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Research Article

Approximate Controllability of Abstract Discrete-Time Systems

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Approximate controllability for semilinear abstract discrete-time systems is considered. Specifically, we consider the semilinear discrete-time system $x_{k+1} = A_k x_k + f(k, x_k) + B_k u_k$, $k \in \mathbb{N}_0$, where A_k are bounded linear operators acting on a Hilbert space X, B_k are X-valued bounded linear operators defined on a Hilbert space U, and f is a nonlinear function. Assuming appropriate conditions, we will show that the approximate controllability of the associated linear system $x_{k+1} = A_k x_k + B_k u_k$ implies the approximate controllability of the semilinear system.

1. Introduction

In this paper we deal with the controllability problem for semilinear distributed discrete-time control systems. In order to specify the class of systems to be considered, we set X for the state space and U for the control space. We assume that X and U are Hilbert spaces. Moreover, throughout this paper we denote by $A_k : X \to X$ bounded linear operators, $B_k : U \to X$, $k \in \mathbb{N}_0$, bounded linear maps that represent the control action, and $f : \mathbb{N}_0 \times X \to X$ a map such that $f(k, \cdot)$ is continuous for each $k \in \mathbb{N}_0$. Furthermore, A_k, B_k, and f satisfy appropriate conditions which will be specified later. We will study the controllability of control systems described by the equation

$$x_{k+1} = A_k x_k + f(k, x_k) + B_k u_k, \quad k \in \mathbb{N}_0,\ x_0 \in X, \tag{1.1}$$

where $x_k \in X$, $u_k \in U$.

The study of controllability is an important topic in systems theory. In particular, the controllability of systems similar to (1.1) has been the object of several works. We only mention here [1–11] and the references cited therein. Specially, Leiva and Uzcategui [5] have studied the exact controllability of the linear and semilinear system. However, it is well known [12–16] that most of continuous distributed systems that arise in concrete situations are not exactly controllable but only approximately controllable. A similar situation has been established in [10] in relation with the discrete wave equation and in [11] in relation with the discrete heat equation (see [17–22]). As mentioned in this paper, the lack of controllability is related to the fact that the spaces in which the solutions of these systems evolve are infinite dimensional.

For this reason, in this paper we study the approximate controllability of system (1.1). Specifically, we will compare the approximate controllability of system (1.1) with the approximate controllability of linear system

$$x_{k+1} = A_k x_k + B_k u_k, \quad k \in \mathbb{N}_0, \ x_0 \in X, \tag{1.2}$$

where $x_k \in X$ and $u_k \in U$.

Throughout this paper, for Hilbert spaces X, Y, we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators from X into Y, and we abbreviate this notation by $\mathcal{L}(X)$ for $X = Y$. Moreover, for a linear operator S we denote by $\mathcal{R}(S)$ the range space of S.

The following property of Hilbert spaces is essential for our treatment of controllability.

Lemma 1.1. *Let X be a Hilbert space, and let Y_1, Y_2 be closed subspaces of X such that $X = Y_1 + Y_2$. Then there exists a bounded linear projection $P : X \to Y_2$ such that for each $x \in X$, $x_1 = x - Px \in Y_1$ and*

$$\|x_1\| = \min\{\|y\| : y \in Y_1, x = y + z, z \in Y_2\}. \tag{1.3}$$

In the next section we study the controllability of systems of type (1.1) when the state space X is a Hilbert space and, in Section 3, we will apply our results to study the controllability of a typical system.

2. Approximate Controllability

Throughout this section, we assume that X and U are Hilbert spaces endowed with an inner product denoted generically by $\langle \cdot \rangle$. In this case, for $n \in \mathbb{N}$, X^n and U^n are also Hilbert spaces. The inner product in X^n is given by $\langle\langle x, y \rangle\rangle = \sum_{i=0}^{n-1} \langle x_i, y_i \rangle$ for $x = (x_i)_{i=0,\dots,n-1}$, $y = (y_i)_{i=0,\dots,n-1}$, and similarly for U^n.

Let Φ be the evolution operator associated to the linear homogeneous equation

$$x_{k+1} = A_k x_k, \quad k \in \mathbb{N}_0, \ x_0 \in X. \tag{2.1}$$

It is well known [4, 5] that

$$\begin{aligned} \Phi(k, j) &= A_{k-1} \cdots A_j, \quad k > j, \\ \Phi(k, k) &= I. \end{aligned} \tag{2.2}$$

Furthermore, the solution of (1.2) is given by

$$x_n = \Phi(n,0)x_0 + \sum_{k=1}^{n} \Phi(n,k) B_{k-1} u_{k-1}, \quad n \in \mathbb{N}. \tag{2.3}$$

We will abbreviate the notation by writing $x(x_0, u)$ for this solution.

We define the bounded linear operator $S_n : U^n \to X$ by

$$S_n(u) = \sum_{k=1}^{n} \Phi(n,k) B_{k-1} u_{k-1}. \tag{2.4}$$

It is clear that $x_n(0,u) = S_n(u)$.

The system (1.2) is said to be exactly controllable (or simply controllable) on $[0,n]$ if $\mathcal{R}(S_n) = X$.

Definition 2.1. System (1.2) is said to be approximately controllable on $[0,n]$ if the space $\mathcal{R}(S_n)$ is dense in X and approximately controllable in finite time if the space $\bigcup_{n\in\mathbb{N}_0} \mathcal{R}(S_n)$ is dense in X.

If the system (1.2) is approximately controllable on $[0,n]$ and X is a finite-dimensional space, then the system (1.2) is controllable on $[0,n]$.

We introduce the reachability set $\mathcal{R}_0(n,x_0)$ of system (1.2) as the set consisting of the values $x_n(x_0,u)$. Clearly, system (1.2) is approximately controllable on $[0,n]$ if and only if $\mathcal{R}_0(n,x_0)$ is dense in X for every $x_0 \in X$. A weaker property of controllability is established in the following definition.

