Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 951764, 9 pages
doi:10.1155/2010/951764

Research Article

A New Approach to q-Bernoulli Numbers and q-Bernoulli Polynomials Related to q-Bernstein Polynomials

Mehmet Açikgöz, Dilek Erdal, and Serkan Araci

Department of Mathematics, Faculty of Science and Arts, University of Gaziantep, 27310 Gaziantep, Turkey

Correspondence should be addressed to Mehmet Açikgöz, acikgoz@gantep.edu.tr

Received 24 November 2010; Accepted 27 December 2010

Academic Editor: Claudio Cuevas

Copyright © 2010 Mehmet Açikgöz et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We present a new generating function related to the q-Bernoulli numbers and q-Bernoulli polynomials. We give a new construction of these numbers and polynomials related to the second-kind Stirling numbers and q-Bernstein polynomials. We also consider the generalized q-Bernoulli polynomials attached to Dirichlet's character χ and have their generating function. We obtain distribution relations for the q-Bernoulli polynomials and have some identities involving q-Bernoulli numbers and polynomials related to the second kind Stirling numbers and q-Bernstein polynomials. Finally, we derive the q-extensions of zeta functions from the Mellin transformation of this generating function which interpolates the q-Bernoulli polynomials at negative integers and is associated with q-Bernstein polynomials.

1. Introduction, Definitions, and Notations

Let $\mathbb{C}$ be the complex number field. We assume that $q \in \mathbb{C}$ with $|q| < 1$ and that the q-number is defined by $[x]_q = (q^x - 1)/(q - 1)$ in this paper.

Many mathematicians have studied q-Bernoulli, q-Euler polynomials, and related topics (see [1–23]). It is known that the Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{for } |t| < 2\pi, \tag{1.1}$$

and that $B_n = B_n(0)$ are called the nth Bernoulli numbers.

The recurrence formula for the classical Bernoulli numbers B_n is as follows,

$$B_0 = 1, \quad (B+1)^n - B_n = 0, \quad \text{if } n > 0 \tag{1.2}$$

(see [1, 3, 23]). The q-extension of the following recurrence formula for the Bernoulli numbers is

$$B_{0,q} = 1, \qquad q(qB+1)^n - B_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.3}$$

with the usual convention of replacing B^n by $B_{n,q}$ (see [5, 7, 14]).

Now, by introducing the following well-known identities

$$[x+y]_q = [x]_q + q^x [y]_q, \qquad [-x]_q = -\frac{1}{q^x}[x]_q, \qquad [xy]_q = [x]_q [y]_{q^x} \tag{1.4}$$

(see [6]).

The generating functions of the second kind Stirling numbers and q-Bernstein polynomials, respectively, can be defined as follows,

$$\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!}, \tag{1.5}$$

$$F_k(x,t;q) = \frac{\left(t[x]_q\right)^k}{k!} e^{t[1-x]_q} = \sum_{n=0}^{\infty} B_{k,n}(x;q)\frac{t^n}{n!}, \quad t \in \mathbb{C},\ k = 0,1,\ldots,n \tag{1.6}$$

(see [2]), where $\lim_{q\to 1} F_k(x,t;q) = F_k(t,x) = ((tx)^k/k!)e^{t(1-x)}$ (see [4]).

Throughout this paper, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of rational integers, the field of rational numbers, the ring p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ such that $|p|_p = p^{-v_p(p)} = 1/p$. If $q \in \mathbb{C}_p$, we normally assume $|q-1|_p < p^{-1/(p-1)}$ or $|1-q|_p < 1$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$ (see [7–19]).

In this study, we present a new generating function related to the q-Bernoulli numbers and q-Bernoulli polynomials and give a new construction of these numbers and polynomials related to the second kind Stirling numbers and q-Bernstein polynomials. We also consider the generalized q-Bernoulli polynomials attached to Dirichlet's character χ and have their generating function. We obtain distribution relations for the q-Bernoulli polynomials and have some identities involving q-Bernoulli numbers and polynomials related to the second kind Stirling numbers and q-Bernstein polynomials. Finally, we derive the q-extensions of zeta functions from the Mellin transformation of this generating function which interpolates the q-Bernoulli polynomials at negative integers and are associated with q-Bernstein polynomials.

