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Research Article

On the Twisted q-Analogs of the Generalized Euler Numbers and Polynomials of Higher Order

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We consider the twisted q-extensions of the generalized Euler numbers and polynomials attached to χ.

1. Introduction and Preliminaries

Let p be an odd prime number. For $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let $C_{p^n} = \{\zeta \mid \zeta^{p^n} = 1\}$ be the cyclic group of order p^n, and let $T_p = \lim_{n\to\infty} C_{p^n} = \bigcup_{n\ge 0} C_{p^n} = C_{p^\infty}$ be the space of locally constant functions in the p-adic number field $\mathbb{C}_p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|1-q|_p < 1$. In this paper, we use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Let d be a fixed positive odd integer. For $N \in \mathbb{N}$, we set

$$X = X_d = \frac{\lim_{\overleftarrow{N}} \mathbb{Z}}{dp^{\mathbb{N}}\mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p,$$

$$X^{*} = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$
$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a(\bmod\, dp^n)\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^n$; compared to [1–16].

Let χ be the Dirichlet's character with an odd conductor $d \in \mathbb{N}$. Then the generalized ζ-Euler polynomials attached to χ, $E_{n,\chi,\zeta}(x)$, are defined as

$$\begin{aligned} F_{\chi,\zeta}(x,t) &= \frac{2\sum_{l=0}^{d-1}(-1)^l\chi(l)\zeta^l e^{lt}}{\zeta^d e^{dt}+1}e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,\zeta}(x)\frac{t^n}{n!}, \quad \text{for } \zeta \in T_p. \end{aligned} \tag{1.3}$$

In the special case $x = 0$, $E_{n,\chi,\zeta} = E_{n,\chi,\zeta}(0)$ are called the nth ζ-Euler numbers attached to χ. For $f \in UD(\mathbb{Z}_p)$, the p-adic fermionic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-q}\left(x + p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x\frac{q^x}{[p^N]_{-q}}, \quad \text{(see [7–17])}. \end{aligned} \tag{1.4}$$

Let $I_{-1} = \lim_{q\to 1} = I_{-q}(f)$. Then, we see that

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \int_X f(x)d\mu_{-1}(x). \tag{1.5}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then, we have

$$\int_{\mathbb{Z}_p} f(x+n)d\mu_{-1}(x) = (-1)^n\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l). \tag{1.6}$$

Thus, we have

$$I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l), \quad \text{(see [7–17])}. \tag{1.7}$$

By (1.7), we see that

$$\int_X \chi(y)\zeta^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2\sum_{l=0}^{d-1}(-1)^l \chi(l)\zeta^l e^{lt}}{\zeta^d e^{dt}+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,\zeta}(x)\frac{t^n}{n!}. \tag{1.8}$$

From (1.8), we can derive the Witt's formula for $E_{n,\chi,\zeta}(x)$ as follows:

$$\begin{gathered}\int_X \chi(x) x^n \zeta^x d\mu_{-1}(x) = E_{n,\chi,\zeta}, \\ \int_X \chi(y)(y+x)^n \zeta^y d\mu_{-1}(y) = E_{n,\chi,\zeta}(x), \quad \text{for } \zeta \in T_p, \quad (\text{see [5–17]}).\end{gathered} \tag{1.9}$$

The nth generalized ζ-Euler polynomials of order k, $E_{n,\chi,\zeta}^{(k)}$, are defined as

$$\left(\frac{2\sum_{l=0}^{d-1}\zeta^l(-1)^l \chi(l)e^{lt}}{\zeta^d e^{dt}+1} e^{xt}\right)^k = \sum_{n=0}^{\infty} E_{n,\chi,\zeta}^{(k)}(x)\frac{t^n}{n!}. \tag{1.10}$$

In the special case $x = 0$, $E_{n,\chi,\zeta}^{(k)} = E_{n,\chi,\zeta}^{(k)}(0)$ are called the nth ζ-Euler numbers of order k attached to χ.

