Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 496936, 14 pages
doi:10.1155/2010/496936

Research Article

An Extension of the Invariance Principle for a Class of Differential Equations with Finite Delay

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Received 7 October 2010; Accepted 16 December 2010

Academic Editor: Binggen Zhang

An extension of the uniform invariance principle for ordinary differential equations with finite delay is developed. The uniform invariance principle allows the derivative of the auxiliary scalar function V to be positive in some bounded sets of the state space while the classical invariance principle assumes that $\dot{V} \leq 0$. As a consequence, the uniform invariance principle can deal with a larger class of problems. The main difficulty to prove an invariance principle for functional differential equations is the fact that flows are defined on an infinite dimensional space and, in such spaces, bounded solutions may not be precompact. This difficulty is overcome by imposing the vector field taking bounded sets into bounded sets.

1. Introduction

The invariance principle is one of the most important tools to study the asymptotic behavior of differential equations. The first effort to establish invariance principle results for ODEs was likely made by Krasovskiĭ; see [1]. Later, other authors have made important contributions to the development of this theory; in particular, the work of LaSalle is of great importance [2, 3]. Since then, many versions of the classical invariance principle have been given. For instance, this principle has been successfully extended to differential equations on infinite dimensional spaces, [4–7], including functional differential equations (FDEs) and, in particular, retarded functional differential equations (RFDEs). The great advantage of this principle is the possibility of studying the asymptotic behavior of solutions of differential equations without the explicit knowledge of solutions. For this purpose, the invariance principle supposes the existence of a scalar auxiliary function V satisfying $\dot{V} \leq 0$ and studies the implication of the existence of such function on the ω-limit of solutions.

More recently, the invariance principle was successfully extended to allow the derivative of the scalar function V to be positive in some bounded regions and also to take into account parameter uncertainties. For ordinary differential equations, see [8, 9] and for discrete differential systems, see [10]. The main advantage of these extensions is the possibility of applying the invariance theory for a larger class of systems, that is, systems for which one may have difficulties to find a scalar function satisfying $\dot{V} \leq 0$.

The next step along this line of advance is to consider functional differential equations. Let $C([-h,0];\mathbb{R}^n)$ be the space of continuous functions defined on $[-h,0]$ with values in $\mathbb{R}^n$, and Λ a compact subset of $\mathbb{R}^m$. In this paper, an extension of the invariance principle for the following class of autonomous retarded functional differential equations

$$\dot{x}(t) = f(x_t, \lambda), \quad t > 0, \ \lambda \in \Lambda \tag{1.1}$$

is proved.

The main difficulty to prove an invariance principle for functional differential equations is the fact that flows are defined on an infinite dimensional space. It is well known that, in such spaces, boundedness of solutions does not guarantee precompactness of solutions. In order to overcome this difficulty, we will impose conditions on function f to guarantee that solutions of (1.1) belong to a compact set.

The extended invariance principle is useful to obtain uniform estimates of the attracting sets and their basins of attraction, including attractors of chaotic systems. These estimates are obtained as level sets of the auxiliary scalar function V. Despite V is defined on the state space $C([-h,0];\mathbb{R}^n)$, we explore the boundedness of time delay to obtain estimates of the attractor in $\mathbb{R}^n$, which are relevant in practical applications.

This paper is organized as follows. Some preliminary results are discussed in Section 2; an extended invariance principle for functional differential equation with finite delay is proved in Section 3. In Section 4, we present some applications of our results in concrete examples, such as a retarded version of Lorenz system and a retarded version of Rössler system.

2. Preliminary Results

In what follows, $\mathbb{R}^n$ will denote the Euclidean n-dimensional vector space, with norm $\|\cdot\|_{\mathbb{R}^n}$, and $C([-h,0];\mathbb{R}^n)$ will denote the space of continuous functions defined on $[-h,0]$ into $\mathbb{R}^n$, endowed with the norm $\|\phi\|_{C([-h,0];\mathbb{R}^2)} : \sup_{\theta\in[-h,0]}\|\phi(\theta)\|_{\mathbb{R}^n}$.

Let $x : [-h,\alpha) \to \mathbb{R}^n$, $\alpha > 0$, be a continuous function, and, for each $t \in [0,\alpha)$, let x_t be one element of $C([-h,0];\mathbb{R}^n)$ defined as $x_t(\theta) = x(t+\theta)$, $\theta \in [-h,0]$. The element x_t is a segment of the graph of $x(s)$, which is obtained by letting s vary from $t-h$ to t. Let $f : C([-h,0];\mathbb{R}^n) \times \Lambda \to \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^m$, be a continuous function. For a fixed $\lambda \in \Lambda$ and $\phi \in C([-h,0];\mathbb{R}^n)$, consider the following initial value problem:

$$\dot{x}(t) = f(x_t, \lambda), \quad t \geq 0, \tag{2.1}$$

$$x_0 = \phi, \quad \phi \in C([-h,0];\mathbb{R}^n). \tag{2.2}$$

Definition 2.1. A solution of (2.1)-(2.2) is a function $x(t)$ defined and continuous on an interval $[-h,\alpha)$, $\alpha > 0$, such that (2.2) holds and (2.1) is satisfied for all $t \in [0,\alpha)$.

If for each $\phi \in C([-h,0];\mathbb{R}^n)$ and for a fixed λ, the initial value problem (2.1)-(2.2) has a unique solution $x(t,\lambda,\phi)$, then we will denote by $\pi_\lambda(\phi)$ the orbit through ϕ, which is defined as $\pi_\lambda(\phi) := \{x_t(\lambda,\phi),\ t \geq 0\}$. Function ψ belongs to the ω-limit set of $\pi_\lambda(\phi)$, denoted by $\omega_\lambda(\phi)$, if there exists a sequence of real numbers $(t_n)_{n\geq 0}$, with $t_n \to \infty$ as $n \to \infty$, such that $x_{t_n} \to \psi$, with respect to the norm of $C([-h,0];\mathbb{R}^n)$, as $n \to \infty$.

