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Research Article

Error Bounds for Asymptotic Solutions of Second-Order Linear Difference Equations II: The First Case

L. H. Cao[1,2] and J. M. Zhang[3]

[1] *Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong*
[2] *Department of Mathematics, Shenzhen University, Guangdong 518060, China*
[3] *Department of Mathematics, Tsinghua University, Beijin 100084, China*

Correspondence should be addressed to J. M. Zhang, jzhang@math.tsinghua.edu.cn

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We discuss in detail the error bounds for asymptotic solutions of second-order linear difference equation $y(n+2)+n^p a(n)y(n+1)+n^q b(n)y(n)=0$, where p and q are integers, $a(n)$ and $b(n)$ have asymptotic expansions of the form $a(n)\sim\sum_{s=0}^{\infty}(a_s/n^s)$, $b(n)\sim\sum_{s=0}^{\infty}(b_s/n^s)$, for large values of n, $a_0\neq 0$, and $b_0\neq 0$.

1. Introduction

Asymptotic expansion of solutions to second-order linear difference equations is an old subject. The earliest work as we know can go back to 1911 when Birkhoff [1] first deal with this problem. More than eighty years later, this problem was picked up again by Wong and Li [2, 3]. This time two papers on asymptotic solutions to the following difference equations:

$$y(n+2)+a(n)y(n+1)+b(n)y(n)=0 \tag{1.1}$$

$$y(n+2)+n^p a(n)y(n+1)+n^q b(n)y(n)=0 \tag{1.2}$$

were published, respectively, where coefficients $a(n)$ and $b(n)$ have asymptotic properties

$$a(n)\sim\sum_{s=0}^{\infty}\frac{a_s}{n^s},\qquad b(n)\sim\sum_{s=0}^{\infty}\frac{b_s}{n^s}, \tag{1.3}$$

for large values of n, $a_0\neq 0$, $b_0\neq 0$, and $p,q\in Z$.

Unlike the method used by Olver [4] to treat asymptotic solutions of second-order linear differential equations, the method used in Wong and Li's papers cannot give us way to obtain error bounds of these asymptotic solutions. Only order estimations were given in their papers. The estimations of error bounds for these asymptotic solutions to (1.1) were given in [5] by Zhang et al. But the problem of obtaining error bounds for these asymptotic solutions to (1.2) is still open. The purpose of this and the next paper (Error bounds for asymptotic solutions of second-order linear difference equations II: the second case) is to estimate error bounds for solutions to (1.2). The idea used in this paper is similar to that of Olver to obtain error bounds to the Liouville-Green (WKB) asymptotic expansion of solutions to second-order differential equations. It should be pointed out that similar method appeared in some early papers, such as Spigler and Vianello's papers [6–9].

In Wong and Li's second paper [3], two different cases were given according to different values of parameters. The first case is devoted to the situation when $k > 0$, and in the second case as $k < 0$ where $k = 2p - q$. The whole proof of the result is too long to understand, so we divide the estimations into two parts, part I (this paper) and part II (the next paper), which correspond to the different two cases of [3], respectively.

In the rest of this section, we introduce the main results of [3] in the case that k is positive. In the next section, we give two lemmas on estimations of bounds for solutions to a special summation equation and a first order nonlinear difference equation which will be often used later. Section 3 is devoted to the case when $k = 1$. And in Section 4, we discuss the case when $k > 1$. The next paper (Error bounds for asymptotic solutions of second-order linear difference equations II: the second case) is dedicated to the case when $k < 0$.

1.1. The Result in [3] When $k = 1$

When $k = 1$, from [3] we know that (1.2) has two linearly independent solution $y_1(n)$ and $y_2(n)$

$$y_1(n) = [(n-2)!]^{q-p} \rho_1^n n^{\alpha_1} \sum_{s=0}^{\infty} \frac{c_s^{(1)}}{n^s}, \tag{1.4}$$

$$\rho_1 = -\frac{b_0}{a_0}, \qquad \alpha_1 = \frac{b_0}{a_0^2} - \frac{a_1}{a_0} + \frac{b_1}{b_0} - p + q, \qquad c_0^{(1)} \neq 0, \tag{1.5}$$

$$y_2(n) = [(n-2)!]^{p} \rho_2^n n^{\alpha_2} \sum_{s=0}^{\infty} \frac{c_s^{(2)}}{n^s}, \tag{1.6}$$

$$\rho_2 = -a_0, \qquad \alpha_2 = \frac{1}{a_0}\left(a_1 - \frac{b_0}{a_0}\right), \qquad c_0^{(2)} \neq 0, \tag{1.7}$$

for $n \geq 2$.

1.2. The Result in [3] When $k > 1$

When $k > 1$, from [3] we know that (1.2) has two linearly independent solutions $y_1(n)$ and $y_2(n)$

$$y_1(n) = [(n-2)!]^{q-p}\rho_1^n n^{\alpha_1}\sum_{s=0}^{\infty}\frac{c_s^{(1)}}{n^s}, \tag{1.8}$$

$$\rho_1 = -\frac{b_0}{a_0}, \qquad \alpha_1 = \frac{b_1}{b_0} - \frac{a_1}{a_0} - p + q, \qquad c_0^{(1)} \neq 0, \tag{1.9}$$

$$y_2(n) = [(n-2)!]^{p}\rho_2^n n^{\alpha_2}\sum_{s=0}^{\infty}\frac{c_s^{(2)}}{n^s}, \tag{1.10}$$

$$\rho_2 = -a_0, \qquad \alpha_2 = \frac{a_1}{a_0}, \qquad c_0^{(2)} \neq 0. \tag{1.11}$$

In the following sections, we will discuss in detail the error bounds of the proceeding asymptotic solutions of (1.2). Before discussing the error bounds, we consider some lemmas.

