

## Research Article

# Attractor for a Viscous Coupled Camassa-Holm Equation

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The global existence of solution to a viscous coupled Camassa-Holm equation with the periodic boundary condition is investigated. We obtain the compact and bounded absorbing set and the existence of the global attractor for the viscous coupled Camassa-Holm equation in  $H^2$  by uniform prior estimate.

## 1. Introduction

The Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad x \in \mathbb{R}, t > 0, \quad (1.1)$$

has been paid considerable attention due to its rich phenomenology all the time. Its abstract derivation was first discovered by Fuchssteiner and Fokas [1], while in the physical derivation of Camassa and Holm (see [2, 3]), the equation models unidirectional propagation of shallow water waves, and  $u(x, t)$  represents the fluid velocity in the  $x$  direction, equivalently the height of the fluid's free surface above a flat bottom. They also found that the solitary waves interact like solitons. Unlike the Korteweg-de Vries equation (which is an approximation to the equations of motion), this model is obtained by approximating directly in the Hamiltonian for Euler's equations in the shallow water regime (see [3, 4]). Equation (1.1) retains higher-order terms in a small amplitude expansion of incompressible Euler's equations for unidirectional motion of waves at the free surface under the influence of gravity. Dropping these terms leads to the BBM equation, or at the same order, the KdV equation. The Camassa-Holm has quite a few interesting features: it admits solitary waves called "peakons"

with the form of  $u = ce^{-|x-ct|}$ ,  $x \in \mathbb{R}$ ,  $c > 0$ . The peakons of (1.1) are orbitally stable [5]—that is, their shape is stable under small perturbations and therefore these waves are recognized physically. For waves that approximate the peakons in a special way, a stability result was proved by a variation method [6]. This is in sharp contrast to the Korteweg-de Vries equation, where solitary waves are generally smooth. The peaked traveling waves of the Camassa-Holm equation replicate a feature that is characteristic for waves of great height—waves of the largest amplitude that are exact solutions of the governing equations for water waves (see [7, 8]). A breaking wave is a solution which remains bounded but whose slope becomes unbounded in finite time, and, in contrast to the KdV equation, the Camassa-Holm equation models breaking waves [9], as well as a breaking rod (see [4, 10]), since the equation models the propagation of axisymmetric waves in hyperelastic rods. After breaking, the solution can be continued either as a global conservative weak solution or as a global dissipative solution (see [11–13]). Peakons interact “elastically” in the manner typical of all solitons, and their wave dynamics are now well understood (see [3, 14, 15]). Some authors have even argued recently that the Camassa-Holm equation might be relevant to the modeling of tsunamis (see [16, 17]). Moreover, the equation has a bi-Hamiltonian structure [2]. As the Camassa-Holm is completely integrable, it has many conserved quantities. Especially for smooth solutions, the quantities

$$\int_{\mathbb{R}} u \, dx, \quad \int_{\mathbb{R}} (u^2 + u_x^2) \, dx, \quad \int_{\mathbb{R}} (u^3 + uu_x^2) \, dx \quad (1.2)$$

are all time independent [18].

Up to now, great efforts have been already devoted to the Camassa-Holm equation. A. Constantin (see [19–21]) considered the Cauchy problem, inverse spectral problem, and inverse scattering transform for Camassa-Holm equation, proving that the corresponding solution to (1.1) does not exist globally for smooth initial data. Rui et al. (see [22, 23]) employed both bifurcation method and numerical simulation to investigate bounded traveling waves of (1.1) in a general compressible hyperelastic rod. Lenells [24] used the inverse scattering transform to show that a solution of the Camassa-Holm equation is identically zero whenever it vanishes on two horizontal half-lines in the  $x - t$  space. In particular, a solution that has compact support at two different times vanishes everywhere, proving that the Camassa-Holm equation has infinite propagation speed. Cohen et al. [25] presented two new multisymplectic formulations for the Camassa-Holm equation, and the associated local conservation laws were shown to correspond to certain well-known Hamiltonian functionals. The multisymplectic discretisation of each formulation was exemplified by means of the Euler box scheme. Yiping Meng and Lixin Tian [26] investigated the boundary control of the viscous generalized Camassa-Holm equation on  $[0, 1]$ . Long et al. [27] obtained the loop soliton solution and periodic loop soliton solution [28], solitary wave solution and solitary cusp wave solution and smooth periodic wave solution and nonsmooth periodic wave solution of (1.1) and also discussed their dynamic characters and relations by the integral bifurcation method. Moreover, Ding and Tian (see [29, 30]) considered the existence of the global solution to dissipative Camassa-Holm equation and the global attractor of semigroup of solutions of dissipative Camassa-Holm equation in  $H^2$ . Olson [31] showed that the Cauchy problem for a higher-order modification of (1.1) is locally well posed for initial data in  $H^s(\mathbb{R})$  for  $s > s'$ , where  $1/4 \leq s' < 1/2$  and the value of  $s'$  depends on the order of equation, proved the existence and uniqueness of solutions of (1.1) by a contraction mapping argument. Moreover, Zhou and Tian [32] investigated the

initial boundary value problem of a generalized Camassa-Holm equation with dissipation and established local well-posedness of this closed-loop system by using Kato's theorem for abstract quasilinear evolution equation of hyperbolic type. Then they obtained a conservation law that enables us to present a blowup result by using multiplier technique. Lixin Tian et al. [33] discussed optimal control of the viscous Camassa-Holm equation; they deduce that the norm of solution is related to the control item and initial value in the special Hilbert space according to variational method, optimal control theories and distributed parameter system control theories, The optimal control of the viscous Camassa-Holm equation under boundary condition was given, and the existence of optimal solution of the viscous Camassa-Holm equation was proved. Well-posedness problem and scattering problem for DGH equation were also discussed in [34].

On the basis of deformation of bi-Hamiltonian structure of the hydrodynamic type, Chen et al. [35] obtained the following two-component generalization of (1.1):

$$\begin{aligned}u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \rho\rho_x &= 0, \\ \rho_t + (\rho u)_x &= 0.\end{aligned}\tag{1.3}$$

Equation (1.3) is one of many multicomponent generalizations which are integrable (see [35–37]). It has a Lax pair, and it is bi-Hamiltonian. Constantin and Ivanov [36] showed how (1.3) arises in shallow water theory, and it was derived from the Green-Naghdi equations by using expansions in terms of physical parameters. Recently, the infinite propagation speed property for (1.3) was proved in [38]. Escher et al. [39] probed into well-posedness and blowup phenomena of the two-component Camassa-Holm equation in details. Chen et al. [35] obtained solutions of (1.3) by a reciprocal transformation between (1.3) and the first negative flow of the AKNS hierarchy and stated some examples of peakon and multikink solutions of (1.3). Guan and Yin [40] presented a new global existence result and several new blowup results of strong solutions to (1.3) as  $\rho = \bar{\rho} - 1$ , improving considerably earlier results. Jibin Li and Yishen Li [41] obtained the existence of solitary wave solutions, kink and antikink wave solutions, uncountable infinite many breaking wave solutions, and smooth and nonsmooth periodic wave solutions with the method of dynamical systems to the two-component generalization of the Camassa-Holm equation. Yujuan Wang et al. [42] showed that the two-component Camassa-Holm equation possesses a global continuous semigroup of weak conservative solutions for initial data. In [43] a link between central extensions of superconformal algebra and a supersymmetric two-component generalization of the Camassa-Holm equation was concerned. Deformations of superconformal algebra give rise to two compatible bracket structures. For  $\rho \equiv 0$ , the system (1.3) particularizes to the Camassa-Holm equation which is a re-expression of geodesic flow on the diffeomorphism group of the circle (see [44, 45]).

We know that it is of great use to construct an interacting system of equations [37]:

$$\begin{aligned}m_t &= -3m(2u_x + v_x) - m_x(2u + v), & m &= u - u_{xx}, \\ n_t &= -2n(2u_x + v_x) - n_x(2u + v), & n &= v - v_{xx},\end{aligned}\tag{1.4}$$

as  $n = v = 0$  and  $m = u = 0$ , it, respectively, leads to the Degasperis-Procesi equation and Camassa-Holm equation. Three independent conserved quantities have been obtained

as follows:

$$\begin{aligned} & \int (m+n) dx, \quad \int n^\lambda m^{-(1-2\lambda)/3} dx, \\ & \int \left( -9n_x^2 n^{\lambda-2} m^{-(1+2\lambda)/3} + 12n_x m_x n^{\lambda-1} m^{-(4+2\lambda)/3} - 4m_x^2 n^\lambda m^{-(7+2\lambda)/3} \right) dx, \end{aligned} \quad (1.5)$$

here  $\lambda$  is an arbitrary constant.

Ying Fu and Changzheng Qu [46] considered the following coupled Camassa-Holm equation:

$$\begin{aligned} m_t &= 2mu_x + m_x u + (mv)_x + nv_x, & m &= u - u_{xx}, \\ n_t &= 2nv_x + n_x v + (nu)_x + mu_x, & n &= v - v_{xx}, \end{aligned} \quad (1.6)$$

which has peakon solitons in the form of a superposition of multipeakons. It has the following conserved quantities:

$$\begin{aligned} G_1(u) &= \int_R u dx, & G_2(v) &= \int_R v dx, & G_3(u) &= \int_R m dx, \\ G_4(v) &= \int_R n dx, & G_5(u, v) &= \int_R (u^2 + u_x^2 + v^2 + v_x^2) dx. \end{aligned} \quad (1.7)$$

They investigated local well-posedness and blowup solutions of (1.6) by means of Kato's semigroup approach to nonlinear hyperbolic evolution equation and obtained a criterion and condition on the initial data guaranteeing the development of singularities in finite time for strong solutions of (1.6) by energy estimates; moreover, an existence result for a class of local weak solutions was also given. They also showed that the solution of (1.6) is

$$\frac{d}{dt} \int_R (u^2 + \alpha u_x^2 + \beta v^2 + \gamma v_x^2) dx = 0, \quad (1.8)$$

for some positive constants  $\alpha, \beta, \gamma$ .