2. New Approach to q-Bernoulli Numbers and Polynomials

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. For $q \in \mathbb{C}$ with $|q| < 1$, let us define the q-Bernoulli polynomials $B_{n,q}(x)$ as follows,

$$D_q(t,x) = -t\sum_{y=0}^{\infty} q^y e^{[x+y]t} = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

Note that

$$\lim_{q\to 1} D_q(t,x) = \frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \tag{2.2}$$

where $B_n(x)$ are classical Bernoulli polynomials. In the special case $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the nth q-Bernoulli numbers. That is,

$$D_q(t) = D_q(t,0) = -t\sum_{y=0}^{\infty} q^y e^{[y]t} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \tag{2.3}$$

From (2.1) and (2.3), we note that

$$\begin{aligned} qD_q(t,1) - D_q(t) &= qe^t D_q(qt) - D_q(t) \\ &= q\left(\sum_{l=0}^{\infty}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} q^m B_{m,q}\frac{t^m}{m!}\right) - \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!} \\ &= q\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} q^l B_{l,q}\right)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

From (2.1) and (2.3), we can easily derive the following equation:

$$qD_q(t,1) - D_q(t) = 1. \tag{2.5}$$

Equations (2.4) and (2.5), we see that $B_{0,q} = 1$ and

$$\sum_{l=0}^{n}\binom{n}{l} q^{l+1} B_{l,q} - B_{n,q} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases} \tag{2.6}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}^*$, one has*

$$B_{0,q} = 1, \qquad q(qB+1)^n - B_{n,q} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases} \tag{2.7}$$

with the usual convention of replacing B^i and $B_{i,q}$.

From (2.1), *one notes that*

$$\begin{aligned} D_q(t,x) &= e^{[x]_q t} D_q(q^x t) \\ &= \left(\sum_{n=0}^{\infty} [x]_q^n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} q^{nx} B_{n,q} \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} q^{lx} B_{l,q} [x]_q^{n-l}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.8}$$

Therefore, one obtains the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}^*$, one has*

$$B_{n,q}(x) = \sum_{l=0}^{n}\binom{n}{l} q^{lx} B_{l,q} [x]_q^{n-l}. \tag{2.9}$$

By (2.1), one sees that

$$\begin{aligned} D_q(t,x) &= \sum_{n=0}^{\infty}\left(-t\sum_{m=0}^{\infty} q^m [x+m]_q^n\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l+1}{[l+1]_q}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

By (2.1) and (2.10), one obtains the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}^*$, one has*

$$B_{n,q}(x) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l+1}{[l+1]_q}. \tag{2.11}$$

From (2.11) one can derive that, for $s \in \mathbb{N}$,

$$D_q(t,x) = \sum_{a=0}^{s-1} q^a D_{q^s}\left(t[s]_q, \frac{x+a}{s}\right). \tag{2.12}$$

By (2.12), one sees that, for $s \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left([s]_q^n \sum_{a=0}^{s-1} q^a B_{n,q^s}\left(\frac{x+a}{s}\right)\right)\frac{t^n}{n!}. \tag{2.13}$$

Therefore, one obtains the following theorem.