Generally, on infinite dimensional spaces, such as the space $C([-h,0];\mathbb{R}^n)$, the boundedness property of solutions is not sufficient to guarantee compactness of the flow $\pi_\lambda(\phi)$. The compactness of the orbit will be important in the development of our invariance results. In order to guarantee the relatively compactness of set $\pi_\lambda(\phi)$ and, at the same time, the uniqueness of solutions of (2.1)-(2.2), the following assumptions regarding function f are made.

($A1$) For each $r > 0$, there exists a real number $H = H(r) > 0$ such that, $\|f(\phi,\lambda)\|_{\mathbb{R}^n} \leq H$ for all $\|\phi\|_{C([-h,0];\mathbb{R}^n)} \leq r$ and for all $\lambda \in \Lambda$.

($A2$) For each $r > 0$, there exists a real number $L = L(r) > 0$, such that

$$\left\|f(\phi_1,\lambda) - f(\phi_2,\lambda)\right\|_{\mathbb{R}^n} \leq L\left\|\phi_1 - \phi_2\right\|_{C([-h,0];\mathbb{R}^n)}, \tag{2.3}$$

for all $\|\phi_i\|_{C([-h,0];\mathbb{R}^n)} \leq r$, $i = 1,2$ and all $\lambda \in \Lambda$.

Under conditions ($A1$)-($A2$), the problem (2.1)-(2.2) has a unique solution that depends continuously upon ϕ, see [4]. Moreover, one has the following result.

Lemma 2.2 (compacity of solutions [4]). *If $x(t,\lambda,\phi)$ is a solution of* (2.1)-(2.2) *such that $x_t(\lambda,\phi)$ is bounded, with respect to the norm of $C([-h,0];\mathbb{R}^n)$, for $t \geq 0$ and assumptions (A1)-(A2) are satisfied, then $x(t,\lambda,\phi)$ is the unique solution of* (2.1)-(2.2). *Moreover, the flow $\pi_\lambda(\phi)$ through ϕ belongs to a compact subset of $C([-h,0];\mathbb{R}^n)$ for all $t \geq 0$.*

Lemma 2.2 guarantees, under assumptions ($A1$) and ($A2$), that bounded solutions are unique and the orbit is contained in a compact subset of $C([-h,0];\mathbb{R}^n)$.

Let $x(\cdot,\phi,\lambda) : [-h,\infty) \to \mathbb{R}^n$, $\phi \in C([-h,0];\mathbb{R}^n)$, be a solution of problem (2.1)-(2.2) and suppose the existence of a positive constant k in such a manner that

$$\left\|x(t,\lambda,\phi)\right\|_{\mathbb{R}^n} \leq k, \tag{2.4}$$

for every $t \in [-r,\infty)$ and $\lambda \in \Lambda$. The following lemma is a well-known result regarding the properties of ω-limit sets of compact orbits $\pi_\lambda(\phi)$ [6].

Lemma 2.3 (limit set properties). *Let $x(\cdot,\lambda,\phi)$ be a solution of* (2.1)-(2.2) *and suppose that* (2.4) *is satisfied. Then, the ω-limit set of $\pi_\lambda(\phi)$ is a nonempty, compact, connected, invariant set and* $\operatorname{dist}(x_t(\lambda,\phi),\omega_\lambda(\phi)) \to \infty$, *as $t \to \infty$.*

Lyapunov-like functions may provide important information regarding limit-sets of solutions and also provide estimates of attracting sets and their basins of attraction. Thus, it is important to consider the concept of derivative of a function along the solutions of (2.1).

Definition 2.4. Let $V : \mathcal{C}([-h,0];\mathbb{R}^n) \times \Lambda \to \mathbb{R}$ be a continuous scalar function. The derivative of V along solutions of (2.1), which will be denoted by $\dot{V}$, is given by

$$\dot{V}(t) =: \dot{V}(x_t(\lambda,\phi)) = \limsup_{h\to 0^+} \frac{V(x_{t+h}(\lambda,\phi)) - V(x_t)(\lambda,\phi)}{h}. \tag{2.5}$$

Remark 2.5. Function $\dot{V}(t)$ is well defined even when solutions of (2.1) are not unique. In order to be more specific, suppose $x : [-h,\infty) \to \mathbb{R}^n$ and $y : [-h,\infty) \to \mathbb{R}^n$ are solutions of (2.1), satisfying the same initial condition, then it is possible to show [11] that

$$\limsup_{h\to 0^+} \frac{V(x_{t+h}(\lambda,\phi)) - V(x_t)(\lambda,\phi)}{h} = \limsup_{h\to 0^+} \frac{V(y_{t+h}(\lambda,\phi)) - V(y_t)(\lambda,\phi)}{h}. \tag{2.6}$$

Remark 2.6. Generally, if $V(\phi)$ is continuously differentiable and $x(t)$ is a solution of (2.1), then the scalar function $t \to V(x_t)$ is differentiable in the usual sense for $t > 0$. In spite of that, it is possible to guarantee the existence of $\dot{V}(t)$ for $t > 0$ assuming weaker conditions; for example, if $V(\phi)$ is locally Lipschitzian, it is possible to show that $\dot{V}$ is well defined. For more details, we refer the reader to [12].

