1 1 0];A{4}=[1 0 0 0];A{5}=[1 1 0 0];
A{6}=[1 1 1 0];A{7}=[1 1 0 0 0];A{8}=[1 1 1 0 0];A{9}=[1 1 1 0 0 0];
n1=9;
for n=1:n1
        NN(n)=size(A{n},2);
end
s1=1;
for i1=1:n1
                     if NN(i1)==P
                            m1{s1}=[A{i1}];s1=1+s1;
                    elseif NN(i1)>P
                            break
                    end
        for i2=1:n1
                if NN(i1)+NN(i2)==P
                        m1{s1}=[A{i1} A{i2}];s1=1+s1;
                elseif NN(i1)+NN(i2)>P
                        break
                end
        for i3=1:n1
                if NN(i1)+NN(i2)+NN(i3)==P
                        m1{s1}=[A{i1} A{i2} A{i3}];s1=1+s1;
                        elseif NN(i1)+NN(i2)+NN(i3)>P
                        break
                end
        for i4=1:n1
                if NN(i1)+NN(i2)+NN(i3)+NN(i4)==P
                        m1{s1}=[A{i1} A{i2} A{i3} A{i4}];s1=1+s1;
                    elseif NN(i1)+NN(i2)+NN(i3)+NN(i4)>P
                        break
                end
        for i5=1:n1
                if NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)==P
                        m1{s1}=[A{i1} A{i2} A{i3} A{i4} A{i5}];s1=1+s1;
                            elseif NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)>P
                        break
                end
        for i6=1:n1
                if NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)+NN(i6)==P
                        m1{s1}=[A{i1} A{i2} A{i3} A{i4} A{i5} A{i6}];s1=1+s1;
                            elseif NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)+NN(i6)>P
                        break
                end
        for i7=1:n1
                if NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)+NN(i6)+NN(i7)==P
                        m1{s1}=[A{i1} A{i2} A{i3} A{i4} A{i5} A{i6} A{i7}];s1=1+s1;
                    elseif NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)+NN(i6)+NN(i7)>P
                        break
                end
        for i8=1:n1
                if NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)+NN(i6)+NN(i7)+NN(i8)==P
                        m1{s1}=[A{i1} A{i2} A{i3} A{i4} A{i5} A{i6} A{i7} A{i8}];s1=1+s1;
                        elseif NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)+NN(i6)+NN(i7)+NN(i8)>P
                        break
                end
        for i9=1:n1
                if NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)+NN(i6)+NN(i7)+NN(i8)+NN(i9)==P
                        m1{s1}=[A{i1} A{i2} A{i3} A{i4} A{i5} A{i6} A{i7} A{i8} A{i9}];s1=1+s1;
                elseif
NN(i1)+NN(i2)+NN(i3)+NN(i4)+NN(i5)+NN(i6)+NN(i7)+NN(i8)+NN(i9)>P
                break
        end
                end
                end
                end
              end
             end
             end
          end
    end
end
for n=1:size(m1,2)
    M1{n}=m1{n};
    M{n}=[M1{n} M1{n} M1{n} M1{n} M1{n} M1{n} M1{n} M1{n}];
end
K-=[1 1 1 0 1 0 0 0 1 1 0 1 0 1 0 0 1 0 0 1 0 1 0 0 1 0 0 0 1 1 0 1 0 1 1 0 1 0 1 1 0 0 1 0 1 0 0];
K+=[0 0 0 1 0 1 1 0 0 1 0 0 1 1 0 1 0 1 0 1 1 0 1 0 0 0 1 1 1 0 0 1 0 0 1 0 1 1 0 0 1 0 1 1 0 0 1];
```

Algorithm 2