Definition 2.2. System (1.2) is said to be approximately controllable to the origin on $[0,n]$ if $0 \in \overline{\mathcal{R}_0(n,x_0)}$ for every $x_0 \in X$ and approximately controllable to the origin in finite time if $0 \in \overline{\bigcup_{n\in\mathbb{N}_0} \mathcal{R}_0(n,x_0)}$ for every $x_0 \in X$.

On the other hand, for $x_0 \in X$, (1.1) has a unique solution which satisfies the equation

$$x_n = \Phi(n,0)x_0 + \sum_{k=1}^{n} \Phi(n,k)\left[B_{k-1}u_{k-1} + f(k-1, x_{k-1})\right], \quad n \in \mathbb{N}. \tag{2.5}$$

Proceeding as in Definitions 2.1 and 2.2, we next consider the approximate controllability for system (1.1). Let $x = x(x_0, f, u)$ be the solution of (1.1) with initial condition x_0 and control function u. We introduce the reachability set $\mathcal{R}_f(n,x_0)$ of system (1.1) as the set consisting of the values x_n.

Definition 2.3. System (1.1) is said to be

(a) approximately controllable on $[0,n]$ if $\mathcal{R}_f(n,0)$ is dense in X,

(b) approximately controllable in finite time if $\bigcup_{n\in\mathbb{N}} \mathcal{R}_f(n,0)$ is dense in X,

(c) approximately controllable to the origin on $[0,n]$ if $0 \in \overline{\mathcal{R}_f(n,x_0)}$ for every $x_0 \in X$,

(d) approximately controllable to the origin in finite time if $0 \in \overline{\bigcup_{n\in\mathbb{N}} \mathcal{R}_f(n,x_0)}$ for every $x_0 \in X$.

We next introduce some additional notations. The operators $J^n : X^n \to X^n$ and $J_n : X^n \to X$ are given by

$$J^n(x)_k = \sum_{i=1}^{k} \Phi(k,i)x_{i-1}, \quad k = 1,2,\ldots,n-1; \quad J^n(x)_0 = 0,$$
$$J_n(x) = \sum_{i=1}^{n} \Phi(n,i)x_{i-1}. \tag{2.6}$$

It is clear that J^n and J_n are bounded linear operators. We set $\mathcal{N}(n) = \ker(J_n)$. Moreover, we denote by $\widehat{B}^n : U^n \to X^n$ the operator defined by $\widehat{B}^n(u_k) = (B_k u_k)$.

We denote by X_0^n the space consisting of $x \in X^n$ such that $x_0 = 0$.

Next we will show that a modification of an argument of Sukavanam [23] can be applied to compare the approximate controllability of systems (1.1) and (1.2).

For fixed $n \in \mathbb{N}$ and $x \in X^n$, we begin by defining the map $F^n : X_0^n \to X^n$ by $F^n(z_k) = (f(k, x_k + z_k))_{k=0,1,\ldots,n-1}$. It is clear that F^n is a continuous map.

On the other hand, under the assumption that

$$\mathcal{N}(n) + \overline{\mathcal{R}\left(\widehat{B}^n\right)} = X^n, \tag{2.7}$$

we denote by P^n the projection constructed as in Lemma 1.1 with $Y_2 = \mathcal{N}(n)$ and $Y_1 = \overline{\mathcal{R}(\widehat{B}^n)}$. We introduce the space

$$Z = \left\{z \in X_0^n : z = J^n(y),\ y \in \mathcal{N}(n)\right\}, \tag{2.8}$$

and we define the map $\Gamma^n : \overline{Z} \to Z$ by

$$\Gamma^n = J^n \circ P^n \circ F^n. \tag{2.9}$$

We next study the existence of fixed points for Γ^n. In the following statement, we denote $\gamma_n = \|J^n \circ P^n\|$.

Lemma 2.4. *Assume that*

$$\|f(k,y) - f(k,w)\| \le L_k\|y - w\|, \quad k \in \mathbb{N}_0, \tag{2.10}$$

for all $y, w \in X$. If $\gamma_n \max_{k=0,\ldots,n-1} L_k < 1$, then Γ^n has a fixed point.

Proof. It is easy to see that Γ^n is a contraction map. In fact, since J^n and P^n are bounded linear maps, we have

$$\begin{aligned}\|\Gamma^n(z_k)-\Gamma^n(w_k)\|^2 &\le \gamma_n^2\|F^n(z_k)-F^n(w_k)\|^2 \le \gamma_n^2\sum_{k=0}^{n-1}\|f(k,z_k)-f(k,w_k)\|^2\\ &\le \gamma_n^2\sum_{k=0}^{n-1}L_k^2\|z_k-w_k\|^2 \le \gamma_n^2\left(\max_{k=0,\dots,n-1}L_k^2\right)\|(z_k)_k-(w_k)_k\|^2,\end{aligned} \tag{2.11}$$

which implies that Γ^n is a contraction. □

In what follows we always assume that f satisfies the Lipschitz condition (2.10).

Under certain conditions we can modify our hypothesis $\mathcal{N}(n)+\overline{\mathcal{R}(\widehat{B}^n)}=X^n$.

Lemma 2.5. *Assume that $\mathcal{R}(\widehat{B}^n)\subseteq(\mathcal{R}(\widehat{B}^n)\cap\mathcal{N}(n))\oplus\mathcal{N}(n)^\perp$ and the space $\mathcal{N}(n)+\overline{\mathcal{R}(\widehat{B}^n)}$ is dense in X^n. Then $\mathcal{N}(n)+\overline{\mathcal{R}(\widehat{B}^n)}=X^n$.*

Proof. Let $x\in X^n$. There exist sequences $(y_m)_m$ in $\mathcal{N}(n)$ and $(u_m)_m$ in U^n such that $\widehat{B}^nu_m+y_m\to x$ as $m\to\infty$. Let $Q:X^n\to X^n$ be the orthogonal projection on $\mathcal{N}(n)$. Therefore, $(I-Q)\widehat{B}^nu_m+Q\widehat{B}^nu_m+y_m\to x$ as $m\to\infty$. Since $(I-Q)\widehat{B}^nu_m\in\mathcal{R}(\widehat{B})^n\cap\mathcal{N}(n)^\perp$ and $Q\widehat{B}^nu_m+y_m\in\mathcal{N}(n)$, we can assert that the sequence $(I-Q)\widehat{B}^nu_m$ converges to some element $y^1\in\overline{\mathcal{R}(\widehat{B}^n)}$ and the sequence $Q\widehat{B}^nu_m+y_m\in\mathcal{N}(n)$ converges to some element $y^2\in\mathcal{N}(n)$. Consequently, $x=y^2+y^1\in\mathcal{N}(n)+\overline{\mathcal{R}(\widehat{B}^n)}$, which completes the proof. □

