Research Article

Strictly Increasing Solutions of Nonautonomous Difference Equations Arising in Hydrodynamics

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The paper provides conditions sufficient for the existence of strictly increasing solutions of the second-order nonautonomous difference equation

\[ x(n+1) = x(n) + \left( \frac{n}{n+1} \right)^2 \left( x(n) - x(n-1) + h^2 f(x(n)) \right), \quad n \in \mathbb{N}, \]

where \( f \) is supposed to fulfil

\[ \begin{align*}
L_0 &< 0 < L, \quad f \in \text{Lip}_\text{loc}(L_0, \infty), \quad f(L_0) = f(0) = f(L) = 0, \\
x f(x) &< 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \quad f(x) \geq 0 \text{ for } x \in (L, \infty), \\
\exists \overline{B} \in (L_0, 0) \text{ such that } \int_{\overline{B}}^{L} f(z)dz = 0.
\end{align*} \]

The problem is motivated by some models arising in hydrodynamics.

1. Formulation of Problem

We will investigate the following second-order non-autonomous difference equation

\[ x(n+1) = x(n) + \left( \frac{n}{n+1} \right)^2 \left( x(n) - x(n-1) + h^2 f(x(n)) \right), \quad n \in \mathbb{N}, \] (1.1)

where \( f \) is supposed to fulfil

\[ \begin{align*}
L_0 &< 0 < L, \quad f \in \text{Lip}_\text{loc}(L_0, \infty), \quad f(L_0) = f(0) = f(L) = 0, \\
x f(x) &< 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \quad f(x) \geq 0 \text{ for } x \in (L, \infty), \\
\exists \overline{B} \in (L_0, 0) \text{ such that } \int_{\overline{B}}^{L} f(z)dz = 0.
\end{align*} \] (1.2)
Let us note that $f \in \text{Lip}_{\text{loc}}[L_0, \infty)$ means that for each $[L_0, A] \subset [L_0, \infty)$ there exists $K_A > 0$ such that $|f(x) - f(y)| \leq K_A |x - y|$ for all $x, y \in [L_0, A]$. A simple example of a function $f$ satisfying (1.2)–(1.4) is $f(x) = c(x - L_0)x(x - L)$, where $c$ is a positive constant.

A sequence $\{x(n)\}_{n=0}^\infty$ which satisfies (1.1) is called a solution of (1.1). For each values $B, B_1 \in [L_0, \infty)$ there exists a unique solution $\{x(n)\}_{n=0}^\infty$ of (1.1) satisfying the initial conditions

$$x(0) = B, \quad x(1) = B_1.$$  

Then $\{x(n)\}_{n=0}^\infty$ is called a solution of problem (1.1), (1.5).

In [1] we have shown that (1.1) is a discretization of differential equations which generalize some models arising in hydrodynamics or in the nonlinear field theory; see [2–6]. Increasing solutions of (1.1), (1.5) with $B = B_1 \in (L_0, 0)$ has a fundamental role in these models. Therefore, in [1], we have described the set of all solutions of problem (1.1), (1.6), where

$$x(0) = B, \quad x(1) = B, \quad B \in (L_0, 0).$$  

In this paper, using [1], we will prove that for each sufficiently small $h > 0$ there exists at least one $B \in (L_0, 0)$ such that the corresponding solution of problem (1.1), (1.6) fulfils

$$x(0) = x(1), \quad \lim_{n \to \infty} x(n) \not= L, \quad \{x(n)\}_{n=1}^\infty \text{ is increasing}.$$  

Note that an autonomous case of (1.1) was studied in [7]. We would like to point out that recently there has been a huge interest in studying the existence of monotonic and nontrivial solutions of nonlinear difference equations. For papers during last three years see, for example, [8–22]. A lot of other interesting references can be found therein.

2. Four Types of Solutions

Here we present some results of [1] which we need in next sections. In particular, we will use the following definitions and lemmas.

**Definition 2.1.** Let $\{x(n)\}_{n=0}^\infty$ be a solution of problem (1.1), (1.6) such that

$$\{x(n)\}_{n=1}^\infty \text{ is increasing}, \quad \lim_{n \to \infty} x(n) = 0.$$  

Then $\{x(n)\}_{n=0}^\infty$ is called a **damped solution**.