Now, we consider the multivariate p-adic invariant integral on X as follows:

$$\begin{aligned}&\int_X \cdots \int_X \left(\prod_{i=1}^{k}\chi(x_i)\right) e^{(x_1+\cdots+x_k+x)t} \zeta^{x_1+\cdots+x_k} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\ &\quad = \left(\frac{2\sum_{l=0}^{d-1}(-1)^l \chi(l)e^{lt}}{\zeta^d e^{dt}+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,\zeta}^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{1.11}$$

By (1.10) and (1.11), we see the Witt's formula for $E_{n,\chi,\zeta}^{(k)}(x)$ as follows:

$$\int_X \cdots \int_X \left(\prod_{i=1}^{k}\chi(x_i)\right)(x_1+\cdots+x_k+x)^n \zeta^{x_1+\cdots+x_k} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) = E_{n,\chi,\zeta}^{(k)}(x). \tag{1.12}$$

The purpose of this paper is to present a systemic study of some formulas of the twisted q-extension of the generalized Euler numbers and polynomials of order k attached to χ.

2. On the Twisted q-Extension of the Generalized Euler Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $\zeta \in T_p$. For $d \in \mathbb{N}$ with $2 \nmid d$, let χ be the Dirichlet's character with conductor d. For $h \in \mathbb{Z}, k \in \mathbb{N}$, let us consider the twisted (h,q)-extension of the generalized Euler numbers and polynomials of order k attached to χ.

We firstly consider the twisted q-extension of the generalized Euler polynomials of higher order as follows:

$$\sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}(x)\frac{t^n}{n!} = \int_X e^{[x+y]_q t}\zeta^y \chi(y)\,d\mu_{-1}(y)$$
$$= 2\sum_{m=0}^{\infty}\chi(m)(-1)^m\zeta^m e^{[m+x]_q t}. \tag{2.1}$$

By (2.1), we see that

$$\int_X [x+y]_q^n \chi(y)\zeta^y\,d\mu_{-1}(y) = 2\sum_{m=0}^{\infty}\chi(m)(-\zeta)^m e^{[m+x]_q t}$$
$$= 2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{\zeta^{la}q^{l(a+x)}}{1+q^{ld}\zeta^{ld}}. \tag{2.2}$$

From the multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$, we can derive the twisted q-extension of the generalized Euler polynomials of order k attached to χ as follows:

$$\sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(k)}(x)\frac{t^n}{n!} = \int_X\cdots\int_X\left(\prod_{i=1}^{k}\chi(x_i)\right)e^{[x_1+\cdots+x_k+x]_q t}\zeta^{x_1+\cdots+x_k}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k). \tag{2.3}$$

Thus, we have

$$E_{n,\chi,\zeta,q}^{(k)}(x) = \int_X\cdots\int_X\left(\prod_{i=1}^{k}\chi(x_i)\right)[x_1+\cdots+x_k+x]_q^n\zeta^{x_1+\cdots+x_k}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)$$
$$= \sum_{a_1,\ldots,a_k=0}^{d-1}\left(\prod_{i=1}^{k}\chi(a_i)\right)(-\zeta)^{\sum_{j=1}^{k}a_j}\frac{2^k}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{l(x+\sum_{j=1}^{k}a_j)}}{(1+q^{ld}\zeta^d)}$$
$$= 2^k\sum_{a_1,\ldots,a_k=0}^{d-1}\prod_{i=1}^{k}(\chi(a_i))(-\zeta)^{\sum_{j=1}^{k}a_j}\sum_{m=0}^{\infty}\binom{m+k-1}{m}\times\left(-\zeta^d\right)^m[x+a_1+\cdots+a_k+md]_q^n. \tag{2.4}$$

Let $F_{q,\chi,\zeta}^{(k)}(t,x) = \sum_{n=0}^{\infty}E_{n,\chi,\zeta,q}^{(k)}(x)(t^n/n!)$ be the generating function for $E_{n,\chi,\zeta,q}^{(k)}(x)$. By (2.3), we easily see that