2. Lemmas

2.1. The Bounds for Solutions to the Summation Equation

We consider firstly a bound of a special solution for the "summary equation"

$$h(n) = \sum_{j=n}^{\infty}K(n,j)\{R(j) - j^p\phi(j)h(j+1) - j^q\psi(j)h(j)\}. \tag{2.1}$$

Lemma 2.1. *Let $K(n,j)$, $\phi(j)$, $\psi(j)$, $R(j)$ be real or complex functions of integer variables n, j; p and q are integers. If there exist nonnegative constants n_1, θ, ς, N, s, t, β, C_K, C_R, C_ϕ, C_ψ, C_β, C_α which satisfy*

$$-\theta + p - s - \beta \leqslant -\frac{3}{2}, \qquad -\theta + q - t - 2\beta \leqslant -\frac{3}{2}, \tag{2.2}$$

and when $j \geqslant n \geqslant n_1$,

$$\begin{gathered} |K(n,j)| \leqslant C_K\frac{P(n)}{p(j)}j^{-\theta}, \qquad |R(j)| \leqslant C_R p(j)j^{-\varsigma N-1}, \\ |\phi(j)| \leqslant C_\phi j^{-s}, \qquad |\psi(j)| \leqslant C_\psi j^{-t}, \\ \left|\frac{P(j)}{p(j)}\right| \leqslant C_\beta j^{-2\beta}, \qquad \left|\frac{P(j+1)}{p(j)}\right| \leqslant 2C_\alpha|\rho_1|j^{-\beta}, \end{gathered} \tag{2.3}$$

where $P(n)$ and $p(n)$ are positive functions of integer variable n. Let n_0 , n_2 be integers defined by

$$n_2 = \left[\frac{1}{\varsigma}\left\{2C_K\left(2C_\alpha C_\phi|\rho_1|\sup_j\left(1+\frac{1}{j}\right)^{-\varsigma N-\theta-3/2} + C_\psi C_\beta\right) - \theta\right\}\right],$$
$$n_0 = \max\{n_1, n_2\}, \tag{2.4}$$

then (2.1) *has a solution $h(n)$, which satisfies*

$$|h(n)| \leqslant \frac{2C_R C_K P(n) n^{-\varsigma N-\theta-1}}{\varsigma N + \theta - 2C_K\left(2C_\alpha C_\phi|\rho_1|\sup_j(1+1/j)^{-\varsigma N-\theta-3/2} + C_\psi C_\beta\right)}, \tag{2.5}$$

for $N \geqslant n_0$.

Proof. Set

$$h_0(n) = 0,$$
$$h_{s+1}(n) = \sum_{j=n}^{\infty} K(n,j)\{R(j) - j^p\phi(j)h_s(j+1) - j^q\psi(j)h_s(j)\}, \tag{2.6}$$
$$s = 0, 1, 2, \ldots,$$

then

$$|h_1(n)| \leqslant \sum_{j=n}^{\infty}|K(n,j)||R(j)| \leqslant C_K C_R P(n)\left|\sum_{j=n}^{\infty} j^{-\varsigma N-\theta-1}\right|$$
$$\leqslant \frac{2C_R C_K}{\varsigma N+\theta} P(n) n^{-\varsigma N-\theta}. \tag{2.7}$$

The inequality $\sum_{j=n}^{\infty} j^{-p}(2/(p-1)) \leqslant n^{-(p-1)}$, $n \geqslant p-1 > 0$, is used here. Assuming that

$$|h_s(n) - h_{s-1}(n)| \leqslant \frac{2C_R C_K}{\varsigma N+\theta}\lambda^{s-1} P(n) n^{-\varsigma N-\theta-1}, \tag{2.8}$$

where

$$\lambda = \frac{2C_K}{\varsigma N+\theta}\left(2C_\alpha C_\phi|\rho_1|\sup_j\left(1+\frac{1}{j}\right)^{-\varsigma N-\theta-3/2} + C_\psi C_\beta\right); \tag{2.9}$$

then

$$\begin{aligned}
|h_{s+1}(n)-h_s(n)| &\leqslant \sum_{j=n}^{\infty}|K(n,j)|\left[j^p|\phi(j)||h_s(j+1)+h_{s-1}(j+1)|+j^q|\psi(j)||h_s(j)-h_{s-1}(j)|\right] \\
&\leqslant \sum_{j=n}^{\infty}\frac{2C_R C_K^2 P(n)}{(\varsigma N+\theta)p(j)}j^{-\theta}\Big[j^{p-s}C_\phi\lambda^{s-1}P(j+1)(j+1)^{-\varsigma N-\theta-1} \\
&\qquad +j^{q-t}C_\psi\lambda^{s-1}P(j)j^{-\varsigma N-\theta-1}\Big] \\
&\leqslant \frac{2C_R C_K}{\varsigma N+\theta}\lambda^s P(n)n^{-\varsigma N-\theta-1}.
\end{aligned} \tag{2.10}$$

By induction, the inequality holds for any integer s. Hence the series

$$\sum_{s=0}^{\infty}\{h_{s+1}(n)-h_s(n)\}, \tag{2.11}$$

when $\lambda<1$, that is, $N\geqslant n_0=\max\{n_1,n_2\}$, is uniformly convergent in n where

$$n_2=\left[\frac{1}{\varsigma}\left\{2C_K\left(2C_\alpha C_\phi|\rho_1|\sup_j\left(1+\frac{1}{j}\right)^{-\varsigma N-\theta-3/2}+C_\psi C_\beta\right)-\theta\right\}\right]. \tag{2.12}$$

And its sum

$$h(n)=\sum_{s=0}^{\infty}\{h_{s+1}(n)-h_s(n)\} \tag{2.13}$$

satisfies

$$\begin{aligned}
|h(n)| &\leqslant \sum_{s=0}^{\infty}|h_{s+1}(n)-h_s(n)| \leqslant \frac{2C_R C_K}{\varsigma N+\theta}P(n)n^{-\varsigma N-\theta-1}\sum_{s=0}^{\infty}\lambda^s \\
&\leqslant \frac{2C_R C_K P(n)n^{-\varsigma N-\theta-1}}{\varsigma N+\theta-2C_K\left(2C_\alpha C_\phi|\rho_1|\sup_j(1+1/j)^{-\varsigma N-\theta-3/2}+C_\psi C_\beta\right)}.
\end{aligned} \tag{2.14}$$

□

So we get the bound of any solution for the "summary equation" (2.1). Next we consider a nonlinear first-order difference equation.

2.2. The Bound Estimate of a Solution to a Nonlinear First-Order Difference Equation

Lemma 2.2. *If the function $f(n)$ satisfies*

$$f(n) = 1 + \frac{A}{n^2} + f_1(n), \tag{2.15}$$

where $n^3|f_1(n)| \leqslant B$ (A and B are constants), when n is large enough, then the following first-order difference equation

$$\begin{gathered} x(n)x(n+1) = f(n), \\ x(\infty) = 1 \end{gathered} \tag{2.16}$$

has a solution $x(n)$ such that $\sup_n\{n^2|x(n)-1|\}$ is bounded by a constant C_x, when n is big enough.