Related to this result, it is worthwhile to point out that if B_k has a continuous left inverse for each $k\in\mathbb{N}_0$, then the space $\mathcal{R}(\widehat{B}^n)$ is closed. Moreover, if $\ker B_k=\{0\}$ and the range of B_k is a closed subspace, which occurs, for instance, when U is a finite dimensional space, then B_k has a continuous left inverse.

Theorem 2.6. *Assume that $\gamma_n\max_{k=0,\dots,n-1}L_k<1$ and condition (2.7) holds. Then $\mathcal{R}_0(n,x_0)\subseteq\overline{\mathcal{R}_f(n,x_0)}$ for all $x_0\in X$.*

Proof. Let $u=(u_k)_{k=0,1,\dots,n-1}$ be a control vector, and let $x=(x_k)_{k=0,1,\dots,n}$ be the solution of (1.2) with initial condition x_0. In what follows, we apply our construction preceding Lemma 2.4 with the vector $(x_k)_{k=0,1,\dots,n-1}$. Let $z=(z_k)_{k=0,1,\dots,n-1}$ be a fixed point of Γ^n. Clearly $z_0=0$ and $J_n(P^n(F^n(z)))=0$. We set $z_n=0$. We now apply Lemma 1.1 to $F^n(z)$, with respect to spaces $Y_1=\overline{\mathcal{R}(\widehat{B}^n)}$ and $Y_2=\mathcal{N}(n)$. We set $q^n=F^n(z)-P^n(F^n(z))\in\overline{\mathcal{R}(\widehat{B}^n)}$, and we define $y_k=z_k+x_k$, for $k=0,1,\dots,n$. It follows from this construction that $x_n=y_n$, and combining the properties of x and z, we obtain that

$$\begin{aligned}y_k &= \Gamma^n(z)_k+x_k=J^n(P^n\circ F^n(z))_k+x_k\\ &= J^n(F^n(z)-q^n)_k+\Phi(k,0)x_0+J^n\left(\widehat{B}^nu\right)_k\\ &= \Phi(k,0)x_0+\sum_{i=1}^{k}\Phi(k,i)\left[f(i-1,y_{i-1})-q_{i-1}^n+B_{i-1}u_{i-1}\right]\end{aligned} \tag{2.12}$$

for $k = 1, 2, \ldots, n-1$. We can also see directly that (2.12) hods for $k = n$. We select a sequence $v^m \in U^n$ such that $B_k v_k^m \to q^n$ as m goes to infinity and $k = 0, 1, \ldots, n-1$. We denote by y^m the solution of (2.12) when we substitute q^n by $\widehat{B}^n v^m$. Hence, we can write

$$y_k^m = \Phi(k,0)x_0 + \sum_{i=1}^{k} \Phi(k,i)\left[f\left(i-1, y_{i-1}^m\right) + B_{i-1}\left(u_{i-1} - v_{i-1}^m\right)\right]. \tag{2.13}$$

This expression and (2.3) show that y^m is the solution of the equation

$$p_{k+1} = A_k p_k + f\left(k, p_k\right) + B_k\left(u_k - v_k^m\right) \tag{2.14}$$

with initial condition $p_0 = x_0$. Therefore, $y_n^m \in \mathcal{R}_f(n, x_0)$. Since the solution of (2.3) depends continuously on f, we infer that y_n^m converges to y_n as $m \to \infty$. Consequently, $y_n \in \overline{\mathcal{R}_f(n, x_0)}$. Hence, from our previous considerations, we can assert that

$$\mathcal{R}_0(n, x_0) \subseteq \overline{\mathcal{R}_f(n, x_0)}, \tag{2.15}$$

which completes the proof. □

Now we are able to establish the following criteria for the approximate controllability of system (1.1). The next property is an immediate consequence of Theorem 2.6.

Theorem 2.7. *Assume that $\gamma_n \max_{k=0,\ldots,n-1} L_k < 1$, the control system* (1.2) *is approximately controllable on* $[0, n]$ *and the space $\mathcal{N}(n) + \overline{\mathcal{R}(\widehat{B}^n)} = X^n$. Then the system* (1.1) *is approximately controllable on* $[0, n]$.

We are also in a position to establish the following result.

Theorem 2.8. *Assume that the following conditions hold:*

(a) *the control system* (1.2) *is approximately controllable in finite time;*

(b) *for all $n \in \mathbb{N}$, the space $\mathcal{N}(n) + \overline{\mathcal{R}(\widehat{B}^n)} = X^n$;*

(c) *for all $n \in \mathbb{N}$, $\gamma_n \max_{k=0,\ldots,n-1} L_k < 1$.*

Then system (1.1) *is approximately controllable in finite time.*

Proof. Proceeding as in the proof of Theorem 2.6, we can write

$$\bigcup_{n\in\mathbb{N}} \mathcal{R}_0(n, x_0) \subseteq \bigcup_{n\in\mathbb{N}} \overline{\mathcal{R}_f(n, x_0)} \subseteq \overline{\bigcup_{n\in\mathbb{N}} \mathcal{R}_f(n, x_0)}, \tag{2.16}$$

which shows that $\bigcup_{n\in\mathbb{N}} \mathcal{R}_f(n, x_0)$ is dense in X. □

Similar results for approximate controllability to the origin can be established. On the other hand, with appropriate hypotheses we can estimate the controls involved in the strategies of controllability and approximate controllability. This property allows us to compare the controllability in spaces of infinite dimension with the controllability in spaces of finite dimension.