3. Main Result

In this section, we will prove the main result of this work, the uniform invariance principle for differential equations with finite delay. But first, we review a version of the classical invariance principle for differential equations with delay, which has been stated and proven in [4]. Consider the functional differential equation

$$\dot{x} = f(x_t), \quad t \geq 0. \tag{3.1}$$

Theorem 3.1 (the invariance principle). *Let f be a function satisfying assumptions (A1) and (A2) and V a continuous scalar function on $\mathcal{C}([-h,0];\mathbb{R}^n)$. Suppose the existence of positive constants l and k such that $\|\phi(0)\| \leq k$ for all $\phi \in U_l := \{\phi \in \mathcal{C}([-h,0];\mathbb{R}^n); V(\phi) < l\}$. Suppose also that $\dot{V} \leq 0$ for all $\phi \in U_l$. If E is the set of all points in $\overline{U_l}$ where $\dot{V}(\phi) = 0$ and M is the largest invariant set in E, then every solution of* (3.1)*, with initial value in U_l approaches M as $t \to \infty$.*

In Theorem 3.1, constants l and k are chosen in such a manner that the level set, $U_l = \{\phi \in \mathcal{C}([-h,0];\mathbb{R}^n); V(\phi) < l\}$, that is, the set formed by all functions ϕ such that $V(\phi) < l$, is a bounded set in $\mathcal{C}([-h,0];\mathbb{R}^n)$. Using this assumption, it is possible to show that the solution $x_t(\phi,\lambda)$, starting at $t = 0$ with initial condition ϕ, is bounded on $\mathcal{C}([-h,0];\mathbb{R}^n)$ for $t \geq 0$. Now, we are in a position to establish an extension of Theorem 3.1. This extension is uniform with respect to parameters and allows the derivative of V be positive in some bounded sets of $\mathcal{C}([-h,0];\mathbb{R}^n)$. Since in practical applications it is convenient to get information about the behavior of solutions in $\mathbb{R}^n$, our setting is slightly different from that used in Theorem 3.1.

In what follows, let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ be continuous functions and consider, for each $\rho \in \mathbb{R}$, the sets

$$\begin{aligned}
\mathcal{A}_\rho &:= \{\phi \in C([-h,0];\mathbb{R}^n);\ \mathfrak{a}(\phi) < \rho\},\\
\mathcal{A}_\rho(0) &:= \{\phi(0) \in \mathbb{R}^n;\ \phi \in \mathcal{A}_\rho\},\\
\mathfrak{B}_\rho &:= \{\phi \in C([-h,0];\mathbb{R}^n);\ \mathfrak{b}(\phi) < \rho\},\\
\mathfrak{B}_\rho(0) &:= \{\phi(0) \in \mathbb{R}^n;\ \phi \in \mathfrak{B}_\rho\},\\
E_\rho &:= \{\phi \in \mathcal{A}_\rho;\ \mathfrak{c}(\phi) = 0\},\\
E_\rho(0) &:= \{\phi(0) \in \mathbb{R}^n;\ \phi \in E_\rho\},\\
\widetilde{\mathfrak{C}} &:= \{\phi \in \mathcal{A}_\rho;\ \mathfrak{c}(\phi) < 0\}.
\end{aligned} \tag{3.2}$$

Moreover, we assume that the following assumptions are satisfied:

(i) $\mathcal{A}_\rho(0)$ is a bounded set in $\mathbb{R}^n$,

(ii) $\mathfrak{a}(\phi) \le V(\phi,\lambda) \le \mathfrak{b}(\phi)$, for all $(\phi,\lambda) \in C([-h,0];\mathbb{R}^n) \times \Lambda$,

(iii) $-\dot{V}(\phi,\lambda) \ge \mathfrak{c}(\phi)$, for all $(\phi,\lambda) \in C([-h,0];\mathbb{R}^n) \times \Lambda$,

(iv) there is a real number $R > 0$ such that $\sup_{\phi\in\widetilde{\mathfrak{C}}} \mathfrak{b}(\phi) \le R < \rho$.

Under these assumptions, a version of the invariance principle, which is uniform with respect to parameter $\lambda \in \Lambda$, is proposed in Theorem 3.2.

Theorem 3.2 (uniform invariance principle for retarded functional differential equations). *Suppose function f satisfies assumptions (A1)-(A2). Assume the existence of a locally Lipschitzian function $V : C([-h,0];\mathbb{R}^n) \times \Lambda \to \mathbb{R}^n$ and continuous functions $\mathfrak{a},\mathfrak{b},\mathfrak{c} : C([-h,0];\mathbb{R}^n) \to \mathbb{R}$. In addition, assume that function $\mathfrak{b}(\phi)$ takes bounded sets into bounded sets. If conditions (i)–(iv) are satisfied, then, for each fixed $\lambda \in \Lambda$, we have the following.*

(I) *If $\phi \in B_R = \{\phi \in C([-h,0];\mathbb{R}^n); \mathfrak{b}(\phi) \le R\}$. Then,*

(1) *the solution $x(t,\lambda,\phi)$ of* (1.1) *is defined for all $t \ge 0$,*

(2) *$x_t(\cdot,\lambda,\phi) \in A_R$, for $t \ge 0$, where $A_R = \{\phi \in C([-h,0];\mathbb{R}^n); a(\phi) \le R\}$,*

(3) *$x(t,\lambda,\phi) \in A_R(0)$, where $A_R(0) = \{\phi(0) \in \mathbb{R}^n; \phi \in A_R\}$, for all $t \ge 0$,*

(4) *$x_t(\phi,\lambda)$ tends to the largest collection M of invariant sets of* (1.1) *contained in A_R as $t \to \infty$.*

(II) *If $\phi \in \mathfrak{B}_\rho - B_R$. Then,*

(1) *$x(t,\phi,\lambda)$ is defined for all $t \ge 0$,*

(2) *$x_t(\phi,\lambda)$ belongs to $\mathcal{A}_\rho$ for all $t \ge 0$,*

(3) *$x(t,\phi,\lambda)$ belongs to $\mathcal{A}_\rho(0)$ for all $t \ge 0$,*

(4) *$x_t(\phi,\lambda)$ tends to the largest collection M of invariant sets of* (1.1) *contained in $A_R \cup E_\rho$.*