In the field of infinite-dimensional dynamical systems, one of the most important issues is to obtain the existence of global attractors for the semigroups of solutions associated with some concrete partial differential equations. For instance, Yongsheng Li and Xingyu Yan [47] studied the existence and regularity of the global attractor for a weakly damped forced shallow water equation in  $H^1(R)$ . Tian et al. [48] studied the global attractor for the viscous weakly damped forced Korteweg-de Vries equations in  $H^1(R)$ . Yanhong Zhang and Chengkui Zhong [49] investigated the existence of global attractors for a nonlinear wave equation. Lixin Tian and Ruihua Tian [50] studied the attractor for the two-dimensional weakly damped KdV equation in belt field. Ying Xu and Lixin Tian [51] investigated attractor for a coupled nonhomogeneous Camassa-Holm equation with periodic boundary condition. Lixin Tian and Jinglin Fan [52] discussed the global attractors for the viscous Degasperis-Procesi equation in  $H^2$ . Lixin Tian and Ying Gao [53] obtained global attractors for the viscous Fornberg-Whitham equation [54] in  $H^2$ . Here we investigate the existence of global attractor for a viscous coupled Camassa-Holm equation with the periodic boundary condition in  $H^2$

as follows:

$$m_t - \varepsilon m_{xx} = 2mu_x + m_x u + (mv)_x + nv_x, \quad m = u - u_{xx}, \tag{1.9}$$

$$n_t - \varepsilon n_{xx} = 2nv_x + n_x v + (nu)_x + mu_x, \quad n = v - v_{xx}, \tag{1.10}$$

$$u(x, 0) = u_0, \quad v(x, 0) = v_0, \tag{1.11}$$

$$u(0, t) = u(D, t), \quad u'(0, t) = u'(D, t), \quad u''(0, t) = u''(D, t), \tag{1.12}$$

$$v(0, t) = v(D, t), \quad v'(0, t) = v'(D, t), \quad v''(0, t) = v''(D, t), \tag{1.13}$$

where  $t > 0, x \in \Omega, \Omega = [0, D], D > 0, u, v \in H = L^2(\Omega)$ . To the authors' knowledge, the problem of global attractor for (1.9)–(1.13) has not been discussed in previous publications.

Our paper is organized as follows. In Section 2, we give the main definitions and Lemmas. In Section 3, main results are presented, as the core of the paper, and the proofs of the main theorems are completed. Firstly, we prove that (1.9)–(1.13) has a unique solution in infinite time interval then obtain the existence of global solution of (1.9)–(1.13) in  $H^2(\Omega)$  by prior estimates. Meanwhile we obtain that the semigroup of the solution operator has an absorbing set. Finally, we demonstrate the long-time behavior of solution of (1.9)–(1.13) that is described by global attractor. In brief, we obtain the existence of the global attractor for (1.9)–(1.13) in  $H^2(\Omega)$ .

## 2. Preliminaries

*Definition 2.1.* Let  $(,)$  stand for the  $L^2$  inner product and  $\| \cdot \|$  the corresponding  $L^2$  norm. one also denotes

$$\begin{aligned} \|u\|_{L^2(\Omega)} &\triangleq |u|, \|Du\|_{L^2(\Omega)} \triangleq \|u\|, \\ \|D^m u\|_{L^2(\Omega)} &\triangleq |D^m u|, \|u\|_{L^\infty(\Omega)} \triangleq \operatorname{esssup}_{x \in \Omega} |u(x)|, \end{aligned} \tag{2.1}$$

and  $A = -\Delta$ , where  $\Delta$  is Laplace operators and  $A$  is a self-adjoint positive operators with compact inverse. The eigenvalue of  $A$  is  $\lambda_k$  satisfying  $0 < \lambda_1 \leq \lambda_2 \leq \dots \lambda_k, \lambda_k \rightarrow \infty$  as  $k \rightarrow \infty, A\omega_k = \lambda_k \omega_k$ , where  $\omega_k$  is the corresponding eigenvector of  $A$ . For simplicity we will give the following inequalities and only refer to their names wherever necessary.

**Lemma 2.2** (consistent Gronwall inequality). *Assume that  $g(t), y(t),$  and  $h(t)$  are three positive locally integrable functions defined on  $[t_0, +\infty], y'(t)$  is a locally integrable function over  $[t_0, +\infty],$  satisfying*

$$\begin{aligned} y'(t) &\leq g(t)y(t) + h(t), \quad \forall t \geq t_0, \\ \int_t^{t+r} g(s) ds &\leq k_1, \quad \int_t^{t+r} h(s) ds \leq k_2, \quad \int_t^{t+r} y(s) ds \leq k_3, \quad \forall t \geq t_0, \end{aligned} \tag{2.2}$$

where  $r, k_1, k_2,$  and  $k_3$  are positive constants. one can get

$$y(t+r) \leq \left( \frac{k_3}{r} + k_2 \right) \exp(k_1), \quad \forall t \geq t_0. \tag{2.3}$$

**Lemma 2.3** (Sobolev inequality). *Suppose that  $u \in L_q(\Omega) \cap W_0^{m,r}(\Omega)$ ,  $\Omega \subset R^k$ ,  $1 \leq q, r \leq \infty$ ,  $0 \leq j \leq m$ ,  $j/m \leq \alpha < 1$ , and  $1 \leq p < \infty$ , and there exists a constant  $c$ , such that*

$$\|D^j u\|_{L^p} \leq c \|D^m u\|_{L^r}^\alpha \|u\|_{L^q}^{1-\alpha}, \quad \frac{1}{p} = \frac{i}{k} + \left(\frac{1}{r} - \frac{m}{k}\right) + (1-\alpha)\frac{1}{q}. \quad (2.4)$$

**Lemma 2.4** (Young inequality).  *$ab \leq (\varepsilon a^p/p) + (\varepsilon^{-q/p} b^q/q) \leq \varepsilon a^p + \varepsilon^{-q/p} b^q$ , where  $1 < p < \infty$ ,  $(1/p) + (1/q) = 1$ . As  $p = q = 2$ , one has  $ab \leq (\varepsilon a^2/2) + (b^2/2\varepsilon) \leq (\varepsilon a^2) + (1/\varepsilon b^2)$ .*

### 3. Main Results and the Proof of the Theorems

Based on Galerkin procedure, we will show the existence of global solution of (1.9)–(1.13). Suppose that  $\{\omega_k\}_{k=1}^\infty$  is an orthonormal basis in the space  $H$  consisting of eigenfunctions of the operator  $A$ .  $H_m = \text{span}\{\omega_1, \omega_2, \dots, \omega_m\}$ , and  $P_m$  is orthogonal projection from  $H$  to  $H_m$ . By Galerkin procedure [55], (1.9)–(1.13) can be reduced to ordinary differential system

$$m_{mt} - m_{mxx} = 2m_m u_{mx} + m_{mx} u_m + (m_m v_m)_x + n_m v_{mx}, \quad m_m = u_m - u_{mxx}, \quad (3.1)$$

$$n_{mt} - n_{mxx} = 2n_m v_{mx} + n_{mx} v_m + (n_m u_m)_x + m_m u_{mx}, \quad n_m = v_m - v_{mxx}, \quad (3.2)$$

$$u_m(x, 0) = P_m u_0, \quad v_m(x, 0) = P_m v_0. \quad (3.3)$$

By means of existence theory of solution to ordinary differential equations, we know that local smooth solution of (3.1)–(3.3) exists. Now we can establish consistent integral estimate on approximate solution with respect to  $m$  by Galerkin method.

**Theorem 3.1.** *If  $u_0, \rho_0 \in H^l(R)$ ,  $l \geq 2$ , then (1.9)–(1.13) has a global solution in  $H^2(\Omega)$ .*

*Proof.* Taking the inner product of (3.1), and (3.2), respectively, with  $u_m, v_m$  in  $\Omega$  and noting that

$$(u_{mt} - u_{mxx} - \varepsilon(u_m - u_{mxx})_{xx}, u_m) = \frac{1}{2} \frac{d}{dt} (|u_m|^2 + \|u_m\|^2) + \varepsilon (\|u_m\|^2 + |Au_m|^2), \quad (3.4)$$

$$(v_{mt} - v_{mxx} - \varepsilon(v_m - v_{mxx})_{xx}, v_m) = \frac{1}{2} \frac{d}{dt} (|v_m|^2 + \|v_m\|^2) + \varepsilon (\|v_m\|^2 + |Av_m|^2), \quad (3.5)$$

$$P_m [-2(u_{mx} u_{mxx}, u_m) - (u_{mxxx} u_m, u_m)]$$

$$= P_m \left[ \int_{\Omega} (-2u_{mx} u_m u_{mxx}) dx - \int_{\Omega} u_m^2 du_{mxx} \right]$$

$$= P_m \left[ \int_{\Omega} (-2u_m u_{mx} u_{mxx}) dx + \int_{\Omega} u_{mxx} (u_m^2)_x dx \right] = 0,$$

$$P_m [(u_{mx} u_m, v_m) + (u_{mx} v_m + u_m v_{mx}, u_m)] \quad (3.6)$$

$$= P_m \left( 2 \int_{\Omega} u_m u_{mx} v_m dx + \int_{\Omega} u_m^2 dv_m \right)$$

$$= P_m \left( 2 \int_{\Omega} u_m u_{mx} v_m dx - \int_{\Omega} 2v_m u_m u_{mx} dx \right) = 0,$$

with the same reason, we obtain that

$$P_m[-2(v_{mx}v_{mxx}, v_m) - (v_{mxxx}v_m, v_m)] = 0, \quad P_m[(v_{mx}v_m, u_m) + (v_{mx}u_m + v_m u_{mx}, v_m)] = 0. \quad (3.7)$$

By integrating by parts we get

$$\begin{aligned} & P_m [(-u_{mxxx}v_m - u_{mxx}v_{mx} - v_{mxx}v_{mx}, u_m) + (-v_{mxxx}u_m - v_{mxx}u_{mx} - u_{mxx}u_{mx}, v_m)] \\ &= P_m \left( \int_{\Omega} -u_m v_m du_{mxx} - \int_{\Omega} u_m u_{mxx} v_{mx} dx - \int_{\Omega} u_m v_{mxx} v_{mx} dx \right. \\ &\quad \left. - \int_{\Omega} u_m v_m dv_{mxx} - \int_{\Omega} u_{mx} v_{mxx} v_m dx - \int_{\Omega} u_{mxx} v_m u_{mx} dx \right) \\ &= P_m \left( \int_{\Omega} u_{mxx} u_{mx} v_m dx + \int_{\Omega} u_{mxx} u_m v_{mx} dx - \int_{\Omega} u_m u_{mxx} v_{mx} dx \right. \\ &\quad \left. - \int_{\Omega} u_m v_{mxx} v_{mx} dx + \int_{\Omega} v_{mxx} u_{mx} v_m dx + \int_{\Omega} v_{mxx} u_m v_{mx} dx \right. \\ &\quad \left. - \int_{\Omega} u_{mx} v_{mxx} v_m dx - \int_{\Omega} u_{mxx} v_m u_{mx} dx \right) = 0. \end{aligned} \quad (3.8)$$

From (3.4) and (3.5) we obtain that

$$\frac{1}{2} \frac{d}{dt} (|u_m|^2 + \|u_m\|^2 + |v_m|^2 + \|v_m\|^2) + \varepsilon ( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 ) = 0. \quad (3.9)$$

Applying Poincaré inequality, we get

$$\|u_m\|^2 > \lambda_1 |u_m|^2, \quad |Au_m|^2 > \lambda_1 \|u_m\|^2, \quad \|v_m\|^2 > \lambda_1 |v_m|^2, \quad |Av_m|^2 > \lambda_1 \|v_m\|^2. \quad (3.10)$$

Equality (3.9) implies that

$$\frac{d}{dt} (|u_m|^2 + \|u_m\|^2 + |v_m|^2 + \|v_m\|^2) + 2\varepsilon \lambda_1 (|u_m|^2 + \|u_m\|^2 + |v_m|^2 + \|v_m\|^2) \leq 0, \quad (3.11)$$

$$\begin{aligned} & |u_m| + \|u_m\|^2 + |v_m|^2 + \|v_m\|^2 \\ & \leq (|u_m(0)|^2 + \|u_m(0)\|^2 + |v_m(0)|^2 + \|v_m(0)\|^2) \exp\{-2\varepsilon \lambda_1 t\} \\ & \leq |u_m(0)|^2 + \|u_m(0)\|^2 + |v_m(0)|^2 + \|v_m(0)\|^2 \triangleq r_1, \end{aligned} \quad (3.12)$$

where  $r_1$  is nonnegative constant.