Theorem 2.9. *Assume that the control system* (1.2) *is controllable on* $[0, n]$, *condition* (2.7) *holds, each operator* B_k *has a continuous left inverse* C_k, *for* $k = 0, \dots, n-1$, *and* $\gamma_n \max_{k=0,\dots,n-1} L_k < 1$. *Then there exists constants* $M, N > 0$ *such that for every* $\overline{x} \in X$ *and* $\varepsilon > 0$ *there exists a control sequence* w_k, $k = 0, 1, \dots, n-1$, *with* $\|w_k\| \leq M\|\overline{x}\| + N$ *and* $\|p_n - \overline{x}\| \leq \varepsilon$, *where* p_k, $k = 0, 1, \dots, n-1$, *is the solution of* (1.1) *corresponding to* w_k.

Proof. It follows from the controllability of system (1.2) that $S_n : U^n \to X$ is a surjective bounded linear map. We infer that there exists a constant $c_1 > 0$ such that for each $\overline{x} \in X$ there exists $(u_k)_{k:=0,\dots,n-1}$ such that $S(u_k) = \overline{x}$ and $\|(u_k)\| \leq c_1\|\overline{x}\|$. Let x_k, $k = 0, 1, \dots, n-1$, be the solution of (1.2) corresponding to u_k. Since A_k and B_k are uniformly bounded for $k = 0, 1, \dots, n-1$, we can conclude that there exists a constant $c_2 > 0$ such that $\|x_k\| \leq c_2\|\overline{x}\|$ for $k = 0, 1, \dots, n-1$. In the rest of this proof we apply the construction carried out in the proof of Theorem 2.6. Let z be the fixed point of Γ^n. From

$$\left\|f(k, x_k)\right\| \leq L_k c_2\|\overline{x}\| + \left\|f(k, 0)\right\| \tag{2.17}$$

we deduce that

$$\|z\| \leq \|\Gamma^n(z) - \Gamma^n(0)\| + \|\Gamma^n(0)\| \leq \gamma_n \max_{k=0,\dots,n-1} L_k\|z\| + c_3\|\overline{x}\| + c_4, \tag{2.18}$$

which in turn implies that

$$\|z\| \leq \frac{1}{1 - \gamma_n \max_{k=0,\dots,n-1} L_k}(c_3\|\overline{x}\| + c_4) \tag{2.19}$$

which we abbreviate as

$$\|z\| \leq c_5\|\overline{x}\| + c_6. \tag{2.20}$$

Proceeding in a similar way, we can obtain an estimate

$$\|F^n(z)\| \leq c_7\|\overline{x}\| + c_8. \tag{2.21}$$

Hence, $q^n = F^n(z) - P^n F^n(z)$ can also be estimated as

$$\left\|q^n\right\| \leq c_9\|\overline{x}\| + c_{10}. \tag{2.22}$$

We can choose a sequence v_k^m such that $\|q^n - B_k v_k^m\| \leq 1$ and $y_n^m \to y_n = x_n = \overline{x}$ as $m \to \infty$. Therefore, we can take m large enough such that $\|y_n^m - \overline{x}\| \leq \varepsilon$. Since y_n^m is the solution of (1.1) corresponding to controls $w_k = u_k - v_k^m$, to complete the proof we only need to estimate

$$\|v_k^m\| = \|C_k B_k v_k^m\| \le \|C_k(B_k v_k^m - q^n)\| + \|C_k q^n\| \le \|C_k\|(1 + \|q^n\|), \tag{2.23}$$

and the assertion is consequence of (2.22). □

2.1. The Finite-Dimensional Case

Certainly condition (2.7) considered in our previous results is strong. However, the following property holds.

Theorem 2.10. *Assume that X is a space of finite dimension. Then the linear system* (1.2) *is controllable on* $[0, n]$ *if, and only if, condition* (2.7) *holds.*

Proof. Since X has finite dimension, $\overline{\mathcal{R}(\widehat{B}^n)} = \mathcal{R}(\widehat{B}^n)$. Assume initially that system (1.2) is controllable on $[0, n]$. Let $x = (x_0, x_1, \ldots, x_{n-1}) \in X^n$. Using the property of controllability, it follows from [4, Corollary 2.3.1] that there exists $(u_0, u_1, \ldots, u_{n-1}) \in U^n$ such that

$$\sum_{i=1}^{n} \Phi(n,i) B_{i-1} u_{i-1} = \sum_{i=1}^{n} \Phi(n,i) x_{i-1}. \tag{2.24}$$

We define $y_i = x_i - B_i u_i$ for $i = 0, 1, \ldots, n-1$. This implies that

$$J_n(y) = \sum_{i=1}^{n} \Phi(n,i) y_{i-1} = \sum_{i=1}^{n} \Phi(n,i) x_{i-1} - \sum_{i=1}^{n} \Phi(n,i) B_{i-1} u_{i-1} = 0, \tag{2.25}$$

which shows that $x \in \mathcal{N}(n) + \mathcal{R}(\widehat{B}^n)$.

Conversely, assume that condition (2.7) holds; for $z \in X$ we define $x = (0, \ldots, z) \in X^n$. Applying (2.7), we derive the existence of $y \in \mathcal{N}(n)$ and $u = (u_k)_k \in U^n$ such that $x = y + \widehat{B}^n(u)$. The solution of (1.2) is given by

$$x_n(0, u) = S_n(u) = \sum_{i=1}^{n} \Phi(n,i) B_{i-1} u_{i-1} = J_n\left(\widehat{B}^n(u)\right) = J_n(x - y) = J_n(x) = z, \tag{2.26}$$

which completes the proof. □

We will apply Theorem 2.10 to reduce the study of controllability of system (1.1) to the controllability of systems with finite-dimensional state space.

Corollary 2.11. *Assume that X is a space of finite dimension and that the linear system* (1.2) *is controllable on* $[0, n]$. *Then there exists $\varepsilon > 0$ such that nonlinear system* (1.1) *is approximately controllable on* $[0, n]$ *when* $\max_{k=0,\ldots,n-1} L_k < \varepsilon$.