Proof. In order to show (I), we first have to prove that $x_t(\lambda,\phi) \in A_R$, for all $t \geq 0$. Let $x : [0,\omega_+) \to \mathbb{R}^n$ be a solution of (2.1), satisfying the initial condition $\phi \in B_R$ and suppose the existence of $\bar{t} \in [0,\omega_+)$ such that $x_{\bar{t}} \notin A_R$, that is, $\mathfrak{a}(x_{\bar{t}}(\lambda,\phi)) > R$. By assumption, we have $V(\phi,\lambda) \leq \mathfrak{b}(x_0(\lambda,\phi)) = \mathfrak{b}(\phi) < R$ and $V(x_{\bar{t}}(\lambda,\phi)) \geq \mathfrak{a}(x_{\bar{t}}(\lambda,\phi)) > R$. Using the Intermediate Value Theorem [13] and continuity of V with respect to t, it is possible to show the existence of $\tilde{t} \in (0,\bar{t})$ such that $V(x_{\tilde{t}}(\lambda,\phi)) = R$ and $V(x_t(\lambda,\phi)) > R$ for all $t \in (\tilde{t},\bar{t}]$. On the other hand, since function $t \to V(t)$ is nonincreasing on $[\tilde{t},\bar{t}]$ we have $V(x_{\bar{t}}) < V(x_{\tilde{t}})$, but this leads to a contradiction, because $R < a(x_{\bar{t}}(\lambda,\phi)) \leq V(x_{\bar{t}}(\lambda,\phi)) \leq V(x_{\tilde{t}}(\lambda,\phi)) = R$. Therefore, $x_t(\lambda,\phi) \in A_R$, for all $t \geq 0$, which implies that $x_t(\lambda,\phi)$ is bounded and defined for all $t \geq 0$. By definition of $A_R(0)$, we have that $x(t,\lambda,\phi) \in A_R(0) \subset \mathcal{A}_\rho$ for all $t \geq 0$. Since $\mathcal{A}_\rho(0)$ is a bounded set, according to Lemma 2.2, the orbit $\pi_\lambda(\phi)$ belongs to a compact set. As a consequence of Lemma 2.3, the ω-limit set, $\omega_\lambda(\phi)$ of (2.1)-(2.2) is a nonempty invariant subset of A_R. Hence, $x_t(\lambda,\phi)$ tends to the largest collection M of invariant sets of (2.1) contained in A_R.

In order to prove (II), we can suppose that $x_t(\phi,\lambda) \notin B_R$, for all $t \geq 0$. On the contrary, if for some $t \geq 0$, $x_t(\phi,\lambda) \in B_R$, then the result follows trivially from (I). Since $x_t(\phi,\lambda) \notin B_R$, for all $t \geq 0$, $V(x_t(\phi,\lambda))$ is a nonincreasing function of t, which implies that $\mathfrak{a}(x_t(\phi,\lambda)) \leq V(x_t(\phi,\lambda)) \leq V(\phi) \leq \mathfrak{b}(\phi) < \rho$, for all $t > 0$. As a consequence, solution $x_t(\phi,\lambda) \in A_\rho$ for all $t \geq 0$. This implies that $x(t,\lambda,\phi) \in A_\rho(0)$, for all $t \geq 0$, which means that $|x(t,\lambda,\phi)| \leq k$ for some positive constant k, since set $A_\rho(0)$ is bounded by hypothesis. Therefore, the solution $x_t(\lambda,\phi)$ is bounded in $C([-h,0];\mathbb{R}^n)$, which allows us to conclude the existence of a real number l_0 such that $\lim_{t\to\infty} V(x_t(\phi,\lambda)) = l_0$.

By conditions $(A1)$-$(A2)$ and Lemma 2.2, the orbit $\pi_\lambda(\phi)$ lies inside a compact subset of $C([-h,0];\mathbb{R}^n)$. Then, by Lemma 2.3, the $\omega_\lambda(\phi)$ is a nonempty, compact, and connected invariant set.

Next we prove that V is a constant function on the ω-limit set of (2.1)-(2.2). To this end, let $\psi \in \omega_\lambda(\phi)$ be an arbitrary element of $\omega_\lambda(\phi)$. So, there exists a sequence of real numbers t_n, $n \in \mathbb{N}$, $t_n \to \infty$, as $n \to \infty$ such that $x_{t_n}(\phi,\lambda) \to \psi$ as $n \to \infty$. By continuity of $V(\cdot)$, we have that $l_0 = \lim_{t_n\to\infty} V(x_{t_n}(\phi,\lambda)) = V(\psi)$. As a consequence, V is a constant function on $\omega_\lambda(\phi)$. Since $\omega_\lambda(\phi)$ is an invariant set, then $\dot{V}(\psi) = 0$ for all $\psi \in \omega_\lambda(\phi)$. Since $x_t(\phi,\lambda) \notin B_R$ for $t \geq 0$, we have for $\psi \in \omega_\lambda(\phi)$,

$$0 = \dot{V}(\psi) \geq c(\psi) \geq 0, \tag{3.3}$$

which implies that $c(\psi) = 0$ and thus $\omega_\lambda(\phi) \subset E_\rho$. The proof is complete. □

Remark 3.3. Theorem 3.2 provides estimates on both $C([-h,0];\mathbb{R}^n)$ and $\mathbb{R}^n$. For this purpose, we explore the fact that boundeness of $A_\rho(0)$ implies boundedness of x_t in $C([-h,0];\mathbb{R}^n)$.