Theorem 2.4. *For $s \in \mathbb{N}^*$, one has*

$$B_{n,q}(x) = [s]_q^n \sum_{a=0}^{s-1} q^a B_{n,q^s}\left(\frac{x+a}{s}\right). \tag{2.14}$$

In (2.9), substitute $1-x$ instead of x, one obtains

$$\begin{aligned}
B_{n,q}(1-x) &= \sum_{v=0}^{n}\binom{n}{v} B_{v,q} q^{v(1-x)} [1-x]_q^{n-v} \\
&= \sum_{v=0}^{n}\binom{n}{v}[x]_q^v[1-x]_q^{n-v} B_{v,q}\cdot q^{v(1-x)}[\mathrm{x}]_q^{-v} \\
&= \sum_{m=0}^{\infty}\sum_{v=0}^{n} B_{v,n}(x;q)\binom{v+m-1}{m} q^v (1-q)^m [x]_q^{m-v} B_{v,q},
\end{aligned} \tag{2.15}$$

which is the relation between q-Bernoulli polynomials, q-Bernoulli numbers, and q-Bernstein polynomials. In (1.5), substitute $(x\log q)$ instead of t, one gets

$$[x]_q^k = \frac{k!}{(q-1)^k}\sum_{y=0}^{\infty}\frac{S(y,k)(x\log q)^y}{y!}. \tag{2.16}$$

In (2.16), substitute $m-v$ instead of k, and putting the result in (2.15), one has the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}^*$ and $|q| < 1$, one has*

$$\begin{aligned}
B_{n,q}(x) = \sum_{m,y=0}^{\infty}\sum_{v=0}^{n}\sum_{j=0}^{v}\binom{v+m-1}{m}\binom{v}{j}\frac{(-1)^{m-v+j}(m-v)!q^{v+j}}{y!} \\
\times S(y,m-v) B_{n-v,n}(x;q) B_{v,q}(x\log q)^y,
\end{aligned} \tag{2.17}$$

where $S(k,n)$and $B_{k,n}(x;q)$ are the second kind Stirling numbers and q-Bernstein polynomials, respectively.

Let χ be Dirichlet's character with $s \in \mathbb{N}$. Then, one defines the generalized q-Bernoulli polynomials attached to χ as follows,

$$D_{q,\chi}(t,x) = -t\sum_{d=0}^{\infty}\chi(d)q^d e^{[d+x]_q t} = \sum_{n=0}^{\infty}B_{n,\chi,q}(x)\frac{t^n}{n!}. \tag{2.18}$$

In the special case $x = 0$, $B_{\mathrm{n},\chi,q} = B_{n,\chi,q}(0)$ are called the nth generalized q-Bernoulli numbers attached to χ. Thus, the generating function of the generalized q-Bernoulli numbers attached to χ are as follows,

$$\begin{aligned} D_{q,\chi}(t,x) &= -t\sum_{d=0}^{\infty}\chi(d)q^d e^{[d]_q t} \\ &= \sum_{n=0}^{\infty}B_{n,\chi,q}\frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

By (2.1) and (2.18), one sees that

$$\begin{aligned} D_{q,\chi}(t,x) &= \sum_{a=0}^{s-1}q^a\chi(a)D_{q^s}\left(t[s]_q, \frac{x+a}{s}\right) \\ &= \sum_{n=0}^{\infty}\left([s]_q^n\sum_{a=0}^{s-1}q^a\chi(a)B_{n,q^s}\left(\frac{x+a}{s}\right)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.20}$$

Therefore, one obtains the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}^*$ and $s \in \mathbb{N}$, one has*

$$B_{n,\chi,q}(x) = [s]_q^n\sum_{a=0}^{s-1}q^a\chi(a)B_{n,q^s}\left(\frac{x+a}{s}\right). \tag{2.21}$$

By (2.18) and (2.19), one sees that

$$D_{q,\chi}(t,x) = e^{[x]_q t}D_{q,\chi}(q^x t) = \sum_{n=0}^{\infty}\left(\sum_{d=0}^{n}\binom{n}{d}q^{dx}[x]_q^{n-d}B_{d,\chi,q}\right)\frac{t^n}{n!}. \tag{2.22}$$

Hence,

$$B_{n,\chi,q}(x) = \sum_{d=0}^{n}\binom{n}{d}q^{dx}[x]_q^{n-d}B_{d,\chi,q}. \tag{2.23}$$

For $s \in \mathbb{C}$, one now considers the Mellin transformation for the generating function of $D_q(t,x)$. That is,

$$\frac{1}{\Gamma(s)}\int_0^\infty D_q(-t,x)t^{s-2}dt = \sum_{n=0}^{\infty}\frac{q^n}{[x+n]_q^s}, \tag{2.24}$$

for $s \in \mathbb{C}$, and $x \neq 0,-1,-2,\ldots$.