```
FD=FD0(1:2*P);FC=FC0(1:2*P);
n1=1;
n3=size(m1,2);n4=2*P;
D{1}=1:n4;
for n=1:n3
    s{n}=0;
end
for i=1:n4-1
    D{i+1}=D{i}+1;
end
for n=1:n3
    for m=1:n4
            N0{n,m}=M{n}(D{m});
            N1{n,m}=M{n}(D{m}(1));
            N2{n,m}=N{n,m}(1);
            F0{n,m}=find(N{n,m}<FD);
            F1{n,m}=find(N{n,m}>FC);
      if (size(F0{n,m},2)~=0&N1{n,m}==0&((N2{n,m}<FD(1))|((N2{n,m}==FD(1))&......
                all(N{n,m}(1:F0{n,m}(1)-1)==FD(1:F0{n,m}(1)-1)==1))))......
|((size(F1{n,m},2)~=0)&N1{n,m}==1&((N2{n,m}>FC(1))|((N2{n,m}==FC(1))&......
all(N{n,m}(1:F1{n,m}(1)-1)==FC(1:F1{n,m}(1)-1)==1))))
k{n,m}=1;s{n}=s{n}+1;
      else
            k{n,m}=0;
      end
    end
if s{n}==n4
     B{n1}=M1{n};
        n1=n1+1;
end
end
```

Algorithm 3

Appendix

Program 1(a)

Generate the kneading pair (K_+, K_-) for a given Lorenz map (1.3). Find all of the basic 1-strings; (see Algorithm 1).

Program 1(b)

Let P be the period. Find out all of the possible periodic sequences with period of P composed of the basic 1-strings; (see Algorithm 2).

Program 1(c)

Check against the ordering rule of (2.1) and the condition of (2.2) and find out all of the true periodic sequences with the period of P; (see Algorithm 3).

```
n6=1;
for n=1:size(B,2)-1
       s{n}=0;
end
for n=1:size(B,2)-1
              BD=[B{n} B{n}];
       for n1=n+1:size(B,2)
 if
(all(BD(2:P+1)==B{n1})==1)|(all(BD(3:P+2)==B{n1})==1)|(all(BD(4:P+3)==B{n1})==1)|(all(B
D(5:P+4)==B{n1})==1)|......
(all(BD(6:P+5)==B{n1})==1)|(all(BD(7:P+6)==B{n1})==1)|(all(BD(8:P+7)==B{n1})==1)|(all(B
D(9:P+8)==B{n1})==1)|......
(all(BD(10:P+9)==B{n1})==1)|(all(BD(11:P+10)==B{n1})==1)%|(all(BD(12:P+11)==B{n1})==
1)|(all(BD(13:P+12)==B{n1})==1)|......
%                          (all(BD(14:P+13)==B{n1})==1)|(all(BD(15:P+14)==B{n1})==1)
              s{n}=s{n}+1; % Lines from 34 to 41 must corresponds to P in Line 3.
               end
       end
       if s{n}==0
              BB{n6}=B{n};n6=n6+1;
       end
end
%celldisp(B);
sizeBB=size(BB,2);BB{sizeBB+1}=B{size(B,2)};
%celldisp(BB)
sizeB=size(B,2),sizeBB=size(BB,2)
toc
```

Algorithm 4

Program 1(d)

Find out the set of periodic sequences in which no one is the shift map of any other one; (see Algorithm 4).

Program 2(a)

Generate P-dimension data arrays with every element being 0 or 1 and we get 2^P arrays in which some are the shift maps for other ones. Give the order to the arrays; (see Algorithm 5).

Program 2(b)

Check against the ordering rule of (2.1) and the condition of (2.2) and find out all of the true periodic sequences with the period of P.

This is similar to Program 1(c).

Program 2(c)

Find out the set of periodic sequences in which no one is the shift map of any other one.

This is similar to Program 1(d).

```
clear
tic
A=[0 1];P=11; n=1;d=15;
for i1=1:2,for i2=1:2,for i3=1:2,for i4=1:2,for i5=1:2,for i6=1:2,for i7=1:2,......
   for i8=1:2,for i9=1:2,for i10=1:2,for i11=1:2,for i12=1:2,for i13=1:2,for i14=1:2,for i15=1:2
m1{n}=[A(i1),A(i2),A(i3),A(i4),A(i5),A(i6),A(i7),A(i8),A(i9),A(i10),A(i11),A(i12),A(i13),A(i14
),A(i15)];
            n=n+1;
   end, end, end, end, end, end, end, end, end, end, end, end, end, end, end
for n=1:2.^P
      M1{n}=m1{n}(d-P+1:d); M{n}=[M1{n} M1{n} M1{n} M1{n} M1{n} M1{n} M1{n}
M1{n}];
End
```

Algorithm 5

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