$$F_{q,\chi,}^{(k)}(t,x) = 2^k \sum_{a_1,\dots,a_k=0}^{d-1} \left(\prod_{i=1}^{k}\chi(a_i)\right)(-\zeta)^{\sum_{j=1}^{k} a_j} \sum_{m=0}^{\infty}\binom{m+k-1}{m} \times (-\zeta)^{dm} e^{[x+a_1+\cdots+a_k+md]_q t}$$
$$= 2^k \sum_{n_1,\dots,n_k=0}^{\infty} (-\zeta)^{n_1+\cdots+n_k}\left(\prod_{i=1}^{k}\chi(n_i)\right) e^{[n_1+\cdots+n_k+x]_q t}. \tag{2.5}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $k \in \mathbb{N}, n \geq 0$, one has*

$$E_{n,\chi,\zeta,q}^{(k)}(x) = 2^k \sum_{n_1,\dots,n_k=0}^{\infty} (-\zeta)^{n_1+\cdots+n_k}\left(\prod_{i=1}^{k}\chi(n_i)\right)[n_1+\cdots+n_k+x]_q^n$$
$$= \sum_{a_1,\dots,a_r=0}^{d-1}\left(\prod_{i=1}^{k}\chi(a_i)\right)(-\zeta)^{\sum_{j=1}^{k} a_j} \frac{2^k}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{l(x+\sum_{j=1}^{k} a_j)}}{(1+q^{ld}\zeta^d)^n}. \tag{2.6}$$

Let $h \in \mathbb{Z}, r \in \mathbb{N}$. Then we define the extension of $E_{n,\chi,\zeta,q}^{(r)}(x)$ as follows:

$$\sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(h,r)}(x)\frac{t^n}{n!} = \int_X \cdots \int_X q^{\sum_{j=1}^{r}(h-j)x_j}\left(\prod_{i=1}^{k}\chi(x_i)\right) e^{[x+\sum_{j=1}^{r} x_j]_q t} \times \zeta^{x_1+\cdots+x_r} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{2.7}$$

Then, $E_{n,\chi,\zeta,q}^{(r)}(x)$ are called the nth generalized (h,q)-Euler polynomials of order r attached to χ. In the special case $x=0$, $E_{n,\chi,\zeta,q}^{(r)} = E_{n,\chi,\zeta,q}^{(r)}(0)$ are called the nth generalized (h,r)-Euler numbers of order r. By (1.7), we obtain the Witt's formula for $E_{n,\chi,\zeta,q}^{(r)}(x)$ as follows:

$$E_{n,\chi,\zeta,q}^{(h,r)}(x) = \int_X \cdots \int_X q^{\sum_{j=1}^{r}(h-j)x_j}\left(\prod_{i=1}^{k}\chi(x_i)\right)\left[x+\sum_{j=1}^{r}x_j\right]_q^n \zeta^{x_1+\cdots+x_r} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$
$$= \sum_{a_1,\dots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-\zeta)^{\sum_{j=1}^{r} a_j} q^{\sum_{j=1}^{r} a_j(h-j)} \times \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{l(x+\sum_{j=1}^{r} a_j)}}{(-q^{d(h-r+l)}\zeta^d;q^d)_r}, \tag{2.8}$$

where $(a;q)_r = (1-a)(1-aq)\cdots(1-aq^{r-1})$.

Let $\binom{n}{k}_q = ([n]_q[n-1]_q \cdots [n-k+1]_q)/[k]_q! = [n]_q!/([[k]_q![n-k]_q!)$ where $[k]_q! = [k]_q[k-1]_q \cdots [2]_q[1]_q$. From (2.8), we note that

$$\begin{aligned}
E_{n,\chi,\zeta,q}^{(h,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{i=1}^{r} \chi(a_i) \right) (-\zeta)^{\sum_{j=1}^{r} a_j} q^{\sum_{j=1}^{r}(h-j)a_j} \\
&\quad \times \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^{r} a_j)} \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^d} \left(-\zeta^d\right)^m q^{d(h-r)m} q^{ldm} \\
&= 2^r \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{i=1}^{r-1} \chi(a_i) \right) (-\zeta)^{\sum_{j=1}^{r} a_j} q^{\sum_{j=1}^{r}(h-j)a_j} \\
&\quad \times \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^d} \left(-\zeta^d\right)^m q^{d(h-r)m} \frac{1}{(1-q)^n} \left(1 - q^{d(m+(x+\sum_{j=1}^{k} a_j)/d)}\right)^n \\
&= 2^r [d]_q^n \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^d} \left(-\zeta^d\right)^m q^{d(h-r)m} \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{i=1}^{r-1} \chi(a_i) \right) \\
&\quad \times (-\zeta)^{\sum_{j=1}^{r} a_j} q^{\sum_{j=1}^{r}(h-j)a_j} \left[m + \frac{x + \sum_{j=1}^{d-1} a_j}{d} \right]_{q^d}^n .
\end{aligned} \tag{2.9}$$