Proof. The assertion is an immediate consequence of Theorems 2.10 and 2.7. □

Next we specialize our developments to consider systems where the associated linear system is invariant. Specifically, we will assume that $A_k = A$ and $B_k = B$ for $k \in \mathbb{N}_0$. That is to say, we will be concerned with the nonlinear system

$$x_{k+1} = Ax_k + f(k, x_k) + Bu_k, \quad k \geq 0, \tag{2.27}$$

with linear part

$$x_{k+1} = Ax_k + Bu_k, \quad k \geq 0. \tag{2.28}$$

In this situation, the subspaces $\mathcal{R}_0(k, 0)$ are nondecreasing. Hence, we get the following immediate consequence.

Proposition 2.12. *Assume that* X *is a space of finite dimension. If the system* (2.28) *is approximately controllable in finite time, then it is controllable on* $[0, m]$*, for some* $m \in \mathbb{N}$.

Proof. Since $X = \overline{\bigcup_{k\in\mathbb{N}} \mathcal{R}_0(k, 0)} = \bigcup_{k\in\mathbb{N}} \mathcal{R}_0(k, 0)$ and $\mathcal{R}_0(k, 0)$ are closed subspaces, then there is $m \in \mathbb{N}$ such that $\mathcal{R}_0(m, 0) = X$. □

2.2. The Projections P^n

Next we will study a property of projections P^n. We begin with some remarks.

Remark 2.13. Let $x = (x_k)_k \in X^n$. Since

$$J_n(x) = \sum_{i=1}^{n} \Phi(n, i) x_{i-1} = \sum_{i=1}^{n} A^{n-i} x_{i-1}, \tag{2.29}$$

we infer that $x \in \mathcal{N}(n)$ if, and only if,

$$\sum_{i=1}^{n} A^{n-i} x_{i-1} = 0. \tag{2.30}$$

Hence, if $x \in \mathcal{N}(n)$ and we define $\tilde{x} = (x, 0) \in X^{n+1}$ and $\tilde{y} = (0, x) \in X^{n+1}$, then $\tilde{x}, \tilde{y} \in \mathcal{N}(n+1)$.

Lemma 2.14. *Assume that condition* (2.7) *holds for* n *and* $n+1$*. Then*

$$\left\| P^{n+1}(x_0, x_1, \ldots, x_n) - (x_0, x_1, \ldots, x_n) \right\| \leq \| P^n(x_0, 0) - (x_0, 0) \| + \| P^n(x_1, \ldots, x_n) - (x_1, \ldots, x_n) \|, \tag{2.31}$$

where $(x_0, 0) \in X^n$.

Proof. We decompose $x = (x_0, x_1, \ldots, x_n) = (x_0, 0) + (0, \overline{x})$, where $\overline{x} = (x_1, \ldots, x_n)$.

Let $y = (x_0, 0) \in X^{n+1}$ and $z = (x_0, 0) \in X^n$. Then $z = P^n z + q$, where $q \in \overline{\mathcal{R}(\widehat{B}^n)}$. We set $\tilde{p} = (P^n z, 0) \in X^{n+1}$ and $\tilde{q} = (q, 0) \in X^{n+1}$. It follows from Remark 2.13 that $y = \tilde{p} + \tilde{q}$, and $\widetilde{p} \in \mathcal{N}(n+1)$ and $\widetilde{q} \in \overline{\mathcal{R}(\widehat{B}^{n+1})}$. Therefore, using the properties of projections P^n and P^{n+1} established in Lemma 1.1, we get

$$\left\| P^{n+1}y - y \right\| \leq \|\widetilde{q}\| = \|q\| = \|P^n z - z\|. \tag{2.32}$$

Similarly, since $\overline{x} \in X^n$, we can decompose $\overline{x} = P^n\overline{x} + q$, where $q \in \overline{\mathcal{R}(\widehat{B}^n)}$. We set $\widetilde{p} = (0, P^n z) \in X^{n+1}$ and $\widetilde{q} = (0, q) \in X^{n+1}$. It follows from Remark 2.13 that $(0, \overline{x}) = \widetilde{p} + \widetilde{q}$, and $\widetilde{p} \in \mathcal{N}(n+1)$ and $\widetilde{q} \in \overline{\mathcal{R}(\widehat{B}^{n+1})}$. Consequently, we have

$$\left\| P^{n+1}(0, \overline{x}) - (0, \overline{x}) \right\| \leq \|\widetilde{q}\| = \|q\| = \|P^n\overline{x} - \overline{x}\|. \tag{2.33}$$

Collecting these assertions, we get

$$\begin{aligned} \left\| P^{n+1}(x_0, x_1, \ldots, x_n) - (x_0, x_1, \ldots, x_n) \right\| &\leq \left\| P^{n+1}(x_0, 0) - (x_0, 0) \right\| \\ &\quad + \left\| P^{n+1}(0, x_1, \ldots, x_n) - (0, x_1, \ldots, x_n) \right\| \\ &\leq \|P^n(x_0, 0) - (x_0, 0)\| \\ &\quad + \|P^n(x_1, \ldots, x_n) - (x_1, \ldots, x_n)\|. \end{aligned} \tag{2.34}$$

□

We say that a sequence $(Y_n, \pi_n)_{n\in\mathbb{N}}$ is an approximation scheme for X associated to system (2.27) if Y_n are finite-dimensional subspaces of X, $\pi_n : X \to Y_n$ are bounded linear projections with $\mathcal{R}(\pi_n) = Y_n$ and $\ker(\pi_n) = Q_n$, and the following conditions are fulfilled:

(i) the subspaces Y_n and Q_n are invariant under A;

(ii) the projections π_n are uniformly bounded with $\|\pi_n\| \leq \rho$ for all $n \in \mathbb{N}$;

(iii) for all $x \in X$, $\pi_n x \to x$ as $n \to \infty$.

We consider the control systems

$$y_{k+1} = Ay_k + \pi_n f(k, y_k) + \pi_n B u_k, \quad k \geq 0, \tag{2.35}$$

$$y_{k+1} = Ay_k + \pi_n B u_k, \quad k \geq 0, \tag{2.36}$$

in the space Y_n. We set $\beta_n = \|B - \pi_n B\|$.