Proof. Obviously from the conditions of this lemma, we know that infinite products $\prod_{k=0}^{\infty} f(n+2k)$ and $\prod_{k=0}^{\infty} f(n+2k+1)$ are convergent.

$$x(n) = \frac{\prod_{k=0}^{\infty} f(n+2k)}{\prod_{k=0}^{\infty} f(n+2k+1)}. \tag{2.17}$$

is a solution of (2.16) with the infinite condition. Let $g(n,k) = f(n+2k)/f(n+2k+1) - 1$; then when n is large enough,

$$\begin{aligned} g(n,k) &\leqslant \frac{4|A| + 4B}{(n+2k)^3}, \\ n^2|x(n)-1| &= n^2\left|\frac{\prod_{k=0}^{\infty} f(n+2k)}{\prod_{k=0}^{\infty} f(n+2k+1)} - 1\right| \\ &= n^2\prod_{k=0}^{\infty}\{[1+g(n,k)] - 1\} \\ &\leqslant 4|A| + 4B = C_x. \end{aligned} \tag{2.18}$$

□

3. Error Bounds in the Case When $k = 1$

Before giving the estimations of error bounds of solutions to (1.2), we rewrite $y_i(n)$ as

$$y_i(n) = L_N^{(i)}(n) + \varepsilon_N^{(i)}(n), \quad i = 1, 2, \tag{3.1}$$

with

$$L_N^{(1)}(n) = [(n-2)!]^{q-p}\rho_1^n n^{\alpha_1}\sum_{s=0}^{N-1}\frac{c_s^{(1)}}{n^s},$$
$$L_N^{(2)}(n) = [(n-2)!]^{p}\rho_2^n n^{\alpha_2}\sum_{s=0}^{N-1}\frac{c_s^{(2)}}{n^s}, \tag{3.2}$$

and $\varepsilon_N^{(i)}(n)$, $i = 1,2$, being error terms. Then $\varepsilon_N^{(i)}(n)$, $i = 1,2$, satisfy inhomogeneous second-order linear difference equations

$$\varepsilon_N^{(i)}(n+2) + n^p a(n)\varepsilon_N^{(i)}(n+1) + n^q b(n)\varepsilon_N^{(i)}(n) = R_N^{(i)}(n), \quad i = 1,2, \tag{3.3}$$

where

$$R_N^{(i)}(n) = -\Big(L_N^{(i)}(n+2) + n^p a(n)L_N^{(i)}(n+1) + n^q b(n)L_N^{(i)}(n)\Big), \quad i = 1,2. \tag{3.4}$$

We know from [3] that

$$C_{R_1} = \sup_n\left\{n^N\left|\frac{R_N^{(1)}(n)}{(n!)^{q-p}\rho_1^n n^{\alpha_1}}\right|\right\}, \qquad C_{R_2} = \sup_n\left\{n^{N+1}\left|\frac{R_N^{(2)}(n)}{(n!)^{p}\rho_2^n n^{\alpha_2}}\right|\right\}. \tag{3.5}$$

3.1. The Error Bound for the Asymptotic Expansion of $y_1(n)$

Now we firstly estimate the error bound of the asymptotic expansion of $y_1(n)$ in the case $k = 1$. Let

$$x(n) = -\frac{(1-1/n)\rho_2^2(1+2/n)^{\alpha_2} - n^{-2}\rho_1^2(1+2/n)^{\alpha_1}}{\left[\rho_2(1-1/n)(1+1/n)^{\alpha_2} - \rho_1 n^{-1}(1+1/n)^{\alpha_1}\right](a_0+a_1/n)},$$
$$l(n) = -\frac{n(1-1/n)^p\left[\rho_2^2(1+2/n)^{\alpha_2} + (a_0+a_1/n)\rho_2(1+1/n)^{\alpha_2}x(n+1)\right]}{x(n)x(n+1)} - b_0 - \frac{b_1}{n}. \tag{3.6}$$

It can be easily verified that

$$z_1(n) = [(n-2)!]^{q-p}\rho_1^n n^{\alpha_1}\prod_{k=n}^{\infty}x(k),$$
$$z_2(n) = [(n-2)!]^{p}\rho_2^n n^{\alpha_2}\prod_{k=n}^{\infty}x(k) \tag{3.7}$$

are two linear independent solutions of the comparative difference equation

$$z(n+2) + n^p\left(a_0 + \frac{a_1}{n}\right)z(n+1) + n^q\left[b_0 + \frac{b_1}{n} + l(n)\right]z(n) = 0. \tag{3.8}$$

From the definition, we know that the two-term approximation of $x(n)$is

$$x(n) = 1 + \frac{a_0(a_1 - b_0/a_0) - pa_0^2 - (a_1 - pa_0)a_0 + b_0}{a_0^2}\frac{1}{n} + \omega(n), \tag{3.9}$$

where $\omega(n)$ is the reminder and the coefficient of $1/n$ is zero. So $C_x = \sup_n\{n^2|x(n) - 1|\}$ is a constant. And $l(n)$ satisfies $C_l = \sup_n\{n^2|l(n)|\}$ being a constant; here we have made use of the definitions of α_i, ρ_i in (1.5), (1.7), and $2p - q = 1$.

Equation (3.8) is a second-order linear difference equation with two known linear independent solutions. Its coefficients are quite similar to those in (3.3). This reminds us to rewrite (3.3) in the form similar to (3.8).