Remark 3.4. If for each $\phi \in C([-h,0];\mathbb{R}^n) \setminus \overline{\tilde{\mathfrak{C}}}$, $c(\phi) > 0$, or if for all $\phi \in E_\rho \setminus \overline{\tilde{\mathfrak{C}}}$, the solution, $x_t(\phi,\lambda)$ of (2.1) leaves the set E_ρ for sufficiently small $t \geq 0$ and if all conditions of Theorem 3.2 are verified, then we can conclude that solutions of (2.1), with initial condition in A_ρ tend to the largest collection of invariant sets contained in A_R. In this case, A_R is an estimate of the attracting set in $C([-h,0];\mathbb{R}^n)$, in the sense that the attracting set is contained in A_R, and B_ρ is an estimate of the basin of attraction or stability region [8] in $C([-h,0];\mathbb{R}^n)$, while $A_R(0)$ and B_ρ are estimates of the attractor and basin of attraction, respectively, in $\mathbb{R}^n$.

Next, theorem is a global version of Theorem 3.2 that is useful to obtain estimates of global attractors.

Theorem 3.5 (the global uniform invariance principle for functional differential equations). *Suppose function $f : C([-h,0];\mathbb{R}^n) \times \Lambda \to \mathbb{R}^n$ satisfies assumptions (A1)-(A2). Assume the existence of Lipschitzian function $V : C([-h,0];\mathbb{R}^n) \times \Lambda \to \mathbb{R}$ and continuous functions $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} : C([-h,0];\mathbb{R}^n) \to \mathbb{R}$. If conditions (ii)-(iii) are satisfied, the following condition*

$$\sup_{\phi\in\tilde{\mathfrak{C}}} \mathfrak{b}(\phi) \leq R < \infty \tag{3.4}$$

holds and the set $A_R(0)$ is a bounded subset of $C([-h,0];\mathbb{R}^n)$, then for each fixed $\lambda \in \Lambda$ one has the following

(I) *If $\phi \in B_R$, then*

(1) *the solution $x(t,\lambda,\phi)$ of* (1.1) *is defined for all $t \geq 0$,*

(2) *$x_t(\cdot,\lambda,\phi) \in A_R$, for $t \geq 0$,*

(3) *$x(t,\lambda,\phi) \in A_R(0)$ for all $t \geq 0$,*

(4) *$x_t(\phi,\lambda)$ tends to the largest collection M of invariant sets of* (1.1) *contained in A_R as $t \to \infty$.*

(II) *If the solution of* (2.1)-(2.2) *with initial condition $x_0 = \phi \in C([-h,0];\mathbb{R}^n)$ satisfies $\lim_{t\to\infty}\|x_t(\phi,\lambda)\|_{C([-h,0];\mathbb{R}^n)} < \infty$, then $x_t(\lambda,\phi)$ tends to the largest collection of invariant sets contained in $A_R \cup E_\rho$, as $t \to \infty$.*

Proof. The proof of (I) is equal to the proof of (I) of Theorem 3.2. If for some $t_0 \in [0,\infty)$,$x_{t_0}(\phi,\lambda) \in B_R$, then the proof of (II) follows immediately from the proof of item (I). Suppose that $x_t(\phi,\lambda) \notin B_R$ for all $t \geq 0$. So, from (3.4), we have that $x_t(\phi,\lambda) \notin \mathfrak{C}$ and $-\dot{V}(x_t(\phi,\lambda),\lambda) \geq \mathfrak{c}(x_t(\phi,\lambda)) \geq 0$, for all $t \geq 0$. Then, we can conclude that function $t \to V(x_t(\phi,\lambda),\lambda)$ is nonincreasing on $[0,\infty)$ and bounded from below. Therefore, there exists l_o such that $\lim_{t\to\infty}V(x_t(\phi,\lambda),\lambda) = l_0 < \infty$. Using assumptions $(A1)$-$(A2)$ and Lemma 2.3, we can infer that the ω-limit set $\omega_\lambda(\phi)$ of (2.1)-(2.2) is a nonempty, compact, connected, and invariant set. Let $\psi \in \omega_\lambda(\phi)$, then there exists an increasing sequence t_n, $n \in \mathbb{N}$, $t_n \to \infty$, as $n \to \infty$ such that $x_{t_n}(\phi,\lambda) \to \psi$ as $n \to \infty$. Using the continuity of V, we have $V(\psi,\lambda) = l_0$, for all $\psi \in \omega_\lambda(\phi)$. This fact and the invariance of the ω-limit set allow us to conclude that

$$0 = \frac{d}{dt}V(\psi) \geq c(\psi) \geq 0. \tag{3.5}$$

Therefore, $c(\psi) = 0$ for all $\psi \in \omega_\lambda(\phi)$, which implies $\omega_\lambda(\phi) \subset E_\rho$. □

Theorem 3.5 provides information about the location of a global attracting set. More precisely, if the same conditions of Remark 3.4 apply, then A_R is an estimate of the attracting set.

In order to provide estimates of the attractor and the basin of attraction via Theorems 3.2 and 3.5, we have to calculate the maximum of function $\mathfrak{b}$ on the set $\mathfrak{C}$. This is a nonlinear programming problem in $C([-h,0];\mathbb{R}^n)$. In our applications, functions $\mathfrak{b}(\phi)$ and $\mathfrak{c}(\phi)$ are usually convex functions, which allows us to use the next result that simplifies the calculation of the maximum of $\mathfrak{b}$ in practical problems.

Lemma 3.6 (see [14]). *Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|_{\mathbb{X}}$ and let $g : \mathbb{X} \to \mathbb{R}^n$ be a continuous convex function. Suppose that $\Omega \subset \mathbb{X}$ is a bounded, closed, and convex subset in $\mathbb{X}$ such that g attains the maximum at some point $x_0 \in \Omega$. Then, g attains the maximum on $\partial\Omega$, the boundary of the set Ω.*

4. Applications

Example 4.1 (a retarded version of Lorenz system). In this example, we will find a uniform estimate of the chaotic attractor of the following retarded version of the Lorenz system:

$$\begin{aligned}\dot{u} &= \sigma(v(t) - u(t)),\\ \dot{v} &= ru(t) - v(t) - u(t)w(t) - \alpha u(t-h)w(t),\\ \dot{w} &= -bw(t) + u(t)v(t) + \alpha u(t-h)v(t).\end{aligned} \tag{4.1}$$

For $\alpha = 0$, the term with retard disappears and the problem is reduced to the original ODE Lorenz system model.

