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Research Article

Stability of a Jensen Type Logarithmic Functional Equation on Restricted Domains and Its Asymptotic Behaviors

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Let $\mathbb{R}_+$ be the set of positive real numbers, B a Banach space, $f : \mathbb{R}_+ \to B$, and $\epsilon > 0$, $p, q, P, Q \in \mathbb{R}$ with $pqPQ \neq 0$. We prove the Hyers-Ulam stability of the Jensen type logarithmic functional inequality $\|f(x^p y^q) - Pf(x) - Qf(y)\| \le \epsilon$ in restricted domains of the form $\{(x, y) : x > 0, y > 0, x^k y^s \ge d\}$ for fixed $k, s \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$ and $d > 0$. As consequences of the results we obtain asymptotic behaviors of the inequality as $x^k y^s \to \infty$.

1. Introduction

The stability problems of functional equations have been originated by Ulam in 1940 (see [1]). One of the first assertions to be obtained is the following result, essentially due to Hyers [2], that gives an answer for the question of Ulam.

Theorem 1.1. *Suppose that $\langle S, + \rangle$ is an additive semigroup, B is a Banach space, $\epsilon \ge 0$, and $f : S \to B$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \tag{1.1}$$

for all $x, y \in S$. Then there exists a unique function $A : S \to B$ satisfying

$$A(x+y) = A(x) + A(y) \tag{1.2}$$

for which

$$\|f(x) - A(x)\| \le \epsilon \tag{1.3}$$

for all $x \in S$.

In 1950-1951 this result was generalized by the authors Aoki [3] and Bourgin [4, 5]. Unfortunately, no results appeared until 1978 when Th. M. Rassias generalized the Hyers' result to a new approximately linear mappings [6]. Following the Rassias' result, a great number of the papers on the subject have been published concerning numerous functional equations in various directions [6–16]. For more precise descriptions of the Hyers-Ulam stability and related results, we refer the reader to the paper of Moszner [17]. Among the results, the stability problem in a restricted domain was investigated by Skof, who proved the stability problem of the inequality (1.1) in a restricted domain [16]. Developing this result, Jung considered the stability problems in restricted domains for the Jensen functional equation [11] and Jensen type functional equations [14]. The results can be summarized as follows: let X and Y be a real normed space and a real Banach space, respectively. For fixed $d > 0$, if $f : X \to Y$ satisfies the functional inequalities (such as that of Cauchy, Jensen and Jensen type, etc.) for all $x, y \in X$ with $\|x\| + \|y\| \ge d$, the inequalities hold for all $x, y \in X$. We also refer the reader to [18–26] for some interesting results on functional equations and their Hyers-Ulam stabilities in restricted conditions.

Throughout this paper, we denote by $\mathbb{R}_+$ the set of positive real numbers, B a Banach space, $f : \mathbb{R}_+ \to B$, and $p, q, P, Q \in \mathbb{R}$ with $pqPQ \ne 0$. We prove the Hyers-Ulam stability of the Jensen type logarithmic functional inequality

$$\|f(x^p y^q) - Pf(x) - Qf(y)\| \le \epsilon \tag{1.4}$$

in the restricted domains of the form $U_{k,s} = \{(x, y) : x > 0,\ y > 0,\ x^k y^s \ge d\}$ for fixed $k, s \in \mathbb{R}$ with $k \ne 0$ or $s \ne 0$, and $d > 0$. As a result, we prove that if the inequality (1.4) holds for all $(x, y) \in U_{k,s}$, there exists a unique function $L : \mathbb{R}_+ \to B$ satisfying

$$L(xy) - L(x) - L(y) = 0, \quad x, y > 0 \tag{1.5}$$

for which

$$\|f(x) - L(x) - f(1)\| \le 4\epsilon \tag{1.6}$$

for all $x > 0$ if $k/p \ne s/q$,

$$\|f(x) - L(x) - f(1)\| \le \frac{4\epsilon}{|P|} \tag{1.7}$$

for all $x > 0$ if $s \ne 0$, and

$$\|f(x) - L(x) - f(1)\| \le \frac{4\epsilon}{|Q|} \tag{1.8}$$

for all $x > 0$ if $k \neq 0$. As a consequence of the result we obtain the stability of the inequality

$$\|f(px + qy) - Pf(x) - Qf(y)\| \leq \epsilon \tag{1.9}$$

in the restricted domains of the form $\{(x, y) \in \mathbb{R}^2 : kx + sy \geq d\}$ for fixed $k, s \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$, and $d \in \mathbb{R}$. Also we obtain asymptotic behaviors of the inequalities (1.4) and (1.9) as $x^k y^s \to \infty$ and $kx + sy \to \infty$, respectively.