According to the coefficients in (3.8), we rewrite (3.3) as

$$\begin{aligned} &\varepsilon_N^{(1)}(n+2) + n^p\left(a_0 + \frac{a_1}{n}\right)\varepsilon_N^{(1)}(n+1) + n^q\left[b_0 + \frac{b_1}{n} + l(n)\right]\varepsilon_N^{(1)}(n) \\ &\quad = R_N^{(1)}(n) - n^p\left[a(n) - a_0 - \frac{a_1}{n}\right]\varepsilon_N^{(1)}(n+1) - n^q\left[b(n) - b_0 - \frac{b_1}{n} - l(n)\right]\varepsilon_N^{(1)}(n), \end{aligned} \tag{3.10}$$

where $a(n)$ and $b(n)$ are such that

$$C_a = \sup_{j\geqslant n}\left\{j^2\left|a(j) - a_0 - \frac{a_1}{j}\right|\right\}, \qquad C_b = \sup_{j\geqslant n}\left\{j^2\left|b(j) - b_0 - \frac{b_1}{j} - l(j)\right|\right\} \tag{3.11}$$

are finite. Equation (3.10) is a inhomogeneous second-order linear difference equation; its solution takes the form of a particular solution added to an arbitrary linear combination of solutions to the associated homogeneous linear difference equation(3.8).

From [10], any solution of the "summary equation"

$$\begin{aligned} \varepsilon_N^{(1)}(n) = \sum_{j=n}^{\infty} K(n,j)\Bigg\{ & R_N^{(1)}(j) - j^p\left[a(j) - a_0 - \frac{a_1}{j}\right]\varepsilon_N^{(1)}(j+1) \\ & -j^q\left[b(j) - b_0 - \frac{b_1}{j} - l(j)\right]\varepsilon_N^{(1)}(j)\Bigg\} \end{aligned} \tag{3.12}$$

is a solution of (3.10), where

$$K(n,j) = \frac{z_1(j+1)z_2(n) - z_1(n)z_2(j+1)}{z_1(j+2)z_2(j+1) - z_1(j+1)z_2(j+2)}. \tag{3.13}$$

Now we estimate the bound of the function $K(n,j)$.

Firstly we consider the denominator in $K(n,j)$. We get from(3.8)

$$\begin{aligned} &[z_1(n+2)z_2(n+1)-z_1(n+1)z_2(n+2)] \\ &\quad + n^q\left[b_0+\frac{b_1}{n}+l(n)\right][z_1(n)z_2(n+1)-z_1(n+1)z_2(n)]=0. \end{aligned} \tag{3.14}$$

Set the Wronskian of the two solutions of the comparative difference equation as

$$W(n)=z_1(n+1)z_2(n)-z_1(n)z_2(n+1); \tag{3.15}$$

we have

$$W(n+1)=n^q\left(b_0+\frac{b_1}{n}+l(n)\right)W(n). \tag{3.16}$$

From (3.16), we have

$$W(n+1)=W(2)(n!)^q b_0^{n-1}\prod_{k=2}^{n}\left(1+\frac{b_1}{b_0}\frac{1}{k}+\frac{l(k)}{b_0}\right). \tag{3.17}$$

From Lemma 3 of [5], we obtain

$$\begin{aligned} \exp(-k_1)(n+1)^{\mathrm{Re}(b_1/b_0)} &\leqslant \left|\prod_{k=m}^{n}\left(1+\frac{b_1}{b_0}\frac{1}{k}+\frac{l(k)}{b_0}\right)\right| \\ &\leqslant \exp(k_1)(n+1)^{\mathrm{Re}(b_1/b_0)}, \end{aligned} \tag{3.18}$$

where

$$k_1=\left|\frac{b_1}{b_0}\right|\left(\frac{1}{m}+\frac{1}{6m^2}+\frac{1}{60m^4}+\ln m\right)+\frac{\pi^2}{6}\widetilde{\sigma}_0, \tag{3.19}$$

$$\widetilde{\sigma}_0=\sup_k\left\{k^2\left|\ln\left[1+\frac{b_1}{b_0}\frac{1}{k}+\frac{l(k)}{b_0}\right]-\frac{b_1}{b_0}\frac{1}{k}\right|\right\}\quad (m<k<n); \tag{3.20}$$

m is an integer which is large enough such that $1+(b_1/b_0)(1/k)+l(k)/b_0>0$, when $k\geqslant m$.

Let $C^*=|\prod_{k=2}^{m-1}(1+(b_1/b_0)(1/k)+l(k)/b_0)|$, for the property of $l(k)$, we know that C^* is a constant. Then we obtain from (3.18)

$$|W(n+1)|\geqslant|W(2)|(n!)^q\left|b_0^{n-1}\right|C^*\exp(-k_1)(n+1)^{\mathrm{Re}(b_1/b_0)}. \tag{3.21}$$

Now considering the numerator in $K(n,j)$, we get

$$\begin{aligned} &z_1(j+1)z_2(n) - z_1(n)z_2(j+1) \\ &\quad = [(j-1)!]^{p-1}[(n-2)!]^{p-1}\prod_{k=j+1}^{\infty} x(k)\prod_{k=n}^{\infty} x(k) \\ &\quad\quad \times\Big[\rho_1^{j+1}\rho_2^{n}(j+1)^{\alpha_1}n^{\alpha_2}(n-2)! - \rho_1^{n}\rho_2^{j+1}(j+1)^{\alpha_2}n^{\alpha_1}(j-1)!\Big]. \end{aligned} \tag{3.22}$$

Here we have made use of $q-p=p-1$.

From Lemma 2 of [5], we have

$$\left|\prod_{k=j+1}^{\infty} x(k)\prod_{k=n}^{\infty} x(k)\right| \leqslant \exp\left(\frac{2\pi^2}{3}C_x\right), \tag{3.23}$$

where $C_x = \sup_n\{n^2|x(n)-1|\}$ is a constant. For the bound of $K(n,j)$, we set

$$K(n,j) = \frac{[(n-2)!]^{q-p}\rho_1^{n}n^{\alpha_1}}{(j!)^{q-p}\rho_1^{j}j^{\alpha_1}}\widetilde{K}(n,j), \tag{3.24}$$

then

$$\left|\widetilde{K}(n,j)\right| \leqslant |\mathrm{I}| + |\mathrm{II}|, \tag{3.25}$$

where

$$\begin{aligned} |\mathrm{I}| &= \left|\frac{(j!)^{q-p}\rho_1^{j}j^{\alpha_1}}{[(n-2)!]^{q-p}\rho_1^{n}n^{\alpha_1}}\right|\left|\frac{\exp\left((2\pi^2/3)C_x\right)[(j-1)!]^{p-1}[(n-2)!]^{p-1}}{|W(2)|(j!)^{q}\left|b_0^{j}\right|C^{*}\exp(-k_1)(j+1)^{\mathrm{Re}(b_1/b_0)}}\right| \\ &\quad\times\left|\rho_1^{n}\rho_2^{j+1}(j+1)^{\alpha_2}n^{\alpha_1}(j-1)!\right| \\ |\mathrm{II}| &= \left|\frac{(j!)^{q-p}\rho_1^{j}j^{\alpha_1}}{[(n-2)!]^{q-p}\rho_1^{n}n^{\alpha_1}}\right|\left|\frac{\exp\left((2\pi^2/3)C_x\right)[(j-1)!]^{p-1}[(n-2)!]^{p-1}}{|W(2)|(j!)^{q}\left|b_0^{j}\right|C^{*}\exp(-k_1)(j+1)^{\mathrm{Re}(b_1/b_0)}}\right| \\ &\quad\times\left|\rho_1^{j+1}\rho_2^{n}(j+1)^{\alpha_1}n^{\alpha_2}(n-2)!\right|. \end{aligned} \tag{3.26}$$