From (2.24), one defines the zeta type function as follows,

$$\boldsymbol{\zeta}_q^\star(s,x) = \sum_{n=0}^{\infty}\frac{q^n}{[x+n]_q^s}, \quad s \in \mathbb{C},\ x \neq 0,-1,-2,\ldots. \tag{2.25}$$

Note that $\boldsymbol{\zeta}_q^\star(s,x)$ is an analytic function in the whole complex s-plane. Using the Laurent series and the Cauchy residue theorem, one has

$$-n\boldsymbol{\zeta}_q^\star(1-n,x) = B_{n,q}(x), \quad \text{for } n \in \mathbb{N}^*. \tag{2.26}$$

By the same method, one can also obtain the following equations:

$$\frac{1}{\Gamma(s)}\int_0^\infty D_{q,\chi}(-t,x)t^{s-2}dt = \sum_{n=0}^{\infty}\frac{\chi(n)q^n}{[n+x]_q^s}. \tag{2.27}$$

For $s \in \mathbb{C}$,one defines Dirichlet type q-l-function as

$$l_q(s,x \mid \chi) = \sum_{n=0}^{\infty}\frac{\chi(n)q^n}{[n+x]_q^s}, \tag{2.28}$$

where $x \neq 0,-1,-2,\ldots$. Note that $l_q(s,x \mid \chi)$ is also a holomorphic function in the whole complex s-plane. From the Laurent series and the Cauchy residue theorem, one can also derive the following equation:

$$-nl_q(1-n,x \mid \chi) = B_{n,\chi,q}(x). \tag{2.29}$$

In (2.23), substitute $1-x$ instead of x, one obtains

$$\begin{aligned}
B_{n,\chi,q}(1-x) &= \sum_{v=0}^{n}\binom{n}{v}B_{v,\chi,q}q^{v(1-x)}[1-x]_q^{n-v} \\
&= \sum_{v=0}^{n}\binom{n}{v}[x]_q^v[1-x]_q^{n-v}B_{v,\chi,q}\cdot q^{v(1-x)}[x]_q^{-v} \\
&= \sum_{m=0}^{\infty}\sum_{v=0}^{n}B_{v,n}(x;q)\binom{v+m-1}{m}q^v(1-q)^m[x]_q^{m-v}B_{v,\chi,q},
\end{aligned} \tag{2.30}$$

which is the relation between the nth generalized q-Bernoulli numbers and q-Bernoulli polynomials attached to χ and q-Bernstein polynomials. From (2.16), one has the following theorem.

Theorem 2.7. *For $n \in \mathbb{N}^*$ and $|q| < 1$, one has*

$$B_{n,\chi,q}(x) = \sum_{m,y=0}^{\infty} \sum_{v=0}^{n} \sum_{j=0}^{v} \binom{v+m-1}{m} \binom{v}{j} \frac{(-1)^{m-v+j}(m-v)!q^{v+j}}{y!} \times S(y, m-v) B_{n-v,n}(x;q) B_{v,\chi,q}(x \log q)^{y}. \tag{2.31}$$

One now defines particular q-zeta function as follows,

$$H_q(s, a \mid F) = \sum_{m \equiv a (\bmod F)} \frac{q^m}{[m]_q^s}. \tag{2.32}$$

From (2.32), one has

$$H_q(s, a \mid F) = \frac{q^a}{[F]_q^s} \zeta_{q^F}^*\left(s, \frac{a}{F}\right), \tag{2.33}$$

where $\zeta_{q^F}^*(s, a/F)$ is given by (2.25). By (2.26), one has

$$H_q(1-n, a \mid F) = -\frac{q^a [F]_q^{n-1} B_{n,q^F}(a/F)}{n}, \quad n \in \mathbb{N}. \tag{2.34}$$

Therefore, one obtains the following theorem.