Integrating (3.9) over  $[t, t + r]$  yields

$$\varepsilon \int_t^{t+r} ( \|u_m(s)\|^2 + |Au_m(s)|^2 + \|v_m(s)\|^2 + |Av_m(s)|^2 ) ds \leq r_1. \quad (3.13)$$

Taking the inner product of (3.1) and (3.2), respectively, with  $-u_{mxx}, -v_{mxx}$  in  $\Omega$ , we get

$$(u_{mt} - u_{mxtt} - \varepsilon(u_m - u_{mxx})_{xx}, -u_{mxx}) = \frac{1}{2} \frac{d}{dt} (\|u_m\|^2 + |Au_m|^2) + \varepsilon(|Au_m|^2 + |\nabla Au_m|^2), \quad (3.14)$$

$$(v_{mt} - v_{mxtt} - \varepsilon(v_m - v_{mxx})_{xx}, -v_{mxx}) = \frac{1}{2} \frac{d}{dt} (\|v_m\|^2 + |Av_m|^2) + \varepsilon(|Av_m|^2 + |\nabla Av_m|^2) \quad (3.15)$$

since

$$\begin{aligned} |P_m(3u_m u_{mx}, -u_{mxx})| &\leq \frac{3}{2} \|\nabla u_m\|_{L^\infty(\Omega)} \|u_m\|^2, \\ |P_m(3v_m v_{mx}, -v_{mxx})| &\leq \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} \|v_m\|^2, \\ |P_m[(-2u_{mx} u_{mxx}, -u_{mxx}) - (u_{mxxx} u_m, -u_{mxx})]| \\ &= \left| P_m \left[ \int_{\Omega} (2u_{mx} u_{mxx}^2) dx + \int_{\Omega} u_m u_{mxx} du_{mxx} \right] \right| \\ &= \left| P_m \int_{\Omega} \frac{3}{2} u_{mx} u_{mxx}^2 dx \right| \leq \frac{3}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |Au_m|^2, \\ |P_m[(v_{mx} u_m + v_m u_{mx}, -v_{mxx}) + (v_{mxx} v_m, -u_{mxx})]| \\ &= \left| P_m \left( \frac{3}{2} \int_{\Omega} v_{mx}^2 u_{mx} dx \right) \right| \leq \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} \|v_m\|^2, \\ |P_m(-2v_{mx} v_{mxx}, -v_{mxx}) - (v_{mxxx} v_m, -v_{mxx})| &\leq \frac{3}{2} \|v_{mx}\|_{L^\infty(\Omega)} |Av_m|^2. \end{aligned} \quad (3.16)$$

By means of integrating by parts frequently, we obtain that

$$\begin{aligned} &|P_m[(u_{mx} v_m + u_m v_{mx}, -u_{mxx}) + (u_{mxx} u_m, -v_{mxx})]| \\ &= \left| P_m \left( \int_{\Omega} -u_{mx} v_m du_{mx} - \int_{\Omega} u_{mxx} u_m v_{mx} dx - \int_{\Omega} u_m u_{mx} dv_{mx} \right) \right| \\ &= \left| P_m \left( \int_{\Omega} \frac{u_{mx}^2}{2} v_{mx} dx - \int_{\Omega} u_m u_{mxx} dx + \int_{\Omega} u_{mx}^2 v_{mx} dx + \int_{\Omega} u_m u_{mxx} v_{mx} dx \right) \right| \\ &= \left| P_m \left( \frac{3}{2} \int_{\Omega} u_{mx}^2 v_{mx} dx \right) \right| \leq \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} \|u_m\|^2, \\ &|P_m[(-u_{mxxx} v_m - u_{mxx} v_{mx} - v_{mxx} v_{mx}, -u_{mxx}) \\ &\quad + (-v_{mxxx} u_m - v_{mxx} u_{mx} - u_{mxx} u_{mx}, -v_{mxx})]| \end{aligned}$$



$$\begin{aligned}
&= \left| P_m \left( \int_{\Omega} v_m u_{mxx} du_{mxx} + \int_{\Omega} v_{mx} u_{mxx}^2 dx + \int_{\Omega} u_{mxx} v_{mxx} v_{mx} dx \right. \right. \\
&\quad \left. \left. + \int_{\Omega} u_m v_{mxx} dv_{mxx} + \int_{\Omega} u_{mx} v_{mxx}^2 dx + \int_{\Omega} u_{mxx} u_{mx} v_{mxx} dx \right) \right| \\
&= \left| P_m \left( \int_{\Omega} \frac{1}{2} u_{xx}^2 v_x dx + \int_{\Omega} \frac{1}{2} v_{xx}^2 u_x dx \right) \right| + \left| P_m \left[ \int_{\Omega} \frac{1}{2} (u_{mxx}^2 + v_{mxx}^2) v_{mx} dx \right] \right| \\
&\quad + \left| P_m \left[ \int_{\Omega} \frac{1}{2} (u_{mxx}^2 + v_{mxx}^2) u_{mx} dx \right] \right| \\
&= \left| P_m \int_{\Omega} u_{mxx}^2 v_{mx} dx \right| + \left| P_m \int_{\Omega} v_{mxx}^2 u_{mx} dx \right| \\
&\quad + \left| P_m \int_{\Omega} \frac{1}{2} u_{mxx}^2 u_{mx} dx \right| + \left| P_m \int_{\Omega} \frac{1}{2} v_{mxx}^2 v_{mx} dx \right| \\
&\leq \|\nabla v_m\|_{L^\infty(\Omega)} |Au_m|^2 + \|\nabla u_m\|_{L^\infty(\Omega)} |Av_m|^2 \\
&\quad + \frac{1}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |Au_m|^2 + \frac{1}{2} \|\nabla v_m\|_{L^\infty(\Omega)} |Av_m|^2.
\end{aligned} \tag{3.17}$$

Associating all the above inequalities leads to

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
&\quad + \varepsilon \left( |Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2 \right) \\
&\leq \frac{3}{2} \|\nabla u_m\|_{L^\infty(\Omega)} \|u_m\|^2 + \frac{3}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |Au_m|^2 \\
&\quad + \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} \|v_m\|^2 + \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} |Av_m|^2 \\
&\quad + \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} \|u_m\|^2 + \frac{3}{2} \|\nabla u_m\|_{L^\infty(\Omega)} \|v_m\|^2 \\
&\quad + \|\nabla u_m\|_{L^\infty(\Omega)} |Av_m|^2 + \|\nabla v_m\|_{L^\infty(\Omega)} |Au_m|^2 \\
&\quad + \frac{1}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |Au_m|^2 + \frac{1}{2} \|\nabla v_m\|_{L^\infty(\Omega)} |Av_m|^2 \\
&\leq 2 \|\nabla u_m\|_{L^\infty(\Omega)} \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
&\quad + 2 \|\nabla v_m\|_{L^\infty(\Omega)} \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
&\leq 2c_1 \|u_m\|^{1/2} |Au_m|^{1/2} \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
&\quad + 2c_2 \|v_m\|^{1/2} |Av_m|^{1/2} \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right).
\end{aligned} \tag{3.18}$$

Simplifying the above inequality and employing Young inequality, it follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
& \quad + \varepsilon \left( |Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2 \right) \\
& \leq \varepsilon \lambda_1 \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
& \quad + c_3 \|u_m\| |Au_m| \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
& \quad + c_4 \|v_m\| |Av_m| \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right),
\end{aligned} \tag{3.19}$$

where  $c_3 = 2c_1^2/\varepsilon\lambda_1$ ,  $c_4 = 2c_2^2/\varepsilon\lambda_1$ . By means of Poincaré inequality, we obtain that

$$\begin{aligned}
& \frac{d}{dt} \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
& \leq 2(c_3 \|u_m\| |Au_m| + c_4 \|v_m\| |Av_m|) \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
& \leq \left[ c_3 \left( \|u_m\|^2 + |Au_m|^2 \right) + c_4 \left( \|v_m\|^2 + |Av_m|^2 \right) \right] \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
& \leq (c_3 + c_4) \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right)^2.
\end{aligned} \tag{3.20}$$

Let

$$\begin{aligned}
y &= \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2, \\
g &= (c_3 + c_4) \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right);
\end{aligned} \tag{3.21}$$

from (3.13) we get

$$\int_t^{t+r} y(s) \, ds \leq \frac{r_1}{\varepsilon}, \quad \int_t^{t+r} g(s) \, ds \leq (c_3 + c_4) \frac{r_1}{\varepsilon}. \tag{3.22}$$

Finally we obtain that

$$\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \leq \left( \frac{r_1}{\varepsilon r} \right) \exp \left\{ (c_3 + c_4) \frac{r_1}{\varepsilon} \right\} \triangleq r_2. \tag{3.23}$$

Integrating (3.19) over  $[t, t + r]$  yields

$$\begin{aligned}
 & \int_t^{t+r} \varepsilon \left( |Au_m(s)|^2 + |\nabla Au_m(s)|^2 + |Av_m(s)|^2 + |\nabla Av_m(s)|^2 \right) ds \\
 & \leq \int_t^{t+r} \varepsilon \lambda_1 \left( \|u_m(s)\|^2 + |Au_m(s)|^2 + \|v_m(s)\|^2 + |Av_m(s)|^2 \right) ds \\
 & \quad + \int_t^{t+r} (c_3 + c_4) \left( \|u_m(s)\|^2 + |Au_m(s)|^2 + \|v_m(s)\|^2 + |Av_m(s)|^2 \right)^2 ds \\
 & \quad + \left( \|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
 & \leq \varepsilon \lambda_1 r_2 r + (c_3 + c_4) r_2^2 r + r_2 \triangleq r_3,
 \end{aligned} \tag{3.24}$$

where  $c_1, c_2, c_3, c_4, r_2$ , and  $r_3$  are positive constants.