By simple calculations, we get

$$|\mathrm{I}| \leqslant \frac{\exp((2\pi^2/3)C_x + k_1)}{|W(2)|C^*} |\rho_2| \sup_{j \geqslant n} \left(1 + \frac{1}{j}\right)^{\alpha_2 - \mathrm{Re}(b_1/b_0)} j^{-1}. \tag{3.27}$$

Here we have made use of (1.5) and (1.7).

Since $|(n-2)!/(j-1)!(\rho_2/\rho_1)^{n-j}(n/j)^{\alpha_1-\alpha_2}| \leqslant 1$, we have

$$|\mathrm{II}| \leqslant \frac{\exp((2\pi^2/3)C_x + k_1)}{|W(2)|C^*} |\rho_1| \sup_{j \geqslant n} \left(1 + \frac{1}{j}\right)^{\alpha_1 - \mathrm{Re}(b_1/b_0)} j^{-1}. \tag{3.28}$$

Here we also have made use of (1.5) and (1.7).

Let

$$C_K = \frac{\exp((2\pi^2/3)C_x + k_1)}{|W(2)|C^*} \left[|\rho_2| \sup_{j \geqslant n} \left(1 + \frac{1}{j}\right)^{\alpha_2 - \mathrm{Re}(b_1/b_0)} + |\rho_1| \sup_{j \geqslant n} \left(1 + \frac{1}{j}\right)^{\alpha_1 - \mathrm{Re}(b_1/b_0)} \right], \tag{3.29}$$

we have from (3.24) the bound of $K(n,j)$

$$|K(n,j)| \leqslant C_K \left| \frac{[(n-2)!]^{p-1} \rho_1^n n^{\alpha_1}}{(j!)^{p-1} \rho_1^j j^{\alpha_1}} \right| j^{-1}. \tag{3.30}$$

For the bound of $\varepsilon_N^{(1)}(n)$, set $P(n) = [(n-2)!]^{q-p} \rho_1^n n^{\alpha_1}, p(n) = (n!)^{q-p} \rho_1^n n^{\alpha_1}, \theta = 1, R(j) = R_N^{(1)}(j), C_\phi = C_a, C_\psi = C_b, C_R = C_{R_1}, s = t = 2, C_\beta = \sup_{j \geqslant n}(1-1/j)^{1-p}, \beta = p-1, C_\alpha = \sup_{j \geqslant n}(1+1/j)^{\alpha_1}, \varsigma = 1$; we have from Lemma 2.1 that

$$\begin{aligned} \left|\varepsilon_N^{(1)}(n)\right| &\leq \left|[(n-2)!]^{q-p} |\rho_1^n| n^{\alpha_1}\right| n^{-N-1} \\ &\times \left| \frac{2C_{R_1} C_K}{N - 2C_k \left[2C_\alpha C_a |\rho_1| \sup_{j \geq n}(1+1/j)^{-N-5/2} + C_b C_\beta\right]} \right|, \end{aligned} \tag{3.31}$$

when

$$\lambda = \frac{2C_K}{N} \left(2C_\alpha C_a |\rho_1| \sup_{j \geqslant n} (1+1/j)^{-N-5/2} + C_b C_\beta \right) < 1, \tag{3.32}$$

that is, $N \geq n_0 = [2C_K(2C_\alpha C_a|\rho_1| \sup_{j \geqslant n}(1+1/j)^{-N-5/2} + C_b C_\beta) - 1]$ and $j \geq n \geq N \geq n_0$.

3.2. The Error Bound for the Asymptotic Expansion of $y_2(n)$

Now we estimate the error bound of the asymptotic expansion of the linear independent solution $y_2(n)$ to the original difference equation as $k = 1$. Let

$$\varepsilon_N^{(2)}(n) = y_1(n)\delta_N(n). \tag{3.33}$$

From (3.3), we have

$$y_1(n+2)\delta_N(n+2) + n^p a(n) y_1(n+1)\delta_N(n+1) + n^q b(n) y_1(n)\delta_N(n) = R_N^{(2)}(n). \tag{3.34}$$

For $y_1(n)$ being a solution of (1.2), let

$$\Delta_N(n) = \delta_N(n+1) - \delta_N(n); \tag{3.35}$$

then $\Delta_N(n)$ satisfies the first-order linear difference equation

$$y_1(n+2)\Delta_N(n+1) - n^q b(n) y_1(n)\Delta_N(n) = R_N^{(2)}(n). \tag{3.36}$$

The solution of (3.36) is

$$\Delta_N(n) = -\sum_{i=n}^{\infty} \frac{X(n)}{X(i+1)} \frac{R_N^{(2)}(i)}{y_1(i+2)}, \tag{3.37}$$

where $X(n) = X(m)\prod_{j=m}^{n-1}(j^q b(j) y_1(j)/y_1(j+2))$, $X(m)$ is a constant, and m is an integer which is large enough such that when $i \geqslant n \geqslant m$,

$$\left|y_1(i)\right| = [(i-2)!]^{p-1}\left|\rho_1^i\right| i^{\operatorname{Re}\alpha_1}\left|1+\varepsilon_1^{(1)}(i)\right| \geqslant \frac{1}{2}[(i-2)!]^{p-1}\left|\rho_1^i\right| i^{\operatorname{Re}\alpha_1}. \tag{3.38}$$

The two-term approximation of $j^q b(j) y_1(j)/y_1(j+2)$ is

$$\frac{j^q b(j) y_1(j)}{y_1(j+2)} = \frac{b_0 j}{\rho_1^2}\left[1 + \frac{\alpha_2 - \alpha_1}{j} + \sigma(j)\right], \tag{3.39}$$

where $\sigma(j)$ is the reminder and $\sigma_0 = \sup_j\{j^2|\sigma(j)|\}$ is a constant.