Let $F_{q,\chi,\zeta}^{(h,r)}(t,x) = \sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(h,r)}(x)(t^n/n!)$ be the generating function for $E_{n,\chi,\zeta,q}^{(h,r)}(x)$. From (2.8), we can easily derive

$$\begin{aligned}
F_{q,\chi,\zeta}^{(h,r)}(t,x) &= 2^r \sum_{n_1,\ldots,n_r=0}^{\infty} q^{\sum_{j=1}^{r}(h-j)n_j} (-\zeta)^{\sum_{j=1}^{r} n_j} \left(\prod_{j=1}^{r} \chi(n_j) \right) e^{[n_1+\cdots+n_r+x]_q t} \\
&= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q \left(-\zeta^d\right)^m q^{d(h-r)m} \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\prod_{i=1}^{r-1} \chi(a_i) \right) \\
&\quad \times (-\zeta)^{\sum_{j=1}^{r} a_j} q^{\sum_{j=1}^{r}(h-j)a_j} e^{[md+x+\sum_{j=1}^{r} a_j]_q t}.
\end{aligned} \tag{2.10}$$

By (2.10), we obtain the following theorem.

Theorem 2.2. *For $h \in \mathbb{Z}$, $r \in \mathbb{N}$, one has*

$$
\begin{aligned}
E_{n,\chi,\zeta,q}^{(h,r)}(x) &= 2^r \sum_{n_1,\ldots,n_r=0}^{\infty} q^{\sum_{j=1}^{r}(h-j)n_j}(-\zeta)^{\sum_{j=1}^{r} n_j}\left(\prod_{j=1}^{r}\chi(n_j)\right)[n_1+\cdots+n_r+x]_q^n \\
&= 2^r[d]_q^n \sum_{m=0}^{\infty}\binom{m+r-1}{m}_q\left(-\zeta^d\right)^m q^{d(h-r)m}\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r-1}\chi(a_i)\right) \\
&\quad\times(-\zeta)^{\sum_{j=1}^{r}a_j}q^{\sum_{j=1}^{r}(h-j)a_j}\left[m+\frac{x+\sum_{j=1}^{r}a_j}{d}\right]_{q^d}^n \\
&= \sum_{a_1\cdots a_r=0}^{d-1}\left(\prod_{j=1}^{r}\chi(n_j)\right)(-\zeta)^{\sum_{j=1}^{r}a_j}q^{\sum_{j=1}^{r}(h-j)a_j}\times\frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{l(x+\sum_{j=1}^{r}a_j)}}{\left(-q^{d(h-r+l)}\zeta^d;q^d\right)_r}.
\end{aligned}
\tag{2.11}
$$

Let $h=r$. Then we see that

$$
\begin{aligned}
E_{n,\chi,\zeta,q}^{(r,r)}(x) &= \frac{2^r}{(1-q)^n}\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r-1}\chi(a_i)\right)(-\zeta)^{\sum_{i=1}^{r}a_i}q^{\sum_{j=1}^{r}(h-j)a_j}\times\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{l(\sum_{j=1}^{r}a_j+x)}}{\left(-q^{ld}\zeta^d;q^d\right)_r} \\
&= 2^r[d]_q^n\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-\zeta)^m\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right) \\
&\quad\times(-\zeta)^{\sum_{j=1}^{r}a_j}q^{\sum_{j=1}^{r}(r-j)a_j}\left[m+\frac{x+\sum_{j=1}^{r}a_j}{d}\right]_{q^d}^n.
\end{aligned}
\tag{2.12}
$$