Theorem 2.15. *If the system* (2.28) *is approximately controllable in finite time, then the system* (2.36) *is controllable on an interval* $[0, m(n)]$ *for each* $n \in \mathbb{N}$.

Proof. We consider a fixed $n \in \mathbb{N}$. It is immediate from our definition of approximation scheme that if $y_0 = \pi_n x_0$ and we consider the same values of u_k in (2.28) and (2.36), then $y_k = \pi_n x_k$

for all $k \in \mathbb{N}_0$. Let $\overline{y} \in Y_n$. It follows from the previous remark, that if we select u_k such that $x_k \to \overline{y}$ as $k \to \infty$, then

$$\|y_k - \overline{y}\| = \|\pi_n(x_k - \overline{y})\| \le \rho\|x_k - \overline{y}\|, \tag{2.37}$$

which shows that $y_k \to \pi_n \overline{y} = \overline{y}$ as $k \to \infty$. Hence, system (2.36) is approximately controllable in finite time. The assertion is now a consequence of Proposition 2.12. □

To simplify the writing of the text, next we will assume that $\dim(Y_n) = n$ and $Y_n \subseteq Y_{n+1}$. Furthermore, we take an orthonormal basis $\{\varphi_1, \dots, \varphi_n\}$ of Y_n, and π_n is the orthogonal projection. We can establish the following property.

Lemma 2.16. *Assume that condition* (2.7) *holds in* Y_n^n *for all* $n \in \mathbb{N}$. *Then there are constants* $c_n > 0$ *such that*

$$\|P^n(y_0, y_1, \dots, y_{n-1}) - (y_0, y_1, \dots, y_{n-1})\| \le \sum_{i=0}^{n-1}\sum_{j=1}^{n} c_j |\langle y_i, \varphi_j\rangle|, \tag{2.38}$$

for all $y_i \in Y_n$, $i = 0, \dots, n-1$.

Proof. We proceed by using mathematical induction. The assertion holds for $n = 1$. In fact, since $y_0 \in Y_1$ and $\mathcal{N}(1) = \{0\}$, then $y_0 = \langle y_0, \varphi_1\rangle\varphi_1 = \pi_1 B u_0$ and

$$\left\|P^1 y_0 - y_0\right\| \le \|\pi_1 B u_0\| = |\langle y_0, \varphi_1\rangle|. \tag{2.39}$$

Assume now that the assertion is fulfilled for n. We will prove that the assertion holds for $n+1$. For $y_i \in Y_{n+1}$, $i = 0, 1, \dots, n$, we decompose $y_i = \overline{y}_i + \langle y_i, \varphi_{n+1}\rangle\varphi_{n+1}$, where $\overline{y}_i \in Y_n$. We abbreviate the notation by writing $z_i = \langle y_i, \varphi_{n+1}\rangle\varphi_{n+1}$. Consequently, applying Lemma 2.14, we get

$$\begin{aligned}
&\left\|P^{n+1}(y_0, y_1, \dots, y_n) - (y_0, y_1, \dots, y_n)\right\| \\
&\quad\le \left\|P^{n+1}(\overline{y}_0, \overline{y}_1, \dots, \overline{y}_n) - (\overline{y}_0, \overline{y}_1, \dots, \overline{y}_n)\right\| + \left\|P^{n+1}(z_0, z_1, \dots, z_n) - (z_0, z_1, \dots, z_n)\right\| \\
&\quad\le \|P^n(\overline{y}_0, 0) - (\overline{y}_0, 0)\| + \|P^n(\overline{y}_1, \dots, \overline{y}_n) - (\overline{y}_1, \dots, \overline{y}_n)\| \\
&\qquad + \left\|P^{n+1}(z_0, z_1, \dots, z_n) - (z_0, z_1, \dots, z_n)\right\| \\
&\quad\le \sum_{j=1}^{n} c_j |\langle \overline{y}_0, \varphi_j\rangle| + \sum_{i=1}^{n}\sum_{j=1}^{n} c_j |\langle \overline{y}_i, \varphi_j\rangle| + \left\|P^{n+1}(z_0, z_1, \dots, z_n) - (z_0, z_1, \dots, z_n)\right\| \\
&\quad= \sum_{j=1}^{n} c_j |\langle y_0, \varphi_j\rangle| + \sum_{i=1}^{n}\sum_{j=1}^{n} c_j |\langle y_i, \varphi_j\rangle| + \left\|P^{n+1}(z_0, z_1, \dots, z_n) - (z_0, z_1, \dots, z_n)\right\| \\
&\quad= \left\|P^{n+1}(z_0, z_1, \dots, z_n) - (z_0, z_1, \dots, z_n)\right\| + \sum_{i=0}^{n}\sum_{j=1}^{n} c_j |\langle y_i, \varphi_j\rangle|.
\end{aligned} \tag{2.40}$$

On the other hand, since $P^{n+1} - I$ is a bounded linear map on Y_{n+1}^{n+1}, then there exists a constant $c_{n+1} > 0$ such that

$$\begin{aligned}\left\| P^{n+1}(z_0, z_1, \ldots, z_n) - (z_0, z_1, \ldots, z_n) \right\| &\leq c_{n+1} \|(z_0, z_1, \ldots, z_n)\| \\ &= c_{n+1} \left(\sum_{i=0}^{n} \left| \langle z_i, \varphi_{n+1} \rangle \right|^2 \right)^{1/2} \\ &\leq c_{n+1} \sum_{i=0}^{n} \left| \langle z_i, \varphi_{n+1} \rangle \right| \\ &= c_{n+1} \sum_{i=0}^{n} \left| \langle y_i, \varphi_{n+1} \rangle \right|,\end{aligned} \tag{2.41}$$

and substituting these estimates in (2.40), we get that the assertion is fulfilled for $n + 1$. □

Lemma 2.17. *Assume that $\sqrt{2}\|A\| < 1$, condition* (2.7) *holds in Y_n^n for all $n \in \mathbb{N}$, and that the function f in* (2.35) *satisfies the Lipschitz conditions*

$$\left| \langle f(i, y) - f(i, w), \varphi_k \rangle \right| \leq L_{i,k} \|y - w\|, \tag{2.42}$$

where $L_{i,k} > 0$. If

$$\left(\sum_{i=0}^{n-1} \left(\sum_{j=1}^{n} c_j L_{i,j} \right)^2 \right)^{1/2} + \left(\sum_{j=1}^{n} \max_{i:0,\ldots,n-1} L_{i,j}^2 \right)^{1/2} < \frac{\left(1 - 2\|A\|^2\right)^{1/2}}{\sqrt{2}}, \tag{2.43}$$

then the map Γ^n defined in Y_n^n is a contraction.