**Definition 2.2.** Let $\{x(n)\}_{n=0}^\infty$ be a solution of problem (1.1), (1.6) which fulfils

$$\{x(n)\}_{n=1}^\infty \text{ is increasing}, \quad \lim_{n \to \infty} x(n) = L.$$  

Then $\{x(n)\}_{n=0}^\infty$ is called a **homoclinic solution**.
Definition 2.3. Let \( \{x(n)\}_{n=0}^{\infty} \) be a solution of problem (1.1), (1.6). Assume that there exists \( b \in \mathbb{N} \), such that \( \{x(n)\}_{n=1}^{b+1} \) is increasing and

\[
x(b) \leq L < x(b + 1).
\]

Then \( \{x(n)\}_{n=0}^{\infty} \) is called an escape solution.

Definition 2.4. Let \( \{x(n)\}_{n=0}^{\infty} \) be a solution of problem (1.1), (1.6). Assume that there exists \( b \in \mathbb{N}, \, b > 1 \), such that \( \{x(n)\}_{n=1}^{b} \) is increasing and

\[
0 < x(b) < L, \quad x(b + 1) \leq x(b).
\]

Then \( \{x(n)\}_{n=0}^{\infty} \) is called a non-monotonous solution.

Lemma 2.5 (see [1] (on four types of solutions)). Let \( \{x(n)\}_{n=0}^{\infty} \) be a solution of problem (1.1), (1.6). Then \( \{x(n)\}_{n=0}^{\infty} \) is just one of the following four types:

(I) \( \{x(n)\}_{n=0}^{\infty} \) is an escape solution;

(II) \( \{x(n)\}_{n=0}^{\infty} \) is a homoclinic solution;

(III) \( \{x(n)\}_{n=0}^{\infty} \) is a damped solution;

(IV) \( \{x(n)\}_{n=0}^{\infty} \) is a non-monotonous solution.

Lemma 2.6 (see [1] (estimates of solutions)). Let \( \{x(n)\}_{n=0}^{\infty} \) be a solution of problem (1.1), (1.6). Then there exists a maximal \( b \in \mathbb{N} \cup \{\infty\} \) satisfying

\[
x(n) \in [B, L) \quad \text{for } n = 1, \ldots, b, \, \text{if } b \in \mathbb{N},
\]

\[
x(n) \in [B, L] \quad \text{for } n \in \mathbb{N}, \, \text{if } b = \infty.
\]

Further, if \( b > 1 \), then moreover

\[
\{x(n)\}_{n=1}^{b} \text{ is increasing,}
\]

\[
\Delta x(n) < h \sqrt{(L - 2L_0)M_0 + h^2M_0}
\]

for \( n = 1, \ldots, b - 1 \) if \( b \in \mathbb{N} \), and for \( n \in \mathbb{N} \) if \( b = \infty \), where

\[
M_0 = \max\{|f(x)| : x \in [L_0, L]\}.
\]

In [1] we have proved that the set consisting of damped and non-monotonous solutions of problem (1.1), (1.6) is nonempty for each sufficiently small \( h > 0 \). This is contained in the next lemma.

Lemma 2.7 (see [1] (on the existence of non-monotone or damped solutions)). Let \( B \in (\overline{B}, 0) \), where \( \overline{B} \) is defined by (1.4). There exists \( h_B > 0 \) such that if \( h \in (0, h_B] \), then the corresponding solution \( \{x(n)\}_{n=0}^{\infty} \) of problem (1.1), (1.6) is non-monotonous or damped.
In Section 4 of this paper we prove that also the set of escape solutions of problem (1.1), (1.6) is nonempty for each sufficiently small \( h > 0 \). Note that in our next paper [23] we prove this assertion for the set of homoclinic solutions.

3. Properties of Solutions

Now, we provide other properties of solutions important in the investigation of escape solutions.

**Lemma 3.1.** Let \( \{x(n)\}_{n=0}^{\infty} \) be an escape solution of problem (1.1), (1.6). Then \( \{x(n)\}_{n=1}^{\infty} \) is increasing.