Taking the inner product of (3.1) and (3.2), respectively, with  $u_{mxxxx}, v_{mxxxx}$ , in  $\Omega$ , we obtain that

$$\begin{aligned}
 (u_{mt} - u_{mxt} - \varepsilon(u_m - u_{mxx})_{xx}, u_{mxxxx}) &= \frac{1}{2} \frac{d}{dt} \left( |Au_m|^2 + |\nabla Au_m|^2 \right) + \varepsilon \left( |\nabla Au_m|^2 + |A^2 u_m|^2 \right), \\
 (v_{mt} - v_{mxt} - \varepsilon(v_m - v_{mxx})_{xx}, v_{mxxxx}) &= \frac{1}{2} \frac{d}{dt} \left( |Av_m|^2 + |\nabla Av_m|^2 \right) + \varepsilon \left( |\nabla Av_m|^2 + |A^2 v_m|^2 \right), \\
 |P_m(3u_m u_{mx} - 2u_{mx} u_{mxx} - u_m u_{mxxx}, u_{mxxxx})| &\leq 3 \|\nabla u_m\|_{L^\infty(\Omega)} |Au_m|^2 + \frac{5}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |\nabla Au_m|^2, \\
 |P_m(3v_m v_{mx} - 2v_{mx} v_{mxx} - v_m v_{mxxx}, v_{mxxxx})| &\leq 3 \|\nabla v_m\|_{L^\infty(\Omega)} |Av_m|^2 + \frac{5}{2} \|\nabla v_m\|_{L^\infty(\Omega)} |\nabla Av_m|^2.
 \end{aligned} \tag{3.25}$$

By integrating by parts and applying Sobolev inequality, we obtain that

$$\begin{aligned}
 & |P_m[(u_{mx} v_m + u_m v_{mx}, u_{mxxxx}) + (u_m u_{mx}, v_{mxxxx})]| \\
 & = \left| P_m \left[ -(u_{mxxx}, (u_{mxx} v_m + 2u_{mx} v_{mx} + u_m v_{mxx})) - (v_{mxxx}, u_{mx}^2 + u_m u_{mxx}) \right] \right| \\
 & \leq \|u_{mxxx}\| \|u_{mxx} v_m + 2u_{mx} v_{mx} + u_m v_{mxx}\| + \|v_{mxxx}\| \|u_{mx}^2 + u_m u_{mxx}\| \\
 & \leq \frac{\varepsilon}{4} |\nabla Au_m|^2 + \frac{\varepsilon}{4} |\nabla Av_m|^2 + \frac{C_1}{2}, \\
 & |P_m[(v_{mx} u_m + v_m u_{mx}, v_{mxxxx}) + (v_m v_{mx}, u_{mxxxx})]| \\
 & = \left| P_m \left[ -(v_{mxxx}, (v_{mxx} u_m + 2v_{mx} u_{mx} + v_m u_{mxx})) - (u_{mxxx}, v_{mx}^2 + v_m v_{mxx}) \right] \right| \\
 & \leq \|v_{mxxx}\| \|v_{mxx} u_m + 2v_{mx} u_{mx} + v_m u_{mxx}\| + \|u_{mxxx}\| \|v_{mx}^2 + v_m v_{mxx}\| \\
 & \leq \frac{\varepsilon}{4} |\nabla Au_m|^2 + \frac{\varepsilon}{4} |\nabla Av_m|^2 + \frac{C_1}{2},
 \end{aligned} \tag{3.26}$$

where  $C_1$  is a constant depending on  $|u_m|, \|u_m\|, |v_m|$ , and  $\|v_m\|$ . As well as

$$\begin{aligned}
& |P_m[(-u_{mxxx}v_m - u_{mxx}v_{mx} - v_{mxx}v_{mx}, u_{mxxxx}) + (-v_{mxxx}u_m - v_{mxx}u_{mx} - u_{mxx}u_{mx}, v_{mxxxx})]| \\
&= \left| P_m \left( \frac{3}{2} \int_{\Omega} u_{mxxx}^2 v_{mx} dx + \frac{3}{2} \int_{\Omega} v_{mxxx}^2 u_{mx} dx + \frac{1}{2} \int_{\Omega} u_{mxxx} v_{mxx}^2 dx \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \int_{\Omega} v_{mxxx} u_{mxx}^2 dx + \int_{\Omega} u_{mxxx} v_{mxxx} u_{mx} dx + \int_{\Omega} u_{mxxx} v_{mxxx} v_{mx} dx \right) \right| \\
&\leq \left| P_m \left( 2 \int_{\Omega} u_{mxxx}^2 v_{mx} dx + 2 \int_{\Omega} v_{mxxx}^2 u_{mx} dx + \frac{1}{2} \int_{\Omega} u_{mxxx} v_{mxx}^2 dx \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \int_{\Omega} v_{mxxx} u_{mxx}^2 dx \right) \right| + \left| P_m \left( \frac{1}{2} \int_{\Omega} v_{mxxx}^2 v_{mx} dx \right) \right| + \left| P_m \left( \frac{1}{2} \int_{\Omega} u_{mxxx}^2 u_{mx} dx \right) \right| \\
&\leq 2 \|\nabla v_m\|_{L^\infty(\Omega)} |\nabla A u_m|^2 + 2 \|\nabla u_m\|_{L^\infty(\Omega)} |\nabla A v_m|^2 + \frac{\varepsilon}{4} (|\nabla A u_m|^2 + |\nabla A v_m|^2) \\
&\quad + C_2 + \frac{1}{2} \|\nabla v_m\|_{L^\infty(\Omega)} |\nabla A v_m|^2 + \frac{1}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |\nabla A u_m|^2,
\end{aligned} \tag{3.27}$$

where  $C_2$  is a constant depending on  $|u_m|, \|u_m\|, |v_m|, \|v_m\|, |A u_m|$ , and  $|A v_m|$ . Combining all the above inequalities, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (|A u_m|^2 + |\nabla A u_m|^2 + |A v_m|^2 + |\nabla A v_m|^2) \\
&\quad + \frac{\varepsilon}{4} |\nabla A u_m|^2 + \varepsilon |A^2 u_m|^2 + \frac{\varepsilon}{4} |\nabla A v_m|^2 + \varepsilon |A^2 v_m|^2 \\
&\leq 3 \|\nabla u_m\|_{L^\infty(\Omega)} |A u_m|^2 + 3 \|\nabla u_m\|_{L^\infty(\Omega)} |\nabla A u_m|^2 \\
&\quad + 2 \|\nabla u_m\|_{L^\infty(\Omega)} |\nabla A v_m|^2 + 2 \|\nabla v_m\|_{L^\infty(\Omega)} |\nabla A u_m|^2 \\
&\quad + 3 \|\nabla v_m\|_{L^\infty(\Omega)} |A v_m|^2 + 3 \|\nabla v_m\|_{L^\infty(\Omega)} |\nabla A v_m|^2 + C_1 + C_2.
\end{aligned} \tag{3.28}$$

By employing Young inequality, it follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (|A u_m|^2 + |\nabla A u_m|^2 + |A v_m|^2 + |\nabla A v_m|^2) + \frac{\varepsilon}{4} (|\nabla A u_m|^2 + |A^2 u_m|^2 + |\nabla A v_m|^2 + |A^2 v_m|^2) \\
&\leq 3c_5 \|u_m\|^{1/2} |A u_m|^{1/2} (|A u_m|^2 + |\nabla A u_m|^2 + |A v_m|^2 + |\nabla A v_m|^2) \\
&\quad + 3c_6 \|v_m\|^{1/2} |A v_m|^{1/2} (|A v_m|^2 + |\nabla A v_m|^2 + |A u_m|^2 + |\nabla A u_m|^2) + C_1 + C_2
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{4}\varepsilon\lambda_1\left(|Au_m|^2 + |\nabla Au_m|^2 + |\nabla Av_m|^2 + |Av_m|^2\right) \\
 &\quad + c_7\|u_m\||Au_m|\left(|Au_m|^2 + |\nabla Au_m|^2 + |\nabla Av_m|^2 + |Av_m|^2\right) \\
 &\quad + c_8\|v_m\||Av_m|\left(|Au_m|^2 + |\nabla Au_m|^2 + |\nabla Av_m|^2 + |Av_m|^2\right) + C_1 + C_2,
 \end{aligned} \tag{3.29}$$

where  $c_7 = 18c_5^2/\varepsilon\lambda_1$ ,  $c_8 = 18c_6^2/\varepsilon\lambda_1$ . Based on Poincaré inequality we obtain that

$$\begin{aligned}
 &\frac{d}{dt}\left(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2\right) \\
 &\leq c_7\left(\|u_m\|^2 + |Au_m|^2\right)\left(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2\right) \\
 &\quad + c_8\left(\|v_m\|^2 + |Av_m|^2\right)\left(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2\right) + 2C_2 + 2C_1 \\
 &\leq 2(C_1 + C_2) + (c_7 + c_8)\left(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2\right) \\
 &\quad \times \left(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2\right).
 \end{aligned} \tag{3.30}$$

Let

$$\begin{aligned}
 y &= |Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2, \\
 g &= (c_7 + c_8)\left(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2\right), \\
 h &= 2(C_1 + C_2);
 \end{aligned} \tag{3.31}$$

from (3.12) and (3.23), we conclude that  $h$  is bounded, so we suppose that  $\int_t^{t+r} h(s)ds \leq r_4$ ; by (3.24), we have

$$\int_t^{t+r} y(s) ds \leq \frac{r_3}{\varepsilon}, \quad \int_t^{t+r} g(s) ds \leq (c_7 + c_8)r_2r. \tag{3.32}$$

Using Gronwall inequality, we obtain that

$$|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2 \leq \left(\frac{r_3}{\varepsilon} + r_4\right) \exp\{(c_7 + c_8)r_2r\} \triangleq r_5. \tag{3.33}$$

Integrating (3.29) over  $[t, t+r]$ , we obtain that

$$\begin{aligned}
& \int_t^{t+r} \frac{\varepsilon}{4} \left( |\nabla Au_m(s)|^2 + |A^2u_m(s)|^2 + |\nabla Av_m(s)|^2 + |A^2v_m(s)|^2 \right) ds \\
& \leq \int_t^{t+r} \frac{\varepsilon}{4} \lambda_1 \left( |Au_m(s)|^2 + |\nabla Au_m(s)|^2 + |Av_m(s)|^2 + |\nabla Av_m(s)|^2 \right) ds \\
& \quad + \int_t^{t+r} [C_1 + C_2] ds + \left( |Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2 \right) \\
& \quad + \int_t^{t+r} c_7 \|u_m(s)\| |Au_m(s)| \left( |Au_m(s)|^2 + |\nabla Au_m(s)|^2 + |Av_m(s)|^2 + |\nabla Av_m(s)|^2 \right) ds \\
& \quad + \int_t^{t+r} c_8 \|v_m(s)\| |Av_m(s)| \left( |Av_m(s)|^2 + |\nabla Av_m(s)|^2 + |Au_m(s)|^2 + |\nabla Au_m(s)|^2 \right) ds \\
& \leq \frac{\varepsilon}{4} \lambda_1 r_5 r + (c_7 + c_8) r_5 r_2 r + r_4 + r_5 \triangleq r_6,
\end{aligned} \tag{3.34}$$

where  $c_5, c_6, c_7, c_8, r_4, r_5$ , and  $r_6$  are nonnegative constants.