Theorem 2.8. *For $n \in \mathbb{N}$, we have*

$$B_{n,q^F}\left(\frac{a}{F}\right) = -\frac{n H_q(1-n, a \mid F)}{q^a [F]_q^{n-1}}. \tag{2.35}$$

References

[1] M. Açıkgöz and S. Aracı, "The relations between Bernoulli, Bernstein and Euler polynomials," in *Proceedings of the 8th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM '10)*, AIP, Rhodes, Greece, March 2010.

[2] M. Açıkgöz and Y. Şimşek, "A new generating function of q-Bernstein-type polynomials and their interpolation function," *Abstract and Applied Analysis*, vol. 2010, Article ID 769095, 12 pages, 2010.

[3] M. Açıkgöz and Y. Şimşek, "On multiple interpolation functions of the Norlund-type Euler polynomials," *Abstract Applied Analysis*, vol. 2009, Article ID 382574, 14 pages, 2009.

[4] M. Açıkgöz and S. Aracı, "On the generating function of Bernstein polynomials," in *Proceedings of the 8th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM '10)*, AIP, Rhodes, Greece, March 2010.

[5] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, pp. 987–1000, 1948.

[6] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York, NY, USA, 2001.
[7] T. Kim, "A new approach to q-zeta function," *Advanced Studies in Contemporary Mathematics*, vol. 11, no. 2, pp. 157–162, 2005.
[8] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
[9] T. Kim, "On p-adic q-L-functions and sums of powers," *Discrete Mathematics*, vol. 252, no. 1–3, pp. 179–187, 2002.
[10] T. Kim, "q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.
[11] T. Kim, "Non-Archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials," *Russian Journal of Mathematical Physics*, vol. 10, no. 1, pp. 91–98, 2003.
[12] T. Kim, "Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 186–196, 2005.
[13] T. Kim, "On the q-extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.
[14] T. Kim, J. Choi, and Y.-H. Kim, "q-Bernstein polynomials associated with q-Stirling numbers and Carlitz's q-Bernoulli numbers," Abstract and Applied Analysis (in press).
[15] T. Kim, L.-C. Jang, and H. Yi, "A note on the modified q-Bernstein polynomials," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 706483, 12 pages, 2010.
[16] T. Kim, "q-Bernstein polynomials, q-Stirling numbers and q-Bernoulli polynomials," http://arxiv.org/abs/1008.4547.
[17] T. Kim, L.-C. Jang, Y. Kim, and J. Choi, "On p-adic analogue of q-Bernstein polynomials and related integrals," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 179430, 9 pages, 2010.
[18] T. Kim, J. Choi, and Y.-H. Kim, "Some identities on the q-Bernstein polynomials, q-Stirling numbers and q-Bernoulli numbers," *Advanced Studies in Contemporary Mathematics*, vol. 20, pp. 335–341, 2010.
[19] T. Kim, "Analytic continuation of multiple q-zeta functions and their values at negative integers," *Russian Journal of Mathematical Physics*, vol. 11, no. 1, pp. 71–76, 2004.
[20] T. Kim, "Note on the Euler q-zeta functions," *Journal of Number Theory*, vol. 129, no. 7, pp. 1798–1804, 2009.
[21] T. Kim, "Barnes-type multiple q-zeta functions and q-Euler polynomials," *Journal of Physics A*, vol. 43, no. 25, Article ID 255201, 11 pages, 2010.
[22] T. Kim, S.-H. Rim, J.-W. Son, and L.-C. Jang, "On the values of q-analogue of zeta and L-functions. I," *Algebra Colloquium*, vol. 9, no. 2, pp. 233–240, 2002.
[23] H. Rademacher, *Topics in Analytic Number Theory*, Springer, New York, NY, USA, 1973.