Parameters σ, r, and b are considered unknown. The expected values of these parameters are $\overline{\sigma} = 10$, $\overline{r} = 28$, and $\overline{b} = -8/3$. For these nominal parameters, simulations indicate that system (4.1) has a global attracting chaotic set. We assume an uncertainty of 5% in these parameters. More precisely, we assume parameters belong to the following compact set:

$$\Lambda := \left\{(\sigma, r, b) \in \mathbb{R}^3;\ \sigma_m \le \sigma \le \sigma_M,\ r_m \le r \le r_M,\ b_m \le b \le b_M\right\}, \tag{4.2}$$

where $\sigma_m = 9.5$, $\sigma_M = 10.5$, $r_m = 28 - (28/20)$, $r_M = 28 + (28/20)$, $b_m = (8/3) - (8/60)$, and $b_M = (8/3) + (8/60)$. The time delay and parameter α are considered constants. In this example, we have assumed $h = 0.09$ and $\alpha = 0.1$.

Consider the following change of variables:

$$(x, y, z) = \left(u, v, w - \frac{5}{4}r\right). \tag{4.3}$$

In these new variables, system (4.1) becomes

$$\begin{aligned}\dot{x} &= \sigma(y(t) - x(t)),\\ \dot{y} &= rx(t) - y(t) - x(t)\left(z(t) + \frac{5}{4}r\right) - \alpha x(t-h)\left(z(t) + \frac{5}{4}r\right),\\ \dot{z} &= -b\left(z(t) + \frac{5}{4}r\right) + x(t)y(t) + \alpha x(t-h)y(t).\end{aligned} \tag{4.4}$$

In order to write system (4.4) into the form of (1.1), consider $f : C([-h,0];\mathbb{R})^3 \times \Lambda \to \mathbb{R}^3$ defined as $f(\phi,\lambda) = (f_1(\phi,\lambda), f_2(\phi,\lambda), f_3(\phi,\lambda)), f_i : C([-h,0];\mathbb{R}) \times \Lambda \to C([-h,0];\mathbb{R})$, $i = 1,2,3$, where

$$\begin{aligned}
f_1(\phi,\lambda) &:= \sigma\big(\phi_2(0) - \phi_1(0)\big),\\
f_2(\phi,\lambda) &:= r\phi_1(0) - \phi_2(0) - \phi_1(0)\left(\phi_3(0) + \frac{5}{4}r\right) - \alpha\phi_1(-h)\left(\phi_3(0) + \frac{5}{4}r\right),\\
f_3(\phi,\lambda) &:= -b\left(\phi_3(0) + \frac{5}{4}r\right) + \phi_1(0)\phi_2(0) + \alpha\phi_1(-h)\phi_2(0).
\end{aligned} \tag{4.5}$$

Thus, system (4.1) becomes

$$\dot{x}(t) = f(x_t,\lambda), \tag{4.6}$$

with initial condition given by $\phi = (\phi_1,\phi_2,\phi_3)$.

On

$$\mathcal{C}^3 = \{(\phi_1,\phi_2,\phi_3);\ \phi_i \in C([-h,0];\mathbb{R}),\ i = 1,2,3\}, \tag{4.7}$$

consider the Lyapunov-like functional $V : \mathcal{C}^3 \times \Lambda \to \mathbb{R}$ given by

$$V(\phi_1,\phi_2,\phi_3,r,\sigma,b) = r\phi_1^2(0) + 4\sigma\phi_2^2(0) + 4\sigma\phi_3^2(0) + \rho\int_{-h}^{0}\phi_1^2(s)ds. \tag{4.8}$$

Consider $\mathfrak{a}(\phi_1,\phi_2,\phi_3)$ as being

$$\mathfrak{a}(\phi_1,\phi_2,\phi_3) = r_m\phi_1^2(0) + 4\sigma_m\phi_2^2(0) + 4\sigma_m\phi_3^2(0) \tag{4.9}$$

and $\mathfrak{b}(\phi_1,\phi_2,\phi_3)$ as being

$$\mathfrak{b}(\phi_1,\phi_2,\phi_3) = (r_M + h)\|\phi_1\|^2 + 4\sigma_M\|\phi_2\|^2 + 4\sigma_M\|\phi_3\|^2. \tag{4.10}$$

Then, we have

$$\mathfrak{a}(\phi_1,\phi_2,\phi_3) \le V(\phi_1,\phi_2,\phi_3,r,\sigma,b) \le \mathfrak{b}(\phi_1,\phi_2,\phi_3), \tag{4.11}$$

and as a consequence, condition (ii) of Theorem 3.2 is satisfied. Function $\mathfrak{a}(\cdot,\cdot,\cdot)$ is radially unbounded. So, condition (i) is satisfied for any real number ρ. Calculating the derivative of V along the solution we get,

$$\begin{aligned}
-\dot{V}(x_t, y_t, z_t, r, \sigma, b, \rho) &= (2\sigma r - \rho)x_t^2(0) + 8\sigma y_t^2(0) + 10\alpha\sigma r x_t(-h)y_t(0) \\
&\quad + \rho x_t^2(-h) + 8\sigma b z_t^2(0) + 10\sigma b r z_t(0) \\
&\geq (2\sigma_m r_m - \rho)x_t^2(0) + 8\sigma_m y_t^2(0) - 10\alpha\sigma_M r_M |x_t(-h)| \left|y_t(0)\right| \\
&\quad + \rho x_t^2(-h) + 8\sigma_m b_m z_t^2(0) - 10\sigma_M b_M r_M |z_t(0)|.
\end{aligned} \tag{4.12}$$