Respectively, taking the inner product of (3.1) and (3.2) with  $A^3u_m, A^3v_m$  in  $\Omega$ , we can also get  $|\nabla Au_m|^2 + |A^2u_m|^2 + |\nabla Av_m|^2 + |A^2v_m|^2 \leq r_7$ . Connecting (3.12) and (3.23) with (3.33), we can get that

$$\begin{aligned}
|u_m|^2 \leq r_1, \quad \|u_m\|^2 \leq r_1, \quad |Au_m|^2 \leq r_2, \quad |\nabla Au_m|^2 \leq r_5, \quad |A^2u_m|^2 \leq r_7, \\
|v_m|^2 \leq r_1, \quad \|v_m\|^2 \leq r_1, \quad |Av_m|^2 \leq r_2, \quad |\nabla Av_m|^2 \leq r_5, \quad |A^2v_m|^2 \leq r_7,
\end{aligned} \tag{3.35}$$

so  $|m_m|, \|m_m\|, |Am_m|, |n_m|, \|n_m\|$ , and  $|An_m|$  are bounded.

Then we get that  $du_m/dt, dv_m/dt, dm_m/dt, dn_m/dt$  are bounded. Considering Aubin's compactness theorem, we conclude that there is a subsequence  $u'_m, v'_m, m'_m, n'_m$ , so that  $u'_m \rightarrow u, v'_m \rightarrow v, m'_m \rightarrow m$ , and  $n'_m \rightarrow n$ . Now we replace  $u'_m, v'_m, m'_m$ , and  $n'_m$  with  $u, v, m$ , and  $n$ . We will prove that  $u, v, m$ , and  $n$  satisfy (1.9)–(1.10). That is to say, approximate solution of (3.1)–(3.2) is convergent to solution of (1.9)–(1.10).

Let  $\omega \in D(A), |\omega|$  is finite from the above discussion, and by ordinary differential equation (3.1), we have

$$\begin{aligned}
& (m_m(t), \omega) + \varepsilon \int_{t_0}^t (m_m(s), P_m A \omega) ds \\
& = 2 \int_{t_0}^t (m_m(s) \nabla u_m(s), P_m \omega) ds \\
& \quad + \int_{t_0}^t (\nabla m_m(s) u_m(s), P_m \omega) ds + \int_{t_0}^t (\nabla m_m(s) v_m(s), P_m \omega) ds \\
& \quad + \int_{t_0}^t (m_m(s) \nabla v_m(s), P_m \omega) ds + \int_{t_0}^t (n_m(s) \nabla v_m(s), P_m \omega) ds + (m_m(t_0), \omega).
\end{aligned} \tag{3.36}$$

Now it is clear that

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_{t_0}^t (m_m(s), A\omega) ds &= \int_{t_0}^t (m(s), A\omega) ds, \\ \lim_{m \rightarrow \infty} |P_m \omega - \omega| &= 0, \quad \lim_{m \rightarrow \infty} |P_m A\omega - A\omega| = 0, \\ (m_m(t), \omega) &\longrightarrow (m(t), \omega), \quad \int_{t_0}^t (m_m(s), P_m A\omega) ds = \int_{t_0}^t (m(s), A\omega) ds, \quad m \longrightarrow \infty. \end{aligned} \quad (3.37)$$

Note that

$$\begin{aligned} &\left| \int_{t_0}^t (m_m(s) \nabla u_m(s), P_m \omega) ds - \int_{t_0}^t (m(s) \nabla u(s), \omega) ds \right| \\ &= \left| \int_{t_0}^t (m_m(s) \nabla u_m(s), P_m \omega - \omega) ds + \int_{t_0}^t (m_m(s) \nabla u_m(s), \omega) ds - \int_{t_0}^t (m(s) \nabla u(s), \omega) ds \right| \\ &= \left| \int_{t_0}^t (m_m(s) \nabla u_m(s), P_m \omega - \omega) ds + \int_{t_0}^t (m_m(s) - m(s)) \nabla u_m(s), \omega) ds \right. \\ &\quad \left. + \int_{t_0}^t (m(s) (\nabla u_m(s) - \nabla u(s)), \omega) ds \right| \\ &\leq \left| \int_{t_0}^t (m_m(s) \nabla u_m(s), P_m \omega - \omega) ds \right| + \left| \int_{t_0}^t (m_m(s) - m(s)) \nabla u_m(s), \omega) ds \right| \\ &\quad + \left| \int_{t_0}^t (m(s) (\nabla u_m(s) - \nabla u(s)), \omega) ds \right| \\ &\leq \int_{t_0}^t |m_m(s)| \|u_m(s)\| |P_m \omega - \omega| ds + \int_{t_0}^t |m_m(s) - m(s)| \|u_m(s)\| |\omega| ds \\ &\quad + \int_{t_0}^t |m(s)| \|u_m(s) - u(s)\| |\omega| ds \longrightarrow 0. \end{aligned} \quad (3.38)$$

From the above statements we get

$$\lim_{m \rightarrow \infty} \int_{t_0}^t (m_m(s) \nabla u_m(s), P_m \omega) ds = \int_{t_0}^t (m(s) \nabla u(s), \omega) ds. \quad (3.39)$$

Similarly, we have that

$$\begin{aligned}
 \lim_{m \rightarrow \infty} \int_{t_0}^t (u_m(s) \nabla m_m(s), P_m \omega) ds &= \int_{t_0}^t (u(s) \nabla m(s), \omega) ds, \\
 \lim_{m \rightarrow \infty} \int_{t_0}^t (v_m(s) \nabla m_m(s), P_m \omega) ds &= \int_{t_0}^t (v(s) \nabla m(s), \omega) ds, \\
 \lim_{m \rightarrow \infty} \int_{t_0}^t (m_m(s) \nabla v_m(s), P_m \omega) ds &= \int_{t_0}^t (m(s) \nabla v(s), \omega) ds, \\
 \lim_{m \rightarrow \infty} \int_{t_0}^t (n_m(s) \nabla v_m(s), P_m \omega) ds &= \int_{t_0}^t (n(s) \nabla v(s), \omega) ds.
 \end{aligned} \tag{3.40}$$

For all  $\omega \in D(A)$ , we obtain that

$$\begin{aligned}
 (m(t), \omega) + \varepsilon \int_{t_0}^t (m(s), A\omega) ds &= 2 \int_{t_0}^t (m(s) \nabla u(s), \omega) ds + \int_{t_0}^t (\nabla m(s) u(s), \omega) ds \\
 &\quad + \int_{t_0}^t (\nabla m(s) v(s), \omega) ds + \int_{t_0}^t (m(s) \nabla v(s), \omega) ds \\
 &\quad + \int_{t_0}^t (n(s) \nabla v(s), \omega) ds + (m(t_0), \omega).
 \end{aligned} \tag{3.41}$$

From ordinary differential equation (3.2), we know that

$$\begin{aligned}
 (n_m(t), \omega) + \varepsilon \int_{t_0}^t (n_m(s), P_m A\omega) ds \\
 &= 2 \int_{t_0}^t (n_m(s) \nabla v_m(s), P_m \omega) ds + \int_{t_0}^t (\nabla n_m(s) v_m(s), P_m \omega) ds \\
 &\quad + \int_{t_0}^t (\nabla n_m(s) u_m(s), P_m \omega) ds + \int_{t_0}^t (n_m(s) \nabla u_m(s), P_m \omega) ds \\
 &\quad + \int_{t_0}^t (m_m(s) \nabla u_m(s), P_m \omega) ds + (n_m(t_0), \omega),
 \end{aligned} \tag{3.42}$$



as we know that

$$\begin{aligned}
 (n_m(t), \omega) &\longrightarrow (n(t), \omega), \quad \int_{t_0}^t (n_m(s), P_m A \omega) ds = \int_{t_0}^t (n(s), A \omega) ds, \quad m \longrightarrow \infty, \\
 &\left| \int_{t_0}^t (n_m(s) \nabla v_m(s), P_m \omega) ds - \int_{t_0}^t (n(s) \nabla v(s), \omega) ds \right| \\
 &= \left| \int_{t_0}^t (n_m(s) \nabla v_m(s), P_m \omega - \omega) ds + \int_{t_0}^t (n_m(s) \nabla v_m(s), \omega) ds - \int_{t_0}^t (n(s) \nabla v(s), \omega) ds \right| \\
 &= \left| \int_{t_0}^t (n_m(s) \nabla v_m(s), P_m \omega - \omega) ds + \int_{t_0}^t (n_m(s) - n(s)) \nabla v_m(s), \omega) ds \right. \\
 &\quad \left. + \int_{t_0}^t (n(s) (\nabla v_m(s) - \nabla v(s)), \omega) ds \right| \\
 &\leq \left| \int_{t_0}^t (n_m(s) \nabla v_m(s), P_m \omega - \omega) ds \right| + \left| \int_{t_0}^t (n_m(s) - n(s)) \nabla v_m(s), \omega) ds \right| \\
 &\quad + \left| \int_{t_0}^t (n(s) (\nabla v_m(s) - \nabla v(s)), \omega) ds \right| \\
 &\leq \int_{t_0}^t |n_m(s)| |v_m(s)| |P_m \omega - \omega| ds + \int_{t_0}^t |n_m(s) - n(s)| |v_m(s)| |\omega| ds \\
 &\quad + \int_{t_0}^t |n(s)| |v_m(s) - v(s)| |\omega| ds \longrightarrow 0.
 \end{aligned}
 \tag{3.43}$$

From the above discussions we get

$$\lim_{m \rightarrow \infty} \int_{t_0}^t (n_m(s) \nabla v_m(s), P_m \omega) ds = \int_{t_0}^t (n(s) \nabla v(s), \omega) ds.
 \tag{3.44}$$

Simultaneously, we have that

$$\begin{aligned}
 \lim_{m \rightarrow \infty} \int_{t_0}^t (v_m(s) \nabla n_m(s), P_m \omega) ds &= \int_{t_0}^t (v(s) \nabla n(s), \omega) ds, \\
 \lim_{m \rightarrow \infty} \int_{t_0}^t (u_m(s) \nabla n_m(s), P_m \omega) ds &= \int_{t_0}^t (u(s) \nabla n(s), \omega) ds, \\
 \lim_{m \rightarrow \infty} \int_{t_0}^t (n_m(s) \nabla u_m(s), P_m \omega) ds &= \int_{t_0}^t (n(s) \nabla u(s), \omega) ds, \\
 \lim_{m \rightarrow \infty} \int_{t_0}^t (m_m(s) \nabla u_m(s), P_m \omega) ds &= \int_{t_0}^t (m(s) \nabla u(s), \omega) ds.
 \end{aligned}
 \tag{3.45}$$

For all  $\omega \in D(A)$ , we obtain

$$\begin{aligned} (n(t), \omega) + \varepsilon \int_{t_0}^t (n(s), A\omega) ds &= 2 \int_{t_0}^t (n(s) \nabla v(s), \omega) ds + \int_{t_0}^t (\nabla n(s) v(s), \omega) ds \\ &+ \int_{t_0}^t (\nabla n(s) u(s), \omega) ds + \int_{t_0}^t (n(s) \nabla u(s), \omega) ds \\ &+ \int_{t_0}^t (m(s) \nabla u(s), \omega) ds + (n(t_0), \omega). \end{aligned} \quad (3.46)$$

All the above analysis shows that the global solution to (1.9)–(1.13) exists in  $H^2(\Omega)$ .  $\square$

**Theorem 3.2.** Denote  $S(t)$  as the semigroup of the solution operator to (1.9)–(1.13),  $S(t) : H^2(\Omega) \rightarrow H^2(\Omega)$ ,  $u(t) = S(t)u_0$ ,  $\rho(t) = S(t)\rho_0$ . Then  $S(t)$  has an absorbing set in  $H^2(\Omega)$ .