2. Hyers-Ulam Stability in Restricted Domains

We call the functions satisfying (1.5) *logarithmic functions*. As a direct consequence of Theorem 1.1, we obtain the stability of the logarithmic functional equation, viewing $\langle \mathbb{R}_+, \times \rangle$ as a multiplicative group (see also the result of Forti [9]).

Theorem A. *Suppose that* $f : \mathbb{R}_+ \to B$, $\epsilon \geq 0$, *and*

$$\|f(xy) - f(x) - f(y)\| \leq \epsilon \tag{2.1}$$

for all $x, y > 0$. *Then there exists a unique logarithmic function* $L : \mathbb{R}_+ \to B$ *satisfying*

$$\|f(x) - L(x)\| \leq \epsilon \tag{2.2}$$

for all $x > 0$.

We first consider the usual logarithmic functional inequality (2.1) in the restricted domains $U_{k,s}$.

Theorem 2.1. *Let* $\epsilon, d > 0$, $k, s \in \mathbb{R}$ *with* $k \neq 0$ *or* $s \neq 0$. *Suppose that* $f : \mathbb{R}_+ \to B$ *satisfies*

$$\|f(xy) - f(x) - f(y)\| \leq \epsilon \tag{2.3}$$

for all $x, y > 0$, *with* $x^k y^s \geq d$. *Then there exists a unique logarithmic function* $L : \mathbb{R}_+ \to B$ *such that*

$$\|f(x) - L(x)\| \leq 3\epsilon \tag{2.4}$$

for all $x \in \mathbb{R}_+$.

Proof. From the symmetry of the inequality we may assume that $s \neq 0$. For given $x, y \in \mathbb{R}_+$, choose a $z > 0$ such that $x^k y^k z^s \geq d$, $x^k y^s z^s \geq d$, and $y^k z^s \geq d$. Then we have

$$\begin{aligned} \|f(xy) - f(x) - f(y)\| &\leq \|-f(xyz) + f(xy) + f(z)\| \\ &\quad + \|f(xyz) - f(x) - f(yz)\| \\ &\quad + \|f(yz) - f(y) - f(z)\| \\ &\leq 3\epsilon. \end{aligned} \tag{2.5}$$

This completes the proof. □

Now we consider the Hyers-Ulam stability of the Jensen type logarithmic functional inequality (1.4) in the restricted domains $U_{k,s}$.

Theorem 2.2. *Let $\epsilon, d > 0$, $k, s \in \mathbb{R}$, $k/p \neq s/q$. Suppose that $f : \mathbb{R}_+ \to B$ satisfies*

$$\|f(x^p y^q) - Pf(x) - Qf(y)\| \leq \epsilon \tag{2.6}$$

for all $x, y > 0$, with $x^k y^s \geq d$. Then there exists a unique logarithmic function $L : \mathbb{R}_+ \to B$ such that

$$\|f(x) - L(x) - f(1)\| \leq 4\epsilon \tag{2.7}$$

for all $x \in R_+$.

Proof. Replacing x by $x^{1/p}$, y by $y^{1/q}$ in (2.6) we have

$$\left\|f(xy) - Pf\left(x^{1/p}\right) - Qf\left(y^{1/q}\right)\right\| \leq \epsilon \tag{2.8}$$

for all $x, y > 0$, with $x^{k/p} y^{s/q} \geq d$.

For given $x, y \in \mathbb{R}_+$, choose a $z > 0$ such that $x^{k/p} y^{s/q} z^{s/q - k/p} \geq d$, $x^{k/p} z^{s/q - k/p} \geq d$, $y^{s/q} z^{s/q - k/p} \geq d$, and $z^{s/q - k/p} \geq d$. Replacing x by xz^{-1}, y by yz; x by xz^{-1}, y by z; x by z^{-1}, y by yz; x by z^{-1}, y by z in (2.8) we have

$$\begin{aligned} \|f(xy) - f(x) - f(y) + f(1)\| &\leq \left\|f(xy) - Pf\left(x^{1/p} z^{-1/p}\right) - Qf\left((yz)^{1/q}\right)\right\| \\ &\quad + \left\|-f(x) + Pf\left(x^{1/p} z^{-1/p}\right) + Qf\left(z^{1/q}\right)\right\| \\ &\quad + \left\|-f(y) + Pf\left(z^{-1/p}\right) + Qf\left((yz)^{1/q}\right)\right\| \\ &\quad + \left\|f(1) - Pf\left(z^{-1/p}\right) - Qf\left(z^{1/q}\right)\right\| \\ &\leq 4\epsilon. \end{aligned} \tag{2.9}$$

Now by Theorem A, there exists a unique logarithmic function $L : \mathbb{R}_+ \to B$ such that

$$\|f(x) - L(x) - f(1)\| \le 4\epsilon \tag{2.10}$$

for all $x \in \mathbb{R}_+$. This completes the proof. □

As a matter of fact, we obtain that $L = 0$ in Theorem 2.2 provided that $p \ne P$ and p or P is a rational number, or $q \ne Q$ and q or Q is a rational number.