Proof. It follows from our definition that

$$\begin{aligned}\|J^n(y)\|^2 &= \sum_{k=1}^{n-1} \left\| \sum_{i=1}^{k} A^{k-i} y_{i-1} \right\|^2 \\ &\leq \sum_{k=1}^{n-1} \sum_{i=1}^{k} 2^{k-i+1} \left\| A^{k-i} y_{i-1} \right\|^2 \\ &= 2 \sum_{j=0}^{n-2} 2^j \|A\|^{2j} \sum_{i=1}^{n-2-j} \|y_{i-1}\|^2 \\ &\leq 2 \sum_{j=0}^{n-2} 2^j \|A\|^{2j} \|y\|^2 \\ &\leq \frac{2}{1 - 2\|A\|^2} \|y\|^2.\end{aligned} \tag{2.44}$$

On the other hand, since

$$P^n\left(F^n(y) - F^n(w)\right) = (P^n - I)\left(F^n(y) - F^n(w)\right) + \left(F^n(y) - F^n(w)\right) \tag{2.45}$$

applying Lemma 2.16 and the definition of F^n, we have

$$\begin{aligned}
\left\|P^n\left(F^n(y) - F^n(w)\right)\right\| &= \left\|(P^n - I)\left(F^n(y) - F^n(w)\right)\right\| + \left\|\left(F^n(y) - F^n(w)\right)\right\| \\
&\le \sum_{i=0}^{n-1}\sum_{j=1}^{n} c_j \left|\left\langle \pi_n\left(f(i, x_i + y_i) - f(i, x_i + w_i)\right), \varphi_j\right\rangle\right| \\
&\quad + \left(\sum_{i=0}^{n-1}\sum_{j=1}^{n}\left|\left\langle \pi_n\left(f(i, x_i + y_i) - f(i, x_i + w_i)\right), \varphi_j\right\rangle\right|^2\right)^{1/2} \\
&\le \sum_{i=0}^{n-1}\sum_{j=1}^{n} c_j L_{i,j}\|y_i - w_i\| + \left(\sum_{i=0}^{n-1}\sum_{j=1}^{n} L_{i,j}^2 \|y_i - w_i\|^2\right)^{1/2} \\
&\le \left(\sum_{i=0}^{n-1}\left(\sum_{j=1}^{n} c_j L_{i,j}\right)^2\right)^{1/2} \|y - w\| \\
&\quad + \left(\max_{i=0,1\ldots,n-1}\sum_{j=1}^{n} L_{i,j}^2\right)^{1/2} \|y - w\|.
\end{aligned} \tag{2.46}$$

In view of

$$\Gamma^n(y) - \Gamma^n(w) = J^n \circ P^n\left(F^n(y) - F^n(w)\right) \tag{2.47}$$

collecting the above estimate, we get the assertion. □

Using now Theorem 2.15 and Lemma 2.17 we can emphasize the assertion of Corollary 2.11.

Corollary 2.18. *Under the conditions of Lemma 2.17, if the system* (2.28) *is approximately controllable in finite time, then the system* (2.35) *is approximately controllable on* $[0, n]$ *for each* $n \in \mathbb{N}$.

Remark 2.19. Under the above conditions, we can apply Theorem 2.9 in the space Y_n. Consequently, there exist constants M_n and N_n such that for every $\overline{x}^n \in Y_n$ and $\varepsilon > 0$, there exists a sequence of controls u_k^n for $k = 0, 1\ldots, n$ such that $\|u_k^n\| \le M_n\|\overline{x}^n\| + M_n = M_n'$, $\|y_k^n\| \le N_n\|\overline{x}^n\| + N_n = N_n'$, and $\|y_n^n - \overline{x}^n\| \le \varepsilon$, where y_k^n is the solution of (2.35) corresponding to controls u_k^n. Furthermore, we denote $L = \sup_{k\ge 0} L_k < \infty$, where L_k are the constants involved in (2.10), and we assume that

$$\|f(k,y)-\pi_n f(k,y)\| \le \nu_n \|y\|, \quad k=0,1,\ldots,n,\ y\in Y_n. \tag{2.48}$$

Finally, we are in a position to establish the following result of controllability.

Theorem 2.20. *Assume that there exists an approximation scheme* (Y^n,π_n) *and the system* (2.28) *is approximately controllable in finite time. If, in addition,* $\sqrt{2}\|A\|<1$, $\|A\|+L<1$, *and* $\nu_n N_n \to 0$ *and* $\beta_n M_n \to 0$ *as* $n\to\infty$, *then the system* (2.27) *is also approximately controllable in finite time.*

Proof. Let $\overline{x}\in X$ and $\overline{x}^n=\pi_n\overline{x}$. It follows from Corollary 2.18 that system (2.35) is approximately controllable on $[0,n]$. Since $\pi_n\overline{x}\to\overline{x}$ as $n\to\infty$, for $\varepsilon>0$, we chose $n\in\mathbb{N}$ such that $\|\overline{x}-\overline{x}^n\|\le\varepsilon$. It follows from Remark 2.19 that there exists a sequence of controls u_k^n for $k=0,1,\ldots,n$ such that $\|u_k^n\|\le M_n\|\overline{x}^n\|+M_n=M_n'$, $\|y_k^n\|\le N_n\|\overline{x}^n\|+N_n=N_n'$ and $\|y_n^n-\overline{x}^n\|\le\varepsilon$, where y_k^n is the solution of (2.35) corresponding to controls u_k^n.