*Proof.* Taking the inner product of (1.9) and (1.10), respectively, with  $u, v$  in  $\Omega$ , noting that

$$\begin{aligned} (u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx}, u) &= \frac{1}{2} \frac{d}{dt} (|u|^2 + \|u\|^2) + \varepsilon (\|u\|^2 + |Au|^2), \\ (v_t - v_{xxt} - \varepsilon(v - v_{xx})_{xx}, v) &= \frac{1}{2} \frac{d}{dt} (|v|^2 + \|v\|^2) + \varepsilon (\|v\|^2 + |Av|^2), \\ -2(u_x u_{xx}, u) - (u_{xxx} u, u) &= \int_{\Omega} (-2uu_x u_{xx}) dx + \int_{\Omega} u_{xx} (u^2)_x dx = 0, \\ (u_x u, v) + (u_x v + uv_x, u) &= 2 \int_{\Omega} uu_x v dx - \int_{\Omega} 2vuu_x dx = 0, \\ (v_x v, u) + (uv_x + u_x v, v) &= 0, -2(v_x v_{xx}, v) - (v_{xxx} v, v) = 0, \\ (-u_{xxx} v - u_{xx} v_x - v_{xx} v_x, u) &+ (-v_{xxx} u - v_{xx} u_x - u_{xx} u_x, v) = 0, \end{aligned} \quad (3.47)$$

and associating with all the above statements, we obtain

$$\frac{1}{2} \frac{d}{dt} (|u|^2 + \|u\|^2 + |v|^2 + \|v\|^2) + \varepsilon (\|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2) = 0. \quad (3.48)$$

According to the Poincaré inequality, we obtain that

$$\begin{aligned} \frac{d}{dt} (|u|^2 + \|u\|^2 + |v|^2 + \|v\|^2) + 2\varepsilon\lambda_1 (|u|^2 + \|u\|^2 + |v|^2 + \|v\|^2) &\leq 0, \\ |u| + \|u\|^2 + |v|^2 + \|v\|^2 & \\ \leq (|u(0)|^2 + \|u(0)\|^2 + |v(0)|^2 + \|v(0)\|^2) \exp\{-2\varepsilon\lambda_1 t\} & \\ \leq |u(0)|^2 + \|u(0)\|^2 + |v(0)|^2 + \|v(0)\|^2 \triangleq r_8, & \end{aligned} \quad (3.49)$$

where  $r_8$  is nonnegative constant.

Integrating (3.48) over  $[t, t + r]$ , we obtain that

$$\varepsilon \int_t^{t+r} \left( \|v(x, s)\|^2 + |Av(x, s)|^2 + \|u(x, s)\|^2 + |Au(x, s)|^2 \right) ds \leq r_8. \quad (3.50)$$

We will obtain the uniform estimate of (1.9)–(1.13) in  $H^2(\Omega)$  as follows.

Taking the inner product of (1.9) and (1.10), respectively, with  $-u_{xx}, -v_{xx}$ , we have

$$\begin{aligned} (u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx}, -u_{xx}) &= \frac{1}{2} \frac{d}{dt} (\|u\|^2 + |Au|^2) + \varepsilon(|Au|^2 + |\nabla Au|^2), \\ (v_t - v_{xxt} - \varepsilon(v - v_{xx})_{xx}, -v_{xx}) &= \frac{1}{2} \frac{d}{dt} (\|v\|^2 + |Av|^2) + \varepsilon(|Av|^2 + |\nabla Av|^2), \\ |(3uu_x, -u_{xx})| &\leq \frac{3}{2} \|\nabla u\|_{L^\infty(\Omega)} \|u\|^2, \quad |(3vv_x, -v_{xx})| \leq \frac{3}{2} \|\nabla v\|_{L^\infty(\Omega)} \|v\|^2, \\ |(-2u_x u_{xx}, -u_{xx}) - (u_{xxx} u, -u_{xx})| &= \left| \int_\Omega (2u_x u_{xx}^2) dx + \int_\Omega uu_{xx} du_{xx} \right| \\ &= \left| \int_\Omega \frac{3}{2} u_x u_{xx}^2 dx \right| \leq \frac{3}{2} \|\nabla u\|_{L^\infty(\Omega)} |Au|^2, \\ |(-2v_x v_{xx}, -v_{xx}) - (v_{xxx} v, -v_{xx})| &\leq \frac{3}{2} \|\nabla v\|_{L^\infty(\Omega)} |Av|^2, \\ |(u_x v + uv_x, -u_{xx}) + (u_x u, -v_{xx})| \\ &= \left| \int_\Omega \frac{u_x^2}{2} v_x dx - \int_\Omega uu_{xx} v_x dx + \int_\Omega u_x^2 v_x dx + \int_\Omega uu_{xx} v_x dx \right| \\ &= \left| \frac{3}{2} \int_\Omega u_x^2 v_x dx \right| \leq \frac{3}{2} \|\nabla v\|_{L^\infty(\Omega)} \|u\|^2, \\ |(v_x u + vu_x, -v_{xx}) + (v_x v, -u_{xx})| &= \left| \frac{3}{2} \int_\Omega v_x^2 u_x dx \right| \leq \frac{3}{2} \|\nabla u\|_{L^\infty(\Omega)} \|v\|^2; \end{aligned} \quad (3.51)$$

we also obtain

$$\begin{aligned} &|(-u_{xxx} v - u_{xx} v_x - v_{xx} v_x, -u_{xx}) + (-v_{xxx} u - v_{xx} u_x - u_{xx} u_x, -v_{xx})| \\ &\leq \left| \int_\Omega u_{xx}^2 v_x dx \right| + \left| \int_\Omega v_{xx}^2 u_x dx \right| + \left| \int_\Omega \frac{1}{2} u_{xx}^2 u_x dx \right| + \left| \int_\Omega \frac{1}{2} v_{xx}^2 v_x dx \right| \\ &\leq \|\nabla v\|_{L^\infty(\Omega)} |Au|^2 + \|\nabla u\|_{L^\infty(\Omega)} |Av|^2 + \frac{1}{2} \|\nabla u\|_{L^\infty(\Omega)} |Au|^2 + \frac{1}{2} \|\nabla v\|_{L^\infty(\Omega)} |Av|^2. \end{aligned} \quad (3.52)$$

From all the previous statements we obtain that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) + \varepsilon \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) \\
& \leq 2\|\nabla u\|_{L^\infty(\Omega)} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \\
& \quad + 2\|\nabla v\|_{L^\infty(\Omega)} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \\
& \leq 2c_9 \|u\|^{1/2} |Au|^{1/2} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \\
& \quad + 2c_{10} \|v\|^{1/2} |Av|^{1/2} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \\
& \leq \frac{\varepsilon \lambda_1}{2} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) + c_{11} \|u\| |Au| \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \\
& \quad + \frac{\varepsilon \lambda_1}{2} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) + c_{12} \|v\| |Av| \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right),
\end{aligned} \tag{3.53}$$

where  $c_{11} = 2c_9^2/\varepsilon\lambda_1$ ,  $c_{12} = 2c_{10}^2/\varepsilon\lambda_1$ . From Poincaré inequality, we get

$$\begin{aligned}
& \frac{d}{dt} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \\
& \leq \left[ c_{11} \left( \|u\|^2 + |Au|^2 \right) + c_{12} \left( \|v\|^2 + |Av|^2 \right) \right] \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \\
& \leq (c_{11} + c_{12}) \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right)^2.
\end{aligned} \tag{3.54}$$

Let

$$y = \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2, \quad g = (c_{11} + c_{12}) \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \tag{3.55}$$

From (3.50) we have that

$$\begin{aligned}
& \int_t^{t+r} y(s) \, ds \leq \frac{r_8}{\varepsilon}, \quad \int_t^{t+r} g(s) \, ds \leq (c_{11} + c_{12}) \frac{r_8}{\varepsilon}, \\
& \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \leq \left( \frac{r_8}{\varepsilon r} \right) \exp \left\{ (c_{11} + c_{12}) \frac{r_8}{\varepsilon} \right\} \triangleq r_9.
\end{aligned} \tag{3.56}$$

Integrating (3.53) over  $[t, t+r]$ , it follows that

$$\begin{aligned}
& \int_t^{t+r} \varepsilon \left( |Au(x, s)|^2 + |\nabla Au(x, s)|^2 + |Av(x, s)|^2 + |\nabla Av(x, s)|^2 \right) ds \\
& \leq \varepsilon \lambda_1 r_9 r + (c_{11} + c_{12}) r_9^2 r + r_9 \triangleq r_{10},
\end{aligned} \tag{3.57}$$

where  $c_9, c_{10}, c_{11}, c_{12}, r_8, r_9$ , and  $r_{10}$  are nonnegative constants. Taking the inner product of (1.9) and (1.10), respectively, with  $u_{xxxx}, v_{xxxx}$ , we have that

$$\begin{aligned}
 (u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx}, u_{xxxx}) &= \frac{1}{2} \frac{d}{dt} (|Au|^2 + |\nabla Au|^2) + \varepsilon (|\nabla Au|^2 + |A^2u|^2), \\
 (v_t - v_{xxt} - \varepsilon(v - v_{xx})_{xx}, v_{xxxx}) &= \frac{1}{2} \frac{d}{dt} (|Av|^2 + |\nabla Av|^2) + \varepsilon (|\nabla Av|^2 + |A^2v|^2), \\
 |(3uu_x - 2u_xu_{xx} - uu_{xxx}, u_{xxxx})| &\leq 3\|\nabla u\|_{L^\infty(\Omega)}|Au|^2 + \frac{5}{2}\|\nabla u\|_{L^\infty(\Omega)}|\nabla Au|^2, \\
 |(3vv_x - 2v_xv_{xx} - vv_{xxx}, v_{xxxx})| &\leq 3\|\nabla v\|_{L^\infty(\Omega)}|Av|^2 + \frac{5}{2}\|\nabla v\|_{L^\infty(\Omega)}|\nabla Av|^2.
 \end{aligned} \tag{3.58}$$