Theorem 2.3. *Let $\epsilon, d > 0$, $k, s \in \mathbb{R}$, $k/p \ne s/q$. Suppose that $p \ne P$ and p or P is a rational number, or $q \ne Q$ and q or Q is a rational number, and $f : \mathbb{R}_+ \to B$ satisfies*

$$\|f(x^p y^q) - Pf(x) - Qf(y)\| \le \epsilon \tag{2.11}$$

for all $x, y > 0$, with $x^k y^s \ge d$. Then one has

$$\|f(x) - f(1)\| \le 4\epsilon \tag{2.12}$$

for all $x \in \mathbb{R}_+$.

Proof. We prove (2.12) only for the case that $p \ne P$ and p or P is a rational number since the other case is similarly proved. From (2.7) and (2.11), using the triangle inequality we have

$$\|L(x^p y^q) - PL(x) - QL(y)\| \le M \tag{2.13}$$

for all $x, y > 0$, with $x^k y^s \ge d$, where $M = \epsilon(5 + 4|P| + 4|Q|) + |f(1)(1 - P - Q)|$. If $k \ne 0$, putting $y = 1$ in (2.13) we have

$$\|L(x^p) - PL(x)\| \le M \tag{2.14}$$

for all $x > 0$, with $x^k \ge d$. It is easy to see that $L(x^r) = rL(x)$ for all $x > 0$ and all rational numbers r. Thus if p is a rational number, it follows from (2.14) that

$$\|L(x)\| \le \frac{M}{|p - P|} \tag{2.15}$$

for all $x > 0$, with $x^k \ge d$. If there exists $x_0 > 0$ such that $L(x_0) \ne 0$, we can choose a rational number r such that $x_0^{rk} \ge d$ and $\|rL(x_0)\| > M/|p - P|$ (it is realized when r is large if $x_0^k > 1$, and when $-r$ is large if $x_0^k < 1$). Now we have

$$\frac{M}{|p - P|} < \|rL(x_0)\| = \|L(x_0^r)\| \le \frac{M}{|p - P|}. \tag{2.16}$$

Thus it follows that $L = 0$. If P is a rational number, it follows from (2.14) that

$$\left\|L\left(x^{p-P}\right)\right\| \le M \tag{2.17}$$

for all $x > 0$, with $x^k \ge d$, which implies

$$\|L(x)\| \le M \tag{2.18}$$

for all $x > 0$, with $x^{k/(p-P)} \ge d$. Similarly, using (2.18) we can show that $L = 0$. If $k = 0$, choosing $y_0 > 0$ such that $y_0^s \ge d$, putting $y = y_0$ in (2.13) and using the triangle inequality we have

$$\|L(x^p) - PL(x)\| \le M + \left|L\left(y_0^q\right) - QL(y_0)\right| \tag{2.19}$$

for all $x > 0$. Similarly, using (2.19) we can show that $L = 0$. Thus the inequality (2.12) follows from (2.7). This completes the proof. □

Theorem 2.4. *Let $\epsilon, d > 0$, $k, s \in \mathbb{R}$ with $k \ne 0$ or $s \ne 0$. Suppose that $f : \mathbb{R}_+ \to B$ satisfies*

$$\|f(x^p y^q) - Pf(x) - Qf(y)\| \le \epsilon \tag{2.20}$$

for all $x, y > 0$, with $x^k y^s \ge d$. Then there exists a unique logarithmic function $L : \mathbb{R}_+ \to B$ such that

$$\|f(x) - L(x) - f(1)\| \le \frac{4\epsilon}{|P|} \tag{2.21}$$

for all $x \in \mathbb{R}_+$ if $s \ne 0$, and

$$\|f(x) - L(x) - f(1)\| \le \frac{4\epsilon}{|Q|} \tag{2.22}$$

for all $x \in \mathbb{R}_+$ if $k \ne 0$.

Proof. Assume that $s \neq 0$. For given $x, y \in \mathbb{R}_+$, choose a $z > 0$ such that $x^k y^k z^s \geq d$, $x^k y^{ps/q} z^s \geq d$, $y^k z^s \geq d$ and $y^{ps/q} z^s \geq d$. Replacing x by xy, y by z; x by x, y by $y^{p/q}z$; x by y, y by z; x by 1, y by $y^{p/q}z$ in (2.20) we have

$$\begin{aligned}\|Pf(xy) - Pf(x