Rewriting the previous inequation in a matrix form, one obtains

$$\begin{aligned}
-\dot{V} &\geq (2\sigma_m r_m - \rho)x_t^2(0) \\
&\quad + \begin{bmatrix} y_t(0) & x_t(-h) \end{bmatrix} \begin{bmatrix} 8\sigma_m & -5\alpha\sigma_M r_M \\ -5\alpha\sigma_M r_M & \rho \end{bmatrix} \begin{bmatrix} y_t(0) \\ x_t(-h) \end{bmatrix} \\
&\quad + \gamma(|z_t(0)| - \beta)^2 - \eta =: \mathfrak{c}(x_t, y_t, z_t),
\end{aligned} \tag{4.13}$$

where $\gamma = 8\sigma_m b_m$, $\beta = 5\sigma_M b_M r_M / 8\sigma_m b_m$, and $\eta = 25\sigma_M^2 b_M^2 r_M^2 / 8\sigma_m b_m$.

We can make the quadratic term positive definite with an appropriate choice of parameter ρ. Using Sylvester's criterion, we obtain the following estimates for the parameters:

$$\frac{25\alpha^2\sigma_M^2 r_M^2}{8\sigma_m} < \rho < 2\sigma_m r_m. \tag{4.14}$$

The previous inequalities permit us to infer that if we choose α sufficiently small, then the matrix that appears in (4.13) becomes positive definite. This was expected because system (4.1) becomes the classical Lorenz system for $\alpha = 0$. If ρ satisfies the previous inequality, then condition (iii) is satisfied.

In order to estimate the supremum of function $\mathfrak{b}$ in the set $\mathfrak{C}$, consider the set

$$\begin{aligned}
\mathfrak{F} = \Bigg\{ &(x_t, y_t, z_t) \in \mathcal{C}^3 : (2\sigma_m r_m - \rho)x_t^2(0) \\
&+ \begin{bmatrix} y_t(0) & x_t(-h) \end{bmatrix} \begin{bmatrix} 8\sigma_m & -5\alpha\sigma_M r_M \\ -5\alpha\sigma_M r_M & \rho \end{bmatrix} \begin{bmatrix} y_t(0) \\ x_t(-h) \end{bmatrix} \\
&+\gamma(z_t(0) - \beta)^2 - \eta < 0 \Bigg\}.
\end{aligned} \tag{4.15}$$

It can be proved that the supremum of $\mathfrak{b}$ calculated over set $\mathfrak{F}$ is equal to the supremum of function $\mathfrak{b}$ over set $\mathfrak{C}$. Since $\overline{\mathfrak{F}}$ is a closed and convex set in $\mathcal{C}([-h, 0]; \mathbb{R}^n)$, Lemma 3.6

guarantees that the maximum of function $\mathfrak{b}$ is attained on the boundary of $\mathfrak{F}$. With that in mind, consider the following Lagrangian functional:

$$\mathfrak{L}(x_t, y_t, z_t, \mu) := (r_M + \tau)x_1^2 + 4\sigma_M x_2^2 + 4\sigma_M x_3^2 \mu\left[(2\sigma_M r_M - \rho)x_1^2 + 2\sigma_M r_M x_2^2 + \rho x_4^2 + 8\sigma_M b_M\left(x_3 - \frac{5r_m\sigma_m b_m}{4b_M\sigma_M}\right)^2\right]. \tag{4.16}$$

We get the following conditions to calculate the maximum of function $\mathfrak{b}$ in the set $\mathfrak{C}$:

$$\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial x_1} &= 2(r_M + \tau)x_1 + 2\mu(2\sigma_M r_M - \rho)x_1 = 0, \\
\frac{\partial \mathfrak{L}}{\partial x_2} &= 8\sigma_M x_2 + 4\mu\sigma_M r_M x_2 = 0, \\
\frac{\partial \mathfrak{L}}{\partial x_3} &= 8\sigma_M x_3 + 16\mu\sigma_M b_M\left(x_3 - \frac{5r_m\sigma_m b_m}{4b_M\sigma_M}\right) = 0, \\
\frac{\partial \mathfrak{L}}{\partial \mu} &= (2\sigma_M r_M - \rho)x_1^2 + 2\sigma_M r_M x_2^2 + \rho x_4^2 \\
&\quad + 8\sigma_M b_M\left(x_3 - \frac{5r_m\sigma_m b_m}{4b_M\sigma_M}\right)^2 = 0, \\
\frac{\partial \mathfrak{L}}{\partial x_4} &= 2\mu\rho x_4 = 0.
\end{aligned} \tag{4.17}$$

The maximum is attained at $x = -3.2$, $y = 0$, $z = 0$ and

$$\sup_{(\phi_1,\phi_2,\phi_3)\in\tilde{\mathfrak{C}}} \mathfrak{b}(\phi_1, \phi_2, \phi_3) = 5.12, \tag{4.18}$$

which implies that solutions of problem (4.1) are ultimately bounded and enter in the set

$$\left\{(x, y, z) \in \mathbb{R}^3; x^2 + \sigma_m y^2 + \sigma_m z^2 \leq 10.24\right\} \tag{4.19}$$

in finite time. So, by Theorem 3.2, the set

$$B_{10.24} = \left\{(\phi_1, \phi_2, \phi_3) \in C^3; \phi_1(0)^2 + \sigma_M\phi_2(0)^2 + \sigma_M\phi_3(0)^2 \leq 10.24\right\} \tag{4.20}$$

is an estimate for the basin of attraction and the set

$$A_{10.24} = \left\{(\phi_1, \phi_2, \phi_3) \in C^3; \phi_1^2(0) + \sigma_m\phi_2^2(0) + \sigma_m\phi_3^2(0) \leq 10.24\right\} \tag{4.21}$$

is an estimate for the attraction set for the system (4.5).