Integrating by parts and employing Sobolev inequality, we get

$$\begin{aligned}
 |(u_xv + uv_x, u_{xxxx}) + (uu_x, v_{xxxx})| &= \left| -(u_{xxx}, (u_xv + uv_x)_x - (v_{xxx}, u_x^2 + uu_{xx}) \right| \\
 &\leq \frac{\varepsilon}{4}|\nabla Au|^2 + \frac{\varepsilon}{4}|\nabla Av|^2 + \frac{C_3}{2}, \\
 |(v_xu + vu_x, v_{xxxx}) + (vv_x, u_{xxxx})| &= \left| -(v_{xxx}, (v_xu + vu_x)_x - (u_{xxx}, v_x^2 + vv_{xx}) \right| \\
 &\leq \frac{\varepsilon}{4}|\nabla Au|^2 + \frac{\varepsilon}{4}|\nabla Av|^2 + \frac{C_3}{2},
 \end{aligned} \tag{3.59}$$

where  $C_3$  is a constant depending on  $|u|, \|u\|, |v|$ , and  $\|v\|$ .

$$\begin{aligned}
 &|(-u_{xxx}v - u_{xx}v_x - v_{xx}v_x, u_{xxxx}) + (-v_{xxx}u - v_{xx}u_x - u_{xx}u_x, v_{xxxx})| \\
 &= \left| \frac{3}{2} \int_{\Omega} u_{xxx}^2 v_x dx + \frac{3}{2} \int_{\Omega} v_{xxx}^2 u_x dx + \frac{1}{2} \int_{\Omega} u_{xxx} v_{xx}^2 dx \right. \\
 &\quad \left. + \frac{1}{2} \int_{\Omega} v_{xxx} u_{xx}^2 dx + \int_{\Omega} u_{xxx} v_{xxx} u_x dx + \int_{\Omega} u_{xxx} v_{xxx} v_x dx \right| \\
 &\leq \left| 2 \int_{\Omega} u_{xxx}^2 v_x dx + 2 \int_{\Omega} v_{xxx}^2 u_x dx + \frac{1}{2} \int_{\Omega} u_{xxx} v_{xx}^2 dx + \frac{1}{2} \int_{\Omega} v_{xxx} u_{xx}^2 dx \right| \\
 &\quad + \left| \frac{1}{2} \int_{\Omega} v_{xxx}^2 v_x dx \right| + \left| \frac{1}{2} \int_{\Omega} u_{xxx}^2 u_x dx \right| \\
 &\leq 2\|\nabla v\|_{L^\infty(\Omega)}|\nabla Au|^2 + 2\|\nabla u\|_{L^\infty(\Omega)}|\nabla Av|^2 + \frac{\varepsilon}{4} (|\nabla Au|^2 + |\nabla Av|^2) \\
 &\quad + C_4 + \frac{1}{2}\|\nabla v\|_{L^\infty(\Omega)}|\nabla Av|^2 + \frac{1}{2}\|\nabla u\|_{L^\infty(\Omega)}|\nabla Au|^2,
 \end{aligned} \tag{3.60}$$

where  $C_4$  is a constant depending on  $|u|, \|u\|, |v|, \|v_m\|, |Au|$ , and  $|Av|$ . From all the above inequalities we obtain that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) + \varepsilon \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) \\
& \leq 3 \|\nabla u\|_{L^\infty(\Omega)} |Au|^2 + \frac{5}{2} \|\nabla u\|_{L^\infty(\Omega)} |\nabla Au|^2 + 3 \|\nabla v\|_{L^\infty(\Omega)} |Av|^2 \\
& \quad + \frac{5}{2} \|\nabla v\|_{L^\infty(\Omega)} |\nabla Av|^2 + \frac{\varepsilon}{2} |\nabla Au|^2 + \frac{\varepsilon}{2} |\nabla Av|^2 + C_3 \\
& \quad + 2 \|\nabla v\|_{L^\infty(\Omega)} |\nabla Au|^2 + 2 \|\nabla u\|_{L^\infty(\Omega)} |\nabla Av|^2 + \frac{\varepsilon}{4} \left( |\nabla Au|^2 + |\nabla Av|^2 \right) \\
& \quad + C_4 + \frac{1}{2} \|\nabla v\|_{L^\infty(\Omega)} |\nabla Av|^2 + \frac{1}{2} \|\nabla u\|_{L^\infty(\Omega)} |\nabla Au|^2.
\end{aligned} \tag{3.61}$$

Then it follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) + \frac{\varepsilon}{4} \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) \\
& \leq 3c_{13} \|u\|^{1/2} |Au|^{1/2} \left( |Au|^2 + |\nabla Au|^2 + |\nabla Av|^2 + |Av|^2 \right) + C_3 \\
& \quad + 3c_{14} \|v\|^{1/2} |Av|^{1/2} \left( |Av|^2 + |\nabla Av|^2 + |\nabla Au|^2 + |Au|^2 \right) + C_4 \\
& \leq \frac{\varepsilon}{8} \lambda_1 \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) \\
& \quad + c_{15} \|u\| |Au| \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) + C_3 \\
& \quad + \frac{\varepsilon}{8} \lambda_1 \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) \\
& \quad + c_{16} \|v\| |Av| \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) + C_4,
\end{aligned} \tag{3.62}$$

where  $c_{15} = 18c_{13}^2/\varepsilon\lambda_1, c_{16} = 18c_{14}^2/\varepsilon\lambda_1$ . By means of Poincaré inequality, we obtain that

$$\begin{aligned}
& \frac{d}{dt} \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) \\
& \leq c_{15} \left( \|u\|^2 + |Au|^2 \right) \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) \\
& \quad + c_{16} \left( \|v\|^2 + |Av|^2 \right) \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right) + C_3 + C_4 \\
& \leq 2(C_3 + C_4) + (c_{15} + c_{16}) \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 \right) \left( |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \right).
\end{aligned} \tag{3.63}$$

Let

$$\begin{aligned}
 y &= |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2, \\
 g &= (c_{15} + c_{16})\left(\|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2\right), \quad h = 2(C_3 + C_4);
 \end{aligned}
 \tag{3.64}$$

from (3.49) and (3.56), we conclude that  $h$  is bounded; assume that  $\int_t^{t+r} h(s) ds \leq r_{11}$ . From (3.57), we have

$$\int_t^{t+r} y(s)ds \leq \frac{r_{10}}{\varepsilon}, \quad \int_t^{t+r} g(s)ds \leq (c_{15} + c_{16})r_9r.
 \tag{3.65}$$

Then through Gronwall inequality, we obtain

$$|Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \leq \left(\frac{r_{10}}{\varepsilon} + r_{11}\right) \exp\{(c_{15} + c_{16})r_9r\} \triangleq \rho_1.
 \tag{3.66}$$

Integrating (3.62) on  $[t, t + r]$ , we have that

$$\begin{aligned}
 &\int_t^{t+r} \frac{\varepsilon}{4} \left( |\nabla Au(x, s)|^2 + |A^2u(x, s)|^2 + |\nabla Av(x, s)|^2 + |A^2v(x, s)|^2 \right) ds \\
 &\leq \frac{\varepsilon}{4} \lambda_1 \rho_1 r + (c_{15} + c_{16})r_9 \rho_1 r + r_{11} + \rho_1 \triangleq r_{12},
 \end{aligned}
 \tag{3.67}$$

where  $c_{13}, c_{14}, c_{15}, c_{16}, r_{11}, r_{12}$ , and  $r_{13}$  are nonnegative constants. Then from (3.66), we can get  $|Au|^2 \leq \rho_1, |Av|^2 \leq \rho_1$ . In other words, open ball  $B(0, \rho_1)$  is the attracting set of  $S(t)$  in  $H^2(\Omega)$ .  $\square$

**Theorem 3.3.** *Suppose that  $u_0, \rho_0 \in H^1(R), l \geq 2$ , then the semigroup of the solution operator  $S(t)$  to (1.9)–(1.13) has a global attractor in  $H^2(\Omega)$ .*

*Proof.* To obtain the existence of the global attractor, we will prove that it is a compact operator. Taking inner product of (1.9) and (1.10) with  $t^2 \Delta^3 u, t^2 \Delta^3 v$  in  $\Omega$ , we obtain

$$\begin{aligned}
 &\left( u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx}, t^2 u_{xxxxxx} \right) \\
 &= -\frac{1}{2} \frac{d}{dt} \left( |t \nabla Au|^2 + |t A^2 u|^2 \right) + t \left( |\nabla Au|^2 + |A^2 u|^2 \right) - \varepsilon \left( |t \Delta Au|^2 + |\nabla A^2 u|^2 \right),
 \end{aligned}
 \tag{3.68}$$

$$\begin{aligned}
 &\left( v_t - v_{xxt} - \varepsilon(v - v_{xx})_{xx}, t^2 v_{xxxxxx} \right) \\
 &= -\frac{1}{2} \frac{d}{dt} \left( |t \nabla Av|^2 + |t A^2 v|^2 \right) + t \left( |\nabla Av|^2 + |A^2 v|^2 \right) - \varepsilon \left( |t \Delta Av|^2 + |\nabla A^2 v|^2 \right).
 \end{aligned}
 \tag{3.69}$$

By integrating by parts, we obtain that

$$\begin{aligned}
& \left| (3uu_x - 2u_xu_{xx} - uu_{xxx}, t^2u_{xxxxx}) \right| \\
&= t^2 \left| -3 \int_{\Omega} u_x^2 u_{xxxxx} dx - 3 \int_{\Omega} uu_{xx}u_{xxxxx} dx + 2 \int_{\Omega} u_{xx}^2 u_{xxxxx} dx \right. \\
&\quad \left. + 2 \int_{\Omega} u_x u_{xxx} u_{xxxxx} dx + \int_{\Omega} u_x u_{xxx} u_{xxxxx} dx + \int_{\Omega} uu_{xxx} u_{xxxxx} dx \right| \\
&= t^2 \left| \int_{\Omega} 6u_x u_{xx} u_{xxxxx} dx + 3 \int_{\Omega} u_x u_{xx} u_{xxxxx} dx + 3 \int_{\Omega} uu_{xxx} u_{xxxxx} dx \right. \\
&\quad \left. - 4 \int_{\Omega} u_{xx} u_{xxx} u_{xxxxx} dx - 2 \int_{\Omega} u_{xx} u_{xxx} u_{xxxxx} dx - 2 \int_{\Omega} u_x u_{xxxx}^2 dx \right. \\
&\quad \left. - \int_{\Omega} u_{xx} u_{xxx} u_{xxxxx} dx - \int_{\Omega} u_x u_{xxxx}^2 dx - \frac{1}{2} \int_{\Omega} u_x u_{xxxx}^2 dx \right| \\
&= t^2 \left| \frac{21}{2} \int_{\Omega} u_x u_{xxxx}^2 dx + \frac{7}{2} \int_{\Omega} u_{xxx}^3 dx + \frac{7}{2} \int_{\Omega} u_x u_{xxxx}^2 dx \right| \\
&\leq \frac{21}{2} \|\nabla u\|_{L^\infty(\Omega)} |t\nabla Au|^2 + \frac{7}{2} \left( \|\nabla Au\|_{L^\infty(\Omega)} |t\nabla Au|^2 + \|\nabla u\|_{L^\infty(\Omega)} |tA^2u|^2 \right).
\end{aligned} \tag{3.70}$$