From Lemma 3 of [5], we obtain

$$\begin{aligned} &|X(m)|\left|\frac{b_0}{\rho_1^2}\right|^{n-m} \frac{(n-1)!}{(m-1)!} \exp(-k_1) n^{\operatorname{Re}(\alpha_2-\alpha_1)} \\ &\quad \leqslant |X(n)| \leqslant |X(m)|\left|\frac{b_0}{\rho_1^2}\right|^{n-m} \frac{(n-1)!}{(m-1)!} \exp(k_1) n^{\operatorname{Re}(\alpha_2-\alpha_1)}, \end{aligned} \tag{3.40}$$

where

$$k_1 = |\alpha_2 - \alpha_1|\left(\frac{1}{m} + \frac{1}{6m^2} + \frac{1}{60m^4} + \ln m\right) + \frac{\pi^2}{6}\tilde{\sigma}_0,$$
$$\tilde{\sigma}_0 = \sup_j\left\{ j^2\left|\ln\left[1 + \frac{\alpha_2 - \alpha_1}{j} + \sigma(j)\right] - \frac{\alpha_2 - \alpha_1}{j}\right|\right\}, \tag{3.41}$$

are constants.

Substituting (3.38) and (3.40) into (3.37), we get

$$\begin{aligned}|\Delta_N(n)| &\leq \frac{2C_{R_2}e^{2k_1}}{|b_0|}(n-1)!n^{\operatorname{Re}(\alpha_2-\alpha_1)}\left|\frac{\rho_2}{\rho_1}\right|^n \\ &\quad\times\sup_{i\geq n}\left(\frac{i}{i+1}\right)^{\operatorname{Re}\alpha_2}\sup_{i\geq n}\left(\frac{i+1}{i+2}\right)^{\operatorname{Re}\alpha_1}\sum_{i=n}^{\infty} i^{-N-1} \\ &\leqslant \frac{4C_{R_2}e^{2k_1}}{|b_0|}(n-1)!n^{\operatorname{Re}(\alpha_2-\alpha_1)}\left|\frac{\rho_2}{\rho_1}\right|^n \\ &\quad\times\sup_{i\geqslant n}\left(\frac{i}{i+1}\right)^{\operatorname{Re}\alpha_2}\sup_{i\geqslant n}\left(\frac{i+1}{i+2}\right)^{\operatorname{Re}\alpha_1}\frac{n^{-N}}{N}.\end{aligned} \tag{3.42}$$

Let $\mu = (4C_{R_2}e^{2k_1}/|b_0|)\sup_{i\geqslant n}(i/(i+1))^{\operatorname{Re}\alpha_2}\sup_{i\geqslant n}((i+1)/(i+2))^{\operatorname{Re}\alpha_1}1/N$; then

$$|\Delta_N(n)| \leqslant \mu(n-1)!\left|\frac{\rho_2}{\rho_1}\right|^n n^{\operatorname{Re}(\alpha_2-\alpha_1)-N.} \tag{3.43}$$

From (3.35), we have

$$\delta_N(n) = \delta_N(m) + \sum_{i=m}^{n-1}\Delta_N(i) \quad (n \geqslant i \geqslant m), \tag{3.44}$$

where $\delta_N(m)$ is a constant. Let $\delta_N(m) = 0$; we have

$$|\delta_N(n)| \leqslant \sum_{i=m}^{n-1}|\Delta_N(i)| \leqslant \mu\sum_{i=m}^{n-1}(i-1)!\left|\frac{\rho_2}{\rho_1}\right|^i i^{\operatorname{Re}(\alpha_2-\alpha_1)-N}. \tag{3.45}$$

For $\sum_{i=m}^{n-1}(i-1)!|\rho_2/\rho_1|^i i^{\operatorname{Re}(\alpha_2-\alpha_1)-N}$, there exists a positive integer I_0 such that

$$\frac{i!|\rho_2/\rho_1|^{i+1}(i+1)^{\operatorname{Re}(\alpha_2-\alpha_1)-N}}{(i-1)!|\rho_2/\rho_1|^i i^{\operatorname{Re}(\alpha_2-\alpha_1)-N}} = i\left|\frac{\rho_2}{\rho_1}\right|\left(1+\frac{1}{i}\right)^{\operatorname{Re}(\alpha_2-\alpha_1)-N} \geqslant 1, \tag{3.46}$$

when $i \geqslant I_0$. Thus the sequence $\{(i-1)!|\rho_2/\rho_1|^i i^{\operatorname{Re}(\alpha_2-\alpha_1)-N}\}$ is increasing when $i \geqslant I_0 \geqslant m$.

Let $M_0 = \sum_{i=m}^{I_0-1}(i-1)!|\rho_2/\rho_1|^i i^{\mathrm{Re}(\alpha_2-\alpha_1)-N}$; then

$$\begin{aligned}\sum_{i=m}^{n-1}[(i-1)!]\left|\frac{\rho_2}{\rho_1}\right|^i i^{\mathrm{Re}(\alpha_2-\alpha_1)-N} &\leqslant M_0 + n[(n-1)!]\left|\frac{\rho_2}{\rho_1}\right|^n n^{\mathrm{Re}(\alpha_2-\alpha_1)-N} \\ &\leqslant 2[(n-2)!]\left|\frac{\rho_2}{\rho_1}\right|^n n^{\mathrm{Re}(\alpha_2-\alpha_1)-N+1},\end{aligned} \tag{3.47}$$

where $\lim_{n\to\infty}(n-2)!|\rho_2/\rho_1|^n n^{\mathrm{Re}(\alpha_2-\alpha_1)-N+2} = \infty$. Hence

$$|\delta_N(n)| \leqslant 2\mu(n-2)!\left|\frac{\rho_2}{\rho_1}\right|^n n^{\mathrm{Re}(\alpha_2-\alpha_1)-N+2}. \tag{3.48}$$

From (3.33), we obtain

$$\begin{aligned}\left|\varepsilon_N^{(2)}(n)\right| &= |y_1(n)\delta_N(n)| \\ &\leq 2\mu \sup_{n\geqslant m}\left\{[(n-2)!]^{1-p}\left|\rho_1\right|^{-n} n^{-\mathrm{Re}\,\alpha_1}|y_1(n)|\right\} \\ &\quad\times[(n-2)!]^p|\rho_2|^n n^{\mathrm{Re}\,\alpha_2-N+2}.\end{aligned} \tag{3.49}$$

Thus we complete the estimate of error bounds to asymptotic expansions of solutions of (1.2) as $k = 1$.