Example 4.2 (generalized Rössler circuit). Consider the classical Rössler system of ordinary differential equation with bounded time delay in the variable z

$$\begin{aligned}\dot{x}(t) &= -y - z(t-\tau_3),\\ \dot{y}(t) &= x + by,\\ \dot{z}(t) &= b + z(x-\tau_3).\end{aligned} \tag{4.22}$$

Let $\mathfrak{C}^3$ be the space formed by functions $\Phi = (\phi_1, \phi_2, \phi_3)$, such that $\phi_i \in C([-h,0];\mathbb{R}^3)$, $i = 1,2,3$, and $h = \tau_3$ and consider function $f : \mathfrak{C}^3 \to \mathbb{R}^3$ defined by

$$f(\phi_1,\phi_2,\phi_3) = \left(-a\phi_1(0) - a\phi_2(0)\phi_3(-\tau_3), -a\phi_2(0) - \phi_1(0)\phi_3(-\tau_3), -\phi_3(0) + \frac{3b}{2}\right). \tag{4.23}$$

The nominal values of parameters are $\tau_3 := 0.4$ and $b := 2$ and an uncertainty of $\pm 5\%$ is admitted in the determination of these parameters. In this case, we have that $a_m = 0.4 - (0.4/20)$, $a_M = 0.4 + (0.4/20)$, $b_m = 2 - (1/20)$, $b_M = 2 + (1/20)$. In addition, we consider the following set of parameters $\Lambda \subset \mathbb{R}^2$:

$$\Lambda := \left\{(a,b) \in \mathbb{R}^2;\ a_m \le a \le a_M,\ b_m \le b \le b_M\right\}. \tag{4.24}$$

Consider the scalar functions a, b, c, and V as being, respectively,

$$\begin{aligned}V(\phi_1,\phi_2,\phi_3,\lambda) &:= \frac{\phi_1(0)^2}{2} + \frac{\phi_2(0)^2}{2} + \frac{\phi_3(0)^2}{2} + b,\\ \mathfrak{a}(\phi_1,\phi_2,\phi_3) &:= \frac{\phi_1(0)^2}{2} + \frac{\phi_2(0)^2}{2} + \frac{\phi_3(0)^2}{2} + b_m,\\ \mathfrak{b}(\phi_1,\phi_2,\phi_3) &:= \frac{\phi_1(0)^2}{2} + \frac{\phi_2(0)^2}{2} + \frac{\phi_3(0)^2}{2} + b_M.\end{aligned} \tag{4.25}$$

Then, we have

$$\mathfrak{a}(\phi_1,\phi_2,\phi_3) \le V(\phi_1,\phi_2,\phi_3,\lambda) \le \mathfrak{b}(\phi_1,\phi_2,\phi_3) \tag{4.26}$$

and condition (ii) of Theorem 3.2 is satisfied. We also have

$$\begin{aligned}\dot{V}(x_t, y_t, z_t) &= -ax^2(t) - ay^2(t) - z^2(t) + \frac{3b}{2}z(t)\\ &= -ax^2(t) - ay^2(t) - \left(z^2(t) - \frac{3b}{2}z(t) + \frac{9b^2}{16} - \frac{9b^2}{16}\right)\\ &= -ax^2(t) - ay^2(t) - \left(z(t) - \frac{3b}{4}\right)^2 + \frac{9b^2}{16}\\ &\le -a_m x^2(t) - a_m y^2(t) - \left(z(t) - \frac{3b_m}{4}\right)^2 + \frac{9b_M^2}{16}\end{aligned} \tag{4.27}$$

which implies that

$$-\dot{V}(x_t, y_t, z_t) \ge a_m x^2(t) + a_m y^2(t) + \left(z(t) - \frac{3b_m}{4}\right)^2 - \frac{9b_M^2}{16}. \tag{4.28}$$

If we take $\mathfrak{c}(\phi_1, \phi_2, \phi_3) := a_m\phi_1(0)^2 + a_m\phi_2(0)^2 + (\phi_3(0) - (3b_m/4))^2 - (9b_M^2/16)$, then we have

$$-\dot{V}(x_t, y_t, z_t) \ge \mathfrak{c}(x_t, y_t, z_t), \tag{4.29}$$

for all $t \ge 0$.

Next, we will make the association $\phi_1(0)$, $\phi_2(0)$, $\phi_3(0)$ with (x, y, z) and considering the Lagrangian function

$$\mathfrak{L}(x, y, z) := \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + b_M + \mu\left(a_m x^2 + a_m y^2 + \left(z - \frac{3b_m}{4}\right)^2 - \frac{9b_M^2}{16}\right), \tag{4.30}$$

then we have the following extreme conditions:

$$\begin{gathered}\frac{\partial\mathfrak{L}}{\partial x} = x + 2a_m\mu x = 0, \qquad \frac{\partial\mathfrak{L}}{\partial y} = y + 2a_m\mu y = 0, \qquad \frac{\partial\mathfrak{L}}{\partial z} = z + 2\mu\left[z - \frac{3b_m}{4}\right] = 0,\\ \frac{\partial\mathfrak{L}}{\partial\mu} = a_m x^2 + a_m y^2 + \left(z - \frac{3b_m}{4}\right)^2 - \frac{9b_M^2}{16} = 0.\end{gathered} \tag{4.31}$$

After some calculation, we have $x = 0$, $y = 2.3$, and $z = 2.3$, and this implies

$$\sup_{(\phi_1,\phi_2,\phi_3)\in\mathfrak{C}} \mathfrak{b}(\phi_1, \phi_2, \phi_3) \le 7.1. \tag{4.32}$$

As consequence of Theorem 3.2, the generalized Rössler attractor tends to the largest invariant set contained in the set

$$A_R = \left\{ (x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 \leq 14.2 \right\}. \tag{4.33}$$

Acknowledgment

This research was supported by FAPESP Grant no. 07/54247-6.

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