**Proof.** Due to (1.1), \( \{x(n)\}_{n=0}^{\infty} \) fulfills

\[
\Delta x(n) = \left( \frac{n}{n+1} \right)^2 \left( \Delta x(n-1) + h^2 f(x(n)) \right), \quad n \in \mathbb{N}.
\]

(3.1)

According to Definition 2.3 there exists \( b \in \mathbb{N} \), such that \( \{x(n)\}_{n=1}^{b+1} \) is increasing and (2.3) holds. By (1.3) we get \( f(x(b+1)) \geq 0 \). Consequently, by (3.1) and (2.3), \( \Delta x(b+1) \geq (b+1)^2/(b+2)^2 \Delta x(b) > 0 \) and \( f(x(b+2)) \geq 0 \). Similarly \( \Delta x(b+j) \geq (b+j)^2/(b+1+j)^2 \Delta x(b+j-1) \) and

\[
\Delta x(b+j) \geq \left( \frac{b+1}{b+1+j} \right)^2 \Delta x(b), \quad j \in \mathbb{N}.
\]

(3.2)

This yields that \( \{x(n)\}_{n=1}^{\infty} \) is increasing. \( \square \)

**Lemma 3.2.** Assume that \( f(x) = 0 \) for \( x > L \). Choose an arbitrary \( q > 0 \). Let \( B_1, B_2 \in (L_0, 0) \) and let \( \{x(n)\}_{n=0}^{\infty} \) and \( \{y(n)\}_{n=0}^{\infty} \) be a solution of problem (1.1), (1.6) with \( B = B_1 \) and \( B = B_2 \), respectively. Let \( K_L \) be the Lipschitz constant for \( f \) on \( [L_0, L] \). Then

\[
|x(n) - y(n)| \leq |B_1 - B_2| e^{q K_L n},
\]

(3.3)

\[
\left| \frac{\Delta x(n) - \Delta y(n)}{h} \right| \leq |B_1 - B_2| q K_L e^{q K_L n},
\]

(3.4)

where \( n \in \mathbb{N}, n \leq q/h \).

**Proof.** By (3.1) we have

\[
(j+1)^2 \Delta x(j) - j^2 \Delta x(j-1) = h^2 j^2 f(x(j)), \quad j \in \mathbb{N}.
\]

(3.5)

Summing it for \( j = 1, \ldots, k \), we get by (1.6)

\[
\Delta x(k) = h^2 \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(x(j)), \quad k \in \mathbb{N}.
\]

(3.6)
Summing it again for \( k = 1, \ldots, n - 1 \), we get

\[
    x(n) = B_1 + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} f(x(j)), \quad n \in \mathbb{N},
\]

and similarly

\[
    y(n) = B_2 + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} f(y(j)), \quad n \in \mathbb{N}.
\]

From this and by using summation by parts we easily obtain

\[
    |x(n) - y(n)| \leq |B_1 - B_2| + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} |f(x(j)) - f(y(j))| \leq |B_1 - B_2| + (n-1)h^2 K_L \sum_{j=1}^{n-1} |x(j) - y(j)|, \quad n \in \mathbb{N}.
\]

By the discrete analogue of the Gronwall-Bellman inequality (see, e.g., [24, Lemma 4.34]), we get

\[
    |x(n) - y(n)| \leq |B_1 - B_2| e^{(n-1)^2 h^2 K_L} \quad \text{for } n \in \mathbb{N},
\]

which yields (3.3).

By (3.6) and (3.3) we have for \( n \in \mathbb{N}, n \leq \varrho/h, \)

\[
    \left| \frac{\Delta x(n) - \Delta y(n)}{h} \right| \leq h \frac{1}{(n+1)^2} \sum_{j=1}^{n} |f(x(j)) - f(y(j))| \leq h K_L \sum_{j=1}^{n} |x(j) - y(j)| \leq |B_1 - B_2| \varrho K_L e^{\varrho K_L}.
\]

\[\Box\]

**4. Existence of Escape Solutions**

**Lemma 4.1.** Assume that \( C \in (L_0, \overline{B}) \) and \( \{B_k\}_{k=1}^{\infty} \subset (L_0, C) \). Let \( \{x_k(n)\}_{n=0}^{\infty} \) be a solution of problem (1.1), (1.6) with \( B = B_k, k \in \mathbb{N} \). For \( k \in \mathbb{N} \) choose a maximal \( b_k \in \mathbb{N} \) such that \( x_k(n) \in [B_k, L) \) for \( n = 1, \ldots, b_k \) if \( b_k \) is finite, and for \( n \in \mathbb{N} \) if \( b_k = \infty \), and \( \{x_k(n)\}_{n=1}^{b_k} \) is increasing if \( b_k > 1 \). Then there exists \( h^* > 0 \) such that for any \( h \in (0, h^*] \) there exists a unique \( \gamma_k \in \mathbb{N}, \gamma_k < b_k \), such that

\[
    x_k(\gamma_k) \geq C, \quad x_k(\gamma_k - 1) < C.
\]
Moreover, if the sequence \( \{\gamma_k\}_{k=1}^{\infty} \) is unbounded, then there exists \( \ell \in \mathbb{N} \) such that the solution \( \{x_\ell(n)\}_{n=0}^{\infty} \) of problem (1.1), (1.6) with \( B = B_\ell \in (L_0, \overline{B}) \) is an escape solution.