It is easy to see that

$$
\begin{aligned}
&\int_X\cdots\int_X\left(\prod_{i=1}^{k}\chi(x_i)\right)q^{\sum_{j=1}^{r}(h-j)x_j+xm}\zeta^{x_1+\cdots+x_r}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&\quad=\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)q^{mx+\sum_{j=1}^{r}(h-j)a_j}(-\zeta)^{\sum_{j=1}^{r}a_j}\times\int_X\cdots\int_X q^{\sum_{j=1}^{r}(m-j)x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&\quad=\frac{2^r q^{mx}\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{j=1}^{r}\chi(a_j)\right)q^{\sum_{j=1}^{r}(m-j)a_j}(-\zeta)^{\sum_{j=1}^{r}a_j}}{\left(-q^{d(m-r)}\zeta^d;q^d\right)_r}.
\end{aligned}
\tag{2.13}
$$

Thus, we have

$$\frac{2^r q^{mx}\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{j=1}^{r}\chi(a_j)\right)q^{\sum_{j=1}^{r}(m-j)a_j}(-\zeta)^{\sum_{j=1}^{r}a_j}}{\left(-q^{d(m-r)}\zeta^d;q^d\right)_r}$$
$$=\int_X\cdots\int_X\left([x+x_1+\cdots+x_r]_q(q-1)+1\right)^m q^{-\sum_{j=1}^{r}jx_j}\zeta^{x_1+\cdots+x_r}$$
$$\times\left(\prod_{j=1}^{r}\chi(x_j)\right)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$
$$=\sum_{l=0}^{m}\binom{m}{l}(q-1)^l\int_X\cdots\int_X\left(\prod_{j=1}^{r}\chi(x_j)\right)$$
$$\times[x+x_1+\cdots+x_r]_q^l q^{-\sum_{j=1}^{r}jx_j}\zeta^{x_1+\cdots+x_r}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$
$$=\sum_{l=0}^{m}\binom{m}{l}(q-1)^l E_{l,\chi,\zeta,q}^{(0,r)}(x). \tag{2.14}$$

By (2.14), we obtain the following theorem.

Theorem 2.3. *For $d,k\in\mathbb{N}$ with $2\nmid d$, one has*

$$\frac{2^r q^{mx}\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{j=1}^{r}\chi(a_j)\right)q^{\sum_{j=1}^{r}(m-j)a_j}(-\zeta)^{\sum_{j=1}^{r}a_j}}{\left(-q^{d(m-r)}\zeta^d;q^d\right)_r}=\sum_{l=0}^{m}\binom{m}{l}(q-1)^l E_{l,\chi,\zeta,q}^{(0,r)}(x). \tag{2.15}$$

By (1.7), we easily see that

$$\int_X f(x+d)d\mu_{-1}(x)+\int_X f(x)d\mu_{-1}(x)=2\sum_{l=0}^{d-1}(-1)^l f(l). \tag{2.16}$$

Thus,we have

$$q^{d(h-1)}\int_X\cdots\int_X[x+d+x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(r-j)x_j}\zeta^{\sum_{j=1}^{r}x_j}$$
$$\times\left(\prod_{j=1}^{r}\chi(x_j)\right)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$
$$=-\int_X\cdots\int_X[x+x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(r-j)x_j}\zeta^{\sum_{j=1}^{r}x_j}\times\left(\prod_{j=1}^{r}\chi(x_j)\right)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$
$$+2\sum_{l=0}^{d-1}\chi(l)(-\zeta)^l\int_X\cdots\int_X[x+l+x_2+\cdots+x_r]_q^n\left(\prod_{j=1}^{r-1}\chi(x_{j+1})\right)$$
$$\times q^{\sum_{j=1}^{r-1}x_{j+1}(h-1-j)}\zeta^{x_2+x_3+\cdots+x_r}d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r). \tag{2.17}$$

By (2.17), we obtain the following theorem.