4. Error Bounds in Case When $k > 1$

Here we also rewrite $y_i(n)$ as

$$y_i(n) = L_N^{(i)}(n) + \varepsilon_N^{(i)}(n), \quad i = 1, 2, \tag{4.1}$$

with

$$\begin{aligned}L_N^{(1)}(n) &= [(n-2)!]^{q-p}\rho_1^n n^{\alpha_1}\sum_{s=0}^{N-1}\frac{c_s^{(1)}}{n^s}, \\ L_N^{(2)}(n) &= [(n-2)!]^p\rho_2^n n^{\alpha_2}\sum_{s=0}^{N-1}\frac{c_s^{(2)}}{n^s},\end{aligned} \tag{4.2}$$

and $\varepsilon_N^{(i)}(n)$, $i = 1,2$, are error terms. Then $\varepsilon_N^{(i)}(n)$, $i = 1,2$, satisfy the inhomogeneous second-order linear difference equations

$$\varepsilon_N^{(i)}(n+2) + n^p a(n)\varepsilon_N^{(i)}(n+1) + n^q b(n)\varepsilon_N^{(i)}(n) = R_N^{(i)}(n), \quad i = 1, 2, \tag{4.3}$$

where

$$R_N^{(i)}(n) = -\left(L_N^{(i)}(n+2) + n^p a(n) L_N^{(i)}(n+1) + n^q b(n) L_N^{(i)}(n)\right), \qquad i = 1, 2. \tag{4.4}$$

We know from [3] that

$$C_{R_1} = \sup_n \left\{ n^N \left| \frac{R_N^{(1)}(n)}{(n!)^{q-p} \rho_1^n n^{\alpha_1}} \right| \right\}, \qquad C_{R_2} = \sup_n \left\{ n^{N+1} \left| \frac{R_N^{(2)}(n)}{(n!)^{p} \rho_2^n n^{\alpha_2}} \right| \right\}. \tag{4.5}$$

4.1. The Error Bound for the Asymptotic Expansion of $y_1(n)$

Now let us come to the case when $k > 1$. This time a difference equation which has two known linear independent solutions is also constructed for the purpose of comparison for (1.2).

Since

$$-n^{-q} b^{-1}(n) \frac{\rho_2 n^p (n+2)^{\alpha_2}/(n+1)^{\alpha_2} - \rho_1 n^{q-p}(n+2)^{\alpha_1}/(n+1)^{\alpha_1}}{n^{\alpha_2}/\rho_2 (n-1)^p (n+1)^{\alpha_2} - n^{\alpha_1}/\rho_1 (n-1)^{q-p}(n+1)^{\alpha_1}} = 1 + \frac{A}{n^2} + f_1(n), \tag{4.6}$$

where

$$\begin{aligned} A = {} & -1 + \frac{1}{2}(p-q)\left(\frac{2b_1}{b_0} - p + q + 1\right) + \frac{a_1}{2a_0}\left(\frac{a_1}{a_0} - 1\right) \\ & + \frac{1}{2}\left(\frac{b_1}{b_0} - \frac{a_1}{a_0} - p + q\right)\left(\frac{a_1}{a_0} + \frac{b_1}{b_0} - p + q - 3\right), \end{aligned} \tag{4.7}$$

is a constant and $n^3|f_1(n)| \leqslant B$ (B is a constant), from Lemma 2.2, we know the difference equation

$$x(n)x(n+1) = 1 + \frac{A}{n^2} + f_1(n), \tag{4.8}$$

with condition $x(\infty) = 1$ having a solution $x(n)$ such that

$$C_x = \sup_n \left\{ n^2 |x(n) - 1| \right\} \tag{4.9}$$

is a constant. And the function

$$l(n) = -a_0 - \frac{a_1}{n} - \frac{n^{-k} b(n) x(n)}{(1-1/n)^p (1+1/n)^{\alpha_2} \rho_2} - \frac{\rho_2}{x(n+1)}\left(1 + \frac{2}{n}\right)^{\alpha_2}\left(1 + \frac{1}{n}\right)^{-\alpha_2} \tag{4.10}$$

such that

$$C_l = \sup_n \left\{ n^2 |l(n)| \right\} \tag{4.11}$$

is a constant. Here we have made use of the definitions of ρ_1, α_1, ρ_2, α_2 in (1.9), (1.11) and $q - p = p - k$.

Obviously functions

$$\begin{aligned} z_1(n) &= [(n-2)!]^{q-p} \rho_1^n n^{\alpha_1} \prod_{k=n}^{\infty} x(k), \\ z_2(n) &= [(n-2)!]^{p} \rho_2^n n^{\alpha_2} \prod_{k=n}^{\infty} x(k) \end{aligned} \tag{4.12}$$

are two linear independent solutions of the difference equation

$$z(n+2) + n^p \left(a_0 + \frac{a_1}{n} + l(n) \right) z(n+1) + n^q b(n) z(n) = 0. \tag{4.13}$$

This difference equation(4.13) can be regarded as the comparative equation of (4.3). Rewriting (4.3) in the form similar to the comparative difference equation (4.13), we get

$$\begin{aligned} &\varepsilon_N^{(1)}(n+2) + n^p \left(a_0 + \frac{a_1}{n} + l(n) \right) \varepsilon_N^{(1)}(n+1) + n^q b(n) \varepsilon_N^{(1)}(n) \\ &\quad = R_N^{(1)}(n) - n^p \left(a(n) - a_0 - \frac{a_1}{n} - l(n) \right) \varepsilon_N^{(1)}(n+1), \end{aligned} \tag{4.14}$$

where $a(n)$ has the property that $C_a = \sup_n \{ n^2 |a(n) - a_0 - a_1/n - l(n)| \}$ is a constant. Equation (4.14) is an inhomogeneous second-order linear difference equation; its solution takes the form of a particular solution added to an arbitrary linear combination of solutions to the associated homogeneous linear difference equation (4.13).