We denote x_k^n for the solution of system

$$x_{k+1}=Ax_k+f(k,x_k)+Bu_k^n, \quad k=0,\ldots,n-1, \tag{2.49}$$

and we set $z_k^n=x_k^n-y_k^n$. It follows from (2.35) and (2.49) that

$$z_{k+1}^n=Az_k^n+f(k,x_k^n)-\pi_n f(k,y_k^n)+(B-\pi_nB)u_k^n, \tag{2.50}$$

which implies that

$$\begin{aligned}\|z_{k+1}^n\| &\le (\|A\|+L)\|z_k^n\|+\|f(k,y_k^n)-\pi_n f(k,y_k^n)\|+\|(B-\pi_nB)u_k^n\|\\ &\le (\|A\|+L)\|z_k^n\|+\nu_nN_n'+\beta_nM_n'.\end{aligned} \tag{2.51}$$

Consequently, $\|z_k^n\|\le(1/(1-\|A\|-L))(\nu_nN_n'+\beta_nM_n')$. Hence,

$$\begin{aligned}\|x_n^n-\overline{x}\| &\le \frac{1}{1-\|A\|-L}(\nu_nN_n'+\beta_nM_n')+\|y_n^n-\overline{x}^n\|+\|\overline{x}-\overline{x}^n\|\\ &\le \frac{1}{1-\|A\|-L}(\nu_nN_n'+\beta_nM_n')+2\varepsilon.\end{aligned} \tag{2.52}$$

Consequently, $x_n^n\to\overline{x}$ as $n\to\infty$, which completes the proof. □

3. Application

We complete this paper with an application of the results established in Section 2.

In this application we are concerned with a general class of systems that satisfy the conditions considered previously. Specifically, we consider a control system of type (1.1) with state space X of infinite dimension and operators $A_k=A$ and $B_k=B$ for $k\in\mathbb{N}_0$.

We assume that A is a bounded self-adjoint operator with distinct eigenvalues λ_n, $n \in \mathbb{N}$, and $\{\varphi_n : n \in \mathbb{N}\}$ is an orthonormal basis of X consisting of eigenvectors of A corresponding to eigenvalues λ_n, respectively.

We take as control space $U = \mathbb{R}$, and $B : U \to X$ is given by $Bu = bu$, where $b \in X$ is a vector such that $\langle b, \varphi_n \rangle \neq 0$, for all $n \in \mathbb{N}$. It is clear that condition (2.7) does not hold in this case. In fact, since the space $\mathcal{R}(\widehat{B}^n)$ is closed, if we assume that condition (2.7) is fulfilled, then for every $x \in X^n$ there is $\widehat{u} = (u_0, u_1, \ldots, u_{n-1}) \in U^n$ such that $J_n x = J_n \widehat{B}^n(\widehat{u})$. In particular, for an arbitrary $y \in X$ and $x = (0, \ldots, 0, y)$ and applying Remark 2.13, we obtain that $y = \sum_{i=1}^{n} A^{n-i} b u_{i-1}$. However, this means that X is a finite-dimensional space, which is a contradiction. Let $f : \mathbb{N}_0 \times X \to X$ be given by

$$f(k, x) = \sum_{j=1}^{\infty} g_j(k, x) \varphi_j, \tag{3.1}$$

where $g_j : \mathbb{N}_0 \times X \to \mathbb{R}$ are functions such that $g_j(k, 0) = 0$ and the following Lipschitz conditions

$$\left| g_j(k, y) - g_j(k, w) \right| \leq L_{k,j} \| y - w \| \tag{3.2}$$

are verified for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $y, w \in X$. We assume that

$$L = \sup_{0 \leq k < \infty} \left(\sum_{j=1}^{\infty} L_{k,j}^2 \right)^{1/2} < \infty. \tag{3.3}$$

We denote $\nu_n = \sup_{k \geq 0} \left(\sum_{j=n+1}^{\infty} L_{k,j}^2 \right)^{1/2}$.

Let $Y_n = \operatorname{Span}\{\varphi_1, \ldots, \varphi_n\}$, and let $\pi_n : X \to Y_n$ be the orthogonal projection on Y_n. We set $B_n = \pi_n \circ B$. Since Y_n is invariant under A, we can consider the system

$$x_{k+1} = A x_k + B_n u(t), \quad k \in \mathbb{N}_0, \tag{3.4}$$

with $x_k \in Y_n$, which is the restriction of system (1.2) on Y_n. It is well known that system (3.4) is exactly controllable on $[0, \tau]$, for every $\tau > 0$. Furthermore,

$$\beta_n = \| B - \pi_n B \| = \left(\sum_{j=n+1}^{\infty} \left| \langle b, \varphi_j \rangle \right|^2 \right)^{1/2}. \tag{3.5}$$

Let c_j, $j \in \mathbb{N}$, be the constants introduced in Lemma 2.16, and let M_n, N_n be the constants introduced in Remark 2.19. At this point it is worth to note that the constants c_j for $j = 1, \ldots, n$ and M_n, N_n depend on B_n and $g_j(k, \cdot)$ for $k = 0, 1, \ldots, n-1$ and $j = 1, \ldots, n$ while β_n and ν_n depend on $\langle b, \varphi_j \rangle$ and $L_{k,j}$, respectively, for $j \geq n+1$. We can establish the following immediate consequence of Theorem 2.20.

Proposition 3.1. *Assume that the system* (2.28) *is approximately controllable in finite time. If, in addition,* $\sqrt{2}\|A\| < 1$, $\|A\| + L < 1$,

$$\left(\sum_{i=0}^{\infty}\left(\sum_{j=1}^{\infty} c_j L_{i,j}\right)^2\right)^{1/2} + \left(\sum_{j=1}^{\infty} \max_{i\geq 0} L_{i,j}^2\right)^{1/2} < \frac{\left(1-2\|A\|^2\right)^{1/2}}{\sqrt{2}}, \tag{3.6}$$

and $\nu_n N_n \to 0$ *and* $\beta_n M_n \to 0$, *as* $n \to \infty$, *then the system* (2.27) *is also approximately controllable in finite time.*

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