Proof. Choose \( h_0 > 0 \) such that

\[
h_0 \sqrt{(L - 2L_0)M_0 + h_0^2M_0} < |C|. \tag{4.2}
\]

For \( k \in \mathbb{N} \) denote by \( \{x_k(n)\}_{n=0}^{\infty} \) a solution of problem (1.1), (1.6) with \( B = B_k \). The existence of \( b_k \) is guaranteed by Lemma 2.6. By Lemma 2.5, \( \{x_k(n)\}_{n=0}^{\infty} \) is just one of the types (I)-(IV), and if \( h \in (0, h_0] \), then the monotonicity of \( \{x_k(n)\}_{n=0}^{\infty} \) yields a unique \( \gamma_k \in \mathbb{N} \), \( \gamma_k < b_k \), satisfying (4.1).

For \( h \in (0, h_0) \), consider the sequence \( \{\gamma_k\}_{k=1}^{\infty} \) and assume that it is unbounded. Then we have

\[
\lim_{k \to \infty} \gamma_k = \infty \tag{4.3}
\]

(otherwise we take a subsequence.) Assume on the contrary that for any \( k \in \mathbb{N} \), \( \{x_k(n)\}_{n=0}^{\infty} \) is not an escape solution. Choose \( k \in \mathbb{N} \). If \( \{x_k(n)\}_{n=0}^{\infty} \) is damped, then by Definition 2.1, we have \( b_k = \infty \) and

\[
x_k(b_k) := \lim_{k \to \infty} x_k(n) = 0, \quad \Delta x_k(b_k) := \lim_{k \to \infty} \Delta x_k(n) = 0. \tag{4.4}
\]

If \( \{x_k(n)\}_{n=0}^{\infty} \) is homoclinic, then by Definition 2.2, we have \( b_k = \infty \) and

\[
x_k(b_k) := \lim_{k \to \infty} x_k(n) = L, \quad \Delta x_k(b_k) := \lim_{k \to \infty} \Delta x_k(n) = 0. \tag{4.5}
\]

If \( \{x_k(n)\}_{n=0}^{\infty} \) is non-monotonous, then by Definition 2.4, we have \( b_k < \infty \) and

\[
x_k(b_k) \in (0, L), \quad \Delta x_k(b_k) \leq 0. \tag{4.6}
\]

To summarize if \( \{x_k(n)\}_{n=0}^{\infty} \) is not an escape solution, then by (4.4), (4.5), and (4.6), we have

\[
x_k(b_k) \in [0, L], \quad \Delta x_k(b_k) \leq 0. \tag{4.7}
\]

Since \( \Delta x_k(0) = 0 \), there exists \( \overline{\gamma}_k \in \mathbb{N} \) satisfying

\[
\gamma_k \leq \overline{\gamma}_k < b_k, \quad \Delta x_k(\overline{\gamma}_k) = \max\{\Delta x_k(j) : \gamma_k \leq j \leq b_k - 1\}. \tag{4.8}
\]
Consider (3.5) with $x = x_k$. By dividing it by $j^2$, multiplying such obtained equality by $x_k(j + 1) - x_k(j - 1)$ and summing in $j$ from 1 to $n$ we get

$$
(\Delta x_k(n))^2 - h^2 \sum_{j=1}^{n} f(x_k(j))(x_k(j + 1) - x_k(j - 1))
$$

$$
= -\sum_{j=1}^{n} \frac{2j + 1}{j^2} \Delta x_k(j)(x_k(j + 1) - x_k(j - 1)), \quad n \in \mathbb{N}.
$$

(4.9)

Denote

$$
E_k(n + 1) = (\Delta x_k(n))^2 - h^2 \sum_{j=1}^{n} f(x_k(j))(x_k(j + 1) - x_k(j - 1)).
$$

(4.10)