Similarly, we obtain

$$\begin{aligned}
& \left| (3vv_x - 2v_xv_{xx} - vv_{xxx}, t^2v_{xxxxx}) \right| \\
&\leq \frac{21}{2} \|\nabla v\|_{L^\infty(\Omega)} |t\nabla Av|^2 + \frac{7}{2} \left( \|\nabla Av\|_{L^\infty(\Omega)} |t\nabla Av|^2 + \|\nabla v\|_{L^\infty(\Omega)} |tA^2v|^2 \right), \\
& \left| (u_xv + uv_x, t^2u_{xxxxx}) + (uu_x, t^2v_{xxxxx}) \right| \\
&= \left| (t^2u_{xxxxx}, u_{xxx}v + 3u_{xx}v_x + 3u_xv_{xx} + uv_{xxx}) + (t^2v_{xxxxx}, 3u_xu_{xx} + uu_{xxx}) \right| \\
&\leq \frac{\varepsilon}{4} |tA^2u|^2 + \frac{\varepsilon}{4} |tA^2v|^2 + \frac{1}{3} C_5, \\
& \left| (v_xu + vu_x, t^2v_{xxxxx}) + (vv_x, t^2u_{xxxxx}) \right| \\
&= \left| (t^2v_{xxxxx}, v_{xxx}u + 3v_{xx}u_x + 3v_xu_{xx} + vu_{xxx}) + (t^2u_{xxxxx}, 3v_xv_{xx} + vv_{xxx}) \right| \\
&\leq \frac{\varepsilon}{4} |tA^2u|^2 + \frac{\varepsilon}{4} |tA^2v|^2 + \frac{1}{3} C_5,
\end{aligned} \tag{3.71}$$



where  $C_5$  is a constant depending on  $|u|, \|u\|, |v|, \|v_m\|, |Au|, |Av|$ , and  $t$ . Integrating by parts frequently, we obtain

$$\begin{aligned}
 & \left| \left( -u_{xxx}v - u_{xx}v_x - v_{xx}v_x, t^2u_{xxxxxx} \right) + \left( -v_{xxx}u - v_{xx}u_x - u_{xx}u_x, t^2v_{xxxxxx} \right) \right| \\
 & \leq t^2 \left[ \left| 3 \int_{\Omega} v_{xxxx}^2 v_x dx \right| + \left| 3 \int_{\Omega} u_{xxxx}^2 u_x dx \right| + \frac{1}{2} \int_{\Omega} u_{xxxx}^2 v_x dx + \frac{1}{2} \int_{\Omega} v_{xxxx}^2 u_x dx \right] \\
 & \quad + \frac{\varepsilon}{4} |tA^2u|^2 + \frac{\varepsilon}{4} |tA^2v|^2 + \frac{C_5}{3}, \tag{3.72} \\
 & \leq 3 \|v_x\|_{L^\infty(\Omega)} \left( |tA^2v|^2 + |tA^2u|^2 \right) + 3 \|u_x\|_{L^\infty(\Omega)} \left( |tA^2v|^2 + |tA^2u|^2 \right) \\
 & \quad + \frac{\varepsilon}{4} |tA^2u|^2 + \frac{\varepsilon}{4} |tA^2v|^2 + \frac{C_5}{3}.
 \end{aligned}$$

From (3.68) and (3.69), we know that

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left( |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \right) \\
 & \quad + \frac{\varepsilon}{4} |t\Delta Au|^2 + \varepsilon |\nabla A^2u|^2 + \frac{\varepsilon}{4} |t\Delta Av|^2 + \varepsilon |\nabla A^2v|^2 \\
 & \leq \frac{21}{2} \|\nabla u\|_{L^\infty(\Omega)} |t\nabla Au|^2 + \frac{7}{2} \left( \|\nabla Au\|_{L^\infty(\Omega)} |t\nabla Au|^2 + \|\nabla u\|_{L^\infty(\Omega)} |tA^2u|^2 \right) \\
 & \quad + \frac{21}{2} \|\nabla v\|_{L^\infty(\Omega)} |t\nabla Av|^2 + \frac{7}{2} \left( \|\nabla Av\|_{L^\infty(\Omega)} |t\nabla Av|^2 + \|\nabla v\|_{L^\infty(\Omega)} |tA^2v|^2 \right) \\
 & \quad + 3 \|\nabla v\|_{L^\infty(\Omega)} \left( |tA^2v|^2 + |tA^2u|^2 \right) + 3 \|\nabla u\|_{L^\infty(\Omega)} \left( |tA^2v|^2 + |tA^2u|^2 \right) \\
 & \quad + t \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) + C_5 \\
 & \leq \left( 11 \|\nabla u\|_{L^\infty(\Omega)} + 11 \|\nabla v\|_{L^\infty(\Omega)} \right) \left( |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \right) \\
 & \quad + \frac{7}{2} \|\nabla Au\|_{L^\infty(\Omega)} |t\nabla Au|^2 + \frac{7}{2} \|\nabla Av\|_{L^\infty(\Omega)} |t\nabla Av|^2 \\
 & \quad + t \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) + C_5 \\
 & \leq \left( 11c_{17} \|u\|^{1/2} |Au|^{1/2} + 11c_{18} \|v\|^{1/2} |Av|^{1/2} \right) \left( |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \right) \\
 & \quad + \frac{7}{2} c_{19} |\nabla Au|^{1/2} |A^2u|^{1/2} |t\nabla Au|^2 + \frac{7}{2} c_{20} |\nabla Av|^{1/2} |A^2v|^{1/2} |t\nabla Av|^2 \\
 & \quad + t \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) + C_5
 \end{aligned}$$

$$\begin{aligned}
&\leq 11c_{21} \left( \|u\|^{1/2} |Au|^{1/2} + \|v\|^{1/2} |Av|^{1/2} + |\nabla Au|^{1/2} |A^2u|^{1/2} + |\nabla Av|^{1/2} |A^2v|^{1/2} \right) \\
&\quad \times \left( |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \right) \\
&\quad + t \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) + C_5,
\end{aligned} \tag{3.73}$$

where  $c_{21} = \max\{4c_{17}, 4c_{18}, 14c_{19}, 14c_{20}\}$ . Employing Young inequality, we obtain that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left( |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \right) \\
&\quad + \frac{\varepsilon}{4} \left( |t\Delta Au|^2 + |\nabla A^2u|^2 + |t\Delta Av|^2 + |\nabla A^2v|^2 \right) \\
&\leq \frac{\varepsilon}{4} \lambda_1 \left( |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \right) \\
&\quad + c_{22} \left( \|u\| |Au| + \|v\| |Av| + |\nabla Au| |A^2u| + |\nabla Av| |A^2v| \right) \\
&\quad \times \left( |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \right) \\
&\quad + c_{23} \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) + C_5,
\end{aligned} \tag{3.74}$$

where  $c_{22} = 242c_{21}^2/\varepsilon\lambda_1$ ,  $c_{23} = 2/\varepsilon\lambda_1$ . From Poincaré inequality, we obtain that

$$\begin{aligned}
&\frac{d}{dt} \left( |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \right) \\
&\leq 2c_{22} \left( \|u\| |Au| + \|v\| |Av| + |\nabla Au| |A^2u| + |\nabla Av| |A^2v| \right) \\
&\quad \times \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) \\
&\quad + 2c_{23} \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) + 2C_5.
\end{aligned} \tag{3.75}$$

Let

$$\begin{aligned}
y &= |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2, \\
g &= 2c_{22} \left( \|u\| |Au| + \|v\| |Av| + |\nabla Au| |A^2u| + |\nabla Av| |A^2v| \right), \\
h &= 2c_{23} \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) + 2C_5.
\end{aligned} \tag{3.76}$$

Through (3.67), we have

$$\int_t^{t+r} \left( |\nabla Au(x, s)|^2 + |A^2u(x, s)|^2 + |\nabla Av(x, s)|^2 + |A^2v(x, s)|^2 \right) ds \leq \frac{4r_{12}}{\varepsilon}. \quad (3.77)$$

We also get

$$\begin{aligned} & \int_t^{t+r} \left( |t\nabla Au(x, s)|^2 + |tA^2u(x, s)|^2 + |t\nabla Av(x, s)|^2 + |tA^2v(x, s)|^2 \right) ds \leq (t+r)^2 \frac{4r_{12}}{\varepsilon}, \\ & \int_t^{t+r} 2c_{22} \left( \|u\| |Au| + \|v\| |Av| + |\nabla Au| |A^2u| + |\nabla Av| |A^2v| \right) ds \\ & \leq \int_t^{t+r} 2c_{22} \left( \|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2 + |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) ds \\ & \leq 2c_{22} \left( r_9 r + \frac{4r_{12}}{\varepsilon} \right). \end{aligned} \quad (3.78)$$

Through (3.49), (3.56), and (3.66), we assume that

$$\int_t^{t+r} C_5 ds \leq r_{13}. \quad (3.79)$$

Then we have

$$\int_t^{t+r} \left[ 2c_{23} \left( |\nabla Au|^2 + |A^2u|^2 + |\nabla Av|^2 + |A^2v|^2 \right) + 2C_5 \right] ds \leq 2c_{23} \frac{4r_{12}}{\varepsilon} + 2r_{13}, \quad (3.80)$$

and all the analysis indicates that

$$\begin{aligned} & |t\nabla Au|^2 + |tA^2u|^2 + |t\nabla Av|^2 + |tA^2v|^2 \\ & \leq \left[ (t+r)^2 \frac{4r_{12}}{\varepsilon} + 2c_{23} \frac{4r_{12}}{\varepsilon} + 2r_{13} \right] \exp \left\{ 2c_{22} \left( r_9 r + \frac{4r_{12}}{\varepsilon} \right) \right\} \\ & \triangleq E^2(\lambda_1, \rho_1, \varepsilon, t). \end{aligned} \quad (3.81)$$

Finally we obtain that

$$|\nabla Au| \leq \frac{E(\lambda_1, \rho_1, \varepsilon, t)}{t}, \quad |\nabla Av| \leq \frac{E(\lambda_1, \rho_1, \varepsilon, t)}{t}. \quad (3.82)$$

We know that the injection of  $H^3(\Omega)$  into  $H^2(\Omega)$  is compact, then we can conclude that  $S(t)$  is equi-continuity. From Ascoli-Arzela's theorem, we know that  $S(t)$  has the global attractor in  $H^2(\Omega)$ .  $\square$

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