Theorem 2.4. *For $h \in \mathbb{Z}, d \in \mathbb{N}$ with $2 \nmid d$, one has*

$$q^{d(h-1)} E_{n,\chi,\zeta,q}^{(h,r)}(x+d) + E_{n,\chi,\zeta,q}^{(h,r)}(x) = 2\sum_{l=0}^{d-1} \chi(l)(-1)^l E_{n,\chi,\zeta,q}^{(h-1,r-1)}(x+l). \tag{2.18}$$

It is easy to see that

$$q^x E_{n,\chi,\zeta,q}^{(h+1,r)}(x) = (q-1)E_{n+1,\chi,\zeta,q}^{(h,r)} + E_{n,\chi,\zeta,q}^{(h,r)}(x). \tag{2.19}$$

Let $F_{q,\chi,\zeta}^{(h,1)}(t,x) = \sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(h,1)}(x)(t^n/n!)$. Then we note that

$$F_{q,\chi,\zeta}^{(h,1)}(t,x) = 2\sum_{n=0}^{\infty} \chi(n) q^{(h-1)n}(-\zeta)^n e^{[n+x]_q t}. \tag{2.20}$$

From (2.20), we can derive

$$E_{n,\chi,\zeta,q}^{(h,1)}(x) = 2\sum_{m=0}^{\infty} \chi(m) q^{(h-1)m}(-\zeta)^m [m+x]_q^n = \frac{2}{(1-q)^n}\sum_{a=0}^{d-1}\chi(a)(-\zeta)^a \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{l(x+a)}}{(1+q^{ld}\zeta^d)}. \tag{2.21}$$

3. Further Remark

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let χ be the Dirichlet's character with an odd conductor $d \in \mathbb{N}$. From the Mellin transformation of $F_{q,\chi,\zeta}^{(h,r)}(t,x)$ in (2.10), we note that

$$\frac{1}{\Gamma(s)} \oint F_{q,\chi,\zeta}^{(h,r)}(-t,x) t^{s-1} dt = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{q^{\sum_{j=1}^{r}(h-j)m_j}(-\zeta)^{m_1+\cdots+m_r}\left(\prod_{j=1}^{r}\chi(m_j)\right)}{[m_1+\cdots+m_r+x]_q^s}, \tag{3.1}$$

where $h, s \in \mathbb{C}$, $x \neq 0, -1, -2, \ldots$, and $r \in \mathbb{N}$, $\zeta = e^{2\pi i/d}$. By (3.1), we can define the Dirichlet's type multiple (h,q)-l-function as follows.

Definition 3.1. For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $x \neq 0, -1, -2, \ldots$, one defines the Dirichlet's type multiple (h,q)-l-function related to higher order (h,q)-Euler polynomials as

$$l_q^{(h,r)}(s, x \mid \chi) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{q^{\sum_{j=1}^{r}(h-j)m_j}(-\zeta)^{m_1+\cdots+m_r}\left(\prod_{i=1}^{r}\chi(m_i)\right)}{[m_1+\cdots+m_r+x]_q^s}, \tag{3.2}$$

where $s, h \in \mathbb{C}$, $x \neq 0, -1, -2, \cdots$, $r \in \mathbb{N}$, and $\zeta = e^{2\pi i/d}$.

Note that $l_q^{(h,r)}(s, x \mid \chi)$ is analytic continuation in whole complex s-plane. In (2.10), we note that

$$F_{q,\chi,\zeta}^{(h,r)}(t, x) = 2^r \sum_{n_1,\ldots,n_r=0}^{\infty} q^{\sum_{j=1}^{r}(h-j)n_j} (-\zeta)^{n_1+\cdots+n_r} \left(\prod_{j=1}^{r} \chi(n_j)\right) e^{[n_1+\cdots+n_r+x]_q t}$$
$$= \sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(h,r)}(x) \frac{t^n}{n!}. \tag{3.3}$$

By Laurent series and Cauchy residue theorem in (3.1) and (3.3), we obtain the following theorem.

Theorem 3.2. *Let χ be Dirichlet's character with odd conductor $d \in \mathbb{N}$, and let $\zeta = e^{2\pi i/d}$. For $h, s \in \mathbb{C}$, $x \neq 0, -1, -2, \ldots, r \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, one has*

$$l_q^{(h,r)}(-h, x \mid \chi) = E_{n,\chi,\zeta,q}^{(h,r)}(x). \tag{3.4}$$

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