Then we get

$$
E_k(n + 1) = -\sum_{j=1}^{n} \frac{2j + 1}{j^2} \Delta x_k(j)(x_k(j + 1) - x_k(j - 1)), \quad n \in \mathbb{N}.
$$

(4.11)

Let us put $n = y_k - 1$ and $n = b_k - 1$ to (4.11) and subtract. By (4.7) and (4.8) we get

$$
E_k(y_k) - E_k(b_k) = \sum_{j=y_k}^{b_k-1} \frac{2j + 1}{j^2} \Delta x_k(j)(x_k(j + 1) - x_k(j - 1))
$$

$$
\leq 2\frac{2y_k + 1}{y_k^2} \Delta x_k(\bar{y}_k)(L - L_0).
$$

(4.12)

Let us put $n = y_k - 1$ and $n = b_k - 1$ to (4.10) and subtract. We get

$$
E_k(y_k) - E_k(b_k) = (\Delta x_k(y_k - 1))^2 - (\Delta x_k(b_k - 1))^2
$$

$$
+ 2h^2 \sum_{j=y_k}^{b_k-1} f(x_k(j)) \frac{x_k(j + 1) - x_k(j - 1)}{2}.
$$

(4.13)

Choose $\varepsilon > 0$ and $h_1 > 0$ such that

$$
\varepsilon < \frac{1}{2} \int_{C} f(z) \, dz, \quad h_1 M_0 < \sqrt{\varepsilon}.
$$

(4.14)

Let $b_k < \infty$. Then (4.6) holds. Since $\Delta x_k(b_k - 1) > 0$, $f(x_k(b_k)) < 0$ and $\Delta x_k(b_k) \leq 0$, (3.1) yields

$$
\left( \frac{b_k + 1}{b_k} \right)^2 |\Delta x_k(b_k)| + \Delta x_k(b_k - 1) = h^2 |f(x_k(b_k))|,
$$

(4.15)
and hence

\[ 0 < \Delta x_k(b_k - 1) \leq -h^2 f(x_k(b_k)) < h^2 M_0 < h\sqrt{\varepsilon} \quad \text{for} \quad h \in (0,h_1]. \tag{4.16} \]

Clearly, if \( b_k = \infty \), then by (4.4) and (4.5), inequality (4.16) holds, as well. By (1.2), \( f \) is integrable on \([L_0,L]\). So, having in mind (4.1), we can find \( \delta > 0 \) such that if

\[ \frac{x_k(j + 1) - x_k(j - 1)}{2} < \delta, \quad j = y_k, \ldots, b_k - 1, \tag{4.17} \]

then

\[ \left| \sum_{j=y_k}^{b_k-1} f(x_k(j)) \frac{x_k(j + 1) - x_k(j - 1)}{2} - \int_{-1}^{C} f(z)dz \right| < \varepsilon. \tag{4.18} \]

Therefore, due to (1.3) and (4.7),

\[ \sum_{j=y_k}^{b_k-1} f(x_k(j)) \frac{x_k(j + 1) - x_k(j - 1)}{2} > \int_{C}^{L} f(z)dz - \varepsilon \geq \int_{C}^{L} f(z)dz - \varepsilon. \tag{4.19} \]

Let \( h_2 > 0 \) be such that

\[ h_2 \left( \sqrt{(L - 2L_0)M_0 + h_2M_0} \right) < \delta. \tag{4.20} \]

If \( h \in (0,h_2] \), then (2.7) implies (4.17) and hence (4.19) holds.

Now, let us put \( h_3 = \min \{ h_0, h_1, h_2 \} \) and choose \( h \in (0,h^*] \). Then, (4.2), (4.14), (4.20), and (4.13)–(4.19) yield

\[ E_k(y_k) - E_k(b_k) > -h^2 \varepsilon + 2h^2 \left( \int_{C}^{L} f(z)dz - \varepsilon \right) \]

\[ = 2h^2 \left( \int_{C}^{L} f(z)dz - \frac{3}{2} \varepsilon \right) > h^2 \varepsilon > 0. \tag{4.21} \]

Finally, (4.12) and (4.21) imply

\[ 0 < h^2 \varepsilon < E_k(y_k) - E_k(b_k) \leq \frac{2\gamma_k + 1}{\gamma_k^2} \Delta x_k(\overline{y}_k)(L - L_0), \tag{4.22} \]

\[ \frac{h^2 \varepsilon}{2(L - L_0)} \cdot \frac{\gamma_k^2}{2\gamma_k + 1} < \Delta x_k(\overline{y}_k). \]
Letting $k \to \infty$, we obtain, by (4.3), that $\lim_{k \to \infty} \Delta x_k(\bar{\gamma}_k) = \infty$, contrary to (4.17). Therefore an escape solution $\{x_r(n)\}_{n=0}^\infty$ of problem (1.1), (1.6) with $B = B_{\ell} \in (L_0, \bar{B})$ must exist. 