From [10], any solution of the "summary equation"

$$\varepsilon_N^{(1)}(n) = \sum_{j=n}^{\infty} K(n,j) \left\{ R_N^{(1)}(j) - j^p \left[a(j) - a_0 - \frac{a_1}{j} - l(j) \right] \varepsilon_N^{(1)}(j+1) \right\}, \tag{4.15}$$

where

$$K(n,j) = \frac{z_1(j+1) z_2(n) - z_1(n) z_2(j+1)}{z_1(j+2) z_2(j+1) - z_1(j+1) z_2(j+2)}, \tag{4.16}$$

is a solution of (4.14).

Similar to Section 3.1, we have

$$\left|\widetilde{K}(n,j)\right| \leqslant \frac{\exp\left((2\pi^2/3)C_x + k_1\right)}{|W(2)|C^*} j^{-k} \times\left[\left|\rho_2\right|\sup_{j\geqslant n}\left(1+\frac{1}{j}\right)^{\alpha_2-\mathrm{Re}(\mathrm{b}_1/\mathrm{b}_0)} + \left|\rho_1\right|\sup_{j\geqslant n}\left(1+\frac{1}{j}\right)^{\alpha_1-\mathrm{Re}(\mathrm{b}_1/\mathrm{b}_0)}\right]. \tag{4.17}$$

Let

$$C_K = \frac{\exp\left((2\pi^2/3)C_x + k_1\right)}{|W(2)|C^*} \times\left[\left|\rho_2\right|\sup_{j\geqslant n}\left(1+\frac{1}{j}\right)^{\alpha_2-\mathrm{Re}(\mathrm{b}_1/\mathrm{b}_0)} + \left|\rho_1\right|\sup_{j\geqslant n}\left(1+\frac{1}{j}\right)^{\alpha_1-\mathrm{Re}(\mathrm{b}_1/\mathrm{b}_0)}\right]; \tag{4.18}$$

we get

$$\left|K(n,j)\right| \leqslant C_K\left|\frac{[(n-2)!]^{p-k}\rho_1^n n^{\alpha_1}}{(j!)^{p-k}\rho_1^j j^{\alpha_1}}\right|. \tag{4.19}$$

Set $P(n) = [(n-2)!]^{p-k}\rho_1^n n^{\alpha_1}$, $p(n) = (n!)^{p-k}\rho_1^n n^{\alpha_1}$, $\theta = k$, $R(j) = R_N^{(1)}(j)$, $C_\phi = C_a$, $C_\varphi = 0$, $C_R = C_{R_1}$, $s = 2$, $\beta = p-k$, $C_\beta = \sup_{j\geqslant n}(1-1/j)^{-(p-k)}$, $C_\alpha = \sup_{j\geqslant n}(1+1/j)^{\alpha_1}$, $\varsigma = 1$; we have from Lemma 2.1 that

$$\left|\varepsilon_N^{(1)}(n)\right| \leqslant \left|[(n-2)!]^{p-k}\rho_1^n n^{\alpha_1}\right| n^{-N-K}\left|\frac{2C_{R_1}C_K}{N+k-4C_kC_\alpha C_a\left|\rho_1\right|\sup_{j\geqslant n}(1+1/j)^{-N-k-1/2}}\right| \tag{4.20}$$

when

$$\lambda = \frac{4C_K}{N+k}C_\alpha C_a\left|\rho_1\right|\sup_{j\geqslant n}\left(1+\frac{1}{j}\right)^{-N-k-1/2} < 1. \tag{4.21}$$

that is,

$$N \geqslant n_0 = \left[4C_K C_\alpha C_a|\rho_1|\sup_{j\geqslant n}\left(1+\frac{1}{j}\right)^{-N-k-1/2} - k\right], \quad j \geqslant n \geqslant N \geqslant n_0.$$

4.2. The Error Bound for the Asymptotic Expansion of $y_2(n)$

Let

$$\varepsilon_N^{(2)}(n) = y_1(n)\delta_N(n). \tag{4.22}$$

From (3.3), we have

$$y_1(n+2)\delta_N(n+2) + n^p a(n) y_1(n+1)\delta_N(n+1) + n^q b(n) y_1(n)\delta_N(n) = R_N^{(2)}(n). \tag{4.23}$$

Using the method employed in Section 3.2, it is not difficult to obtain

$$\begin{aligned} \left|\varepsilon_N^{(2)}(n)\right| &= \left|y_1(n)\delta_N(n)\right| \\ &\leqslant 2\mu \sup_{n\geqslant m}\left\{[(n-2)!]^k \left|\rho_1\right|^{-n} n^{-\operatorname{Re}\alpha_1}\left|y_1(n)\right|\right\}\left|\rho_2\right|^n n^{\operatorname{Re}\alpha_2 - N+k+1}. \end{aligned} \tag{4.24}$$

Now we completed the estimate of the error bounds for asymptotic solutions to second order linear difference equations in the first case. For the second case, we leave it to the second part of this paper: Error Bound for Asymptotic Solutions of Second-order Linear Difference Equation II: the second case.

In the rest of this paper, we would like to give an example to show how to use the results of this paper to obtain error bounds of asymptotic solutons to second-order linear difference equations. Here the difference equation is

$$y_{n+2} - y_{n+1} + \frac{1}{n+2} y_n = 0. \tag{4.25}$$

It is a special case of the equation

$$(n+1) f_{n+1}^{\alpha}(x) - (n+\alpha) x f_n^{\alpha}(x) + f_{n-1}^{\alpha}(x) = 0, \tag{4.26}$$

$(\alpha = 1, x = 1)$, which is satisfied by Tricomi-Carlitz polynomials. By calculation, the constant C_K in (3.30) is $(1728/5)e^{2\pi^2}$. So (4.25) has a solution

$$y_1(n) = \frac{1}{(n-2)!n^2}(1+\varepsilon_1(n)), \tag{4.27}$$

for $n \geq 3$ with the error term $\varepsilon_1(n)$ satisfing

$$|\varepsilon_1(n)| \leqslant \frac{864}{25} e^{2\pi^2} n^{-1}. \tag{4.28}$$

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