Now, we are in a position to prove the next main result.

**Theorem 4.2** (On the existence of escape solutions). There exists $h^* > 0$ such that for any $h \in (0, h^*)$ the initial value problem (1.1), (1.6) has an escape solution for some $B \in (L_0, \bar{B})$.

**Proof.** We have the following steps.

**Step 1.** Let us define

$$
\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \leq L, \\ 0 & \text{for } x > L, \end{cases} \tag{4.23}
$$

and consider an auxiliary equation

$$
x(n + 1) = x(n) + \left(\frac{n}{n+1}\right)^2 \left( x(n) - x(n-1) + h^2 \tilde{f}(x(n)) \right), \quad n \in \mathbb{N}. \tag{4.24}
$$

Let $h^* > 0$ be the constant of Lemma 4.1 for problem (4.24), (1.6). Choose $h \in (0, h^*], C \in (L_0, \bar{B})$ and let $K_L$ be the Lipschitz constant for $\tilde{f}$ on $[L_0, \infty)$. Consider a sequence $\{B_k\}_{k=1}^\infty \subset (L_0, C)$ such that $\lim_{k \to \infty} B_k = L_0$. Then, for each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that

$$
|B_{k_m} - L_0| < e^{-m^2K_L}(C - L_0). \tag{4.25}
$$

Let $\bar{x}_0(0) = \bar{x}_0(n) = L_0$ for $n \in \mathbb{N}$. Then the sequence $\{\bar{x}_0(n)\}_{n=0}^\infty$ is the unique solution of problem (4.24), (1.6) with $B = L_0$. Let $\{\bar{x}_k(n)\}_{n=0}^\infty$ be a solution of problem (4.24), (1.6) with $B = B_k$, $k \in \mathbb{N}$, and let $\{y_k\}_{k=1}^\infty$ be the sequence corresponding to $\{\bar{x}_k(n)\}_{n=0}^\infty$ by Lemma 4.1. We prove that $\{y_k\}_{k=1}^\infty$ is unbounded. According to Lemma 3.2, for each $m \in \mathbb{N}$,

$$
|\bar{x}_{k_m}(n) - \bar{x}_0(n)| \leq |B_{k_m} - L_0|e^{m^2K_L}, \quad n \leq \frac{m}{h}. \tag{4.26}
$$

Consequently, (4.25) and (4.26) give

$$
|\bar{x}_{k_m}(n) - \bar{x}_0(n)| \leq C - L_0, \quad n \leq \frac{m}{h}, \tag{4.27}
$$

and hence

$$
\bar{x}_{k_m}(n) \leq C, \quad n \leq \frac{m}{h}. \tag{4.28}
$$

Therefore

$$
y_{k_m}(n) \geq \frac{m}{h}, \quad m \in \mathbb{N}, \tag{4.29}
$$
which yields that \( \{y_k\}_{k=1}^{\infty} \) is unbounded. By Lemma 4.1, the auxiliary initial value problem (4.24), (1.6) has an escape solution for some \( B = B_\ell \in (L_0, \bar{B}) \). Denote this solution by \( \{\tilde{x}_\ell(n)\}_{n=0}^{\infty} \).

**Step 2.** By Definition 2.3, there exists \( b \in \mathbb{N} \) such that

\[
\{\tilde{x}(n)\}_{n=1}^{b+1} \text{ is increasing, } \quad \tilde{x}(b) < \tilde{x}(b+1).
\]  

(4.30)

Now, consider the solution \( \{x_\ell(n)\}_{n=0}^{\infty} \) of our original problem (1.1), (1.6) with \( B = B_\ell \). Due to (4.23), \( x_\ell(n) = \tilde{x}_\ell(n) \) for \( n = 0, 1, \ldots, b+1 \). Using (4.30) and Definition 2.3, we get that \( \{x_\ell(n)\}_{n=0}^{\infty} \) is an escape solution of problem (1.1), (1.6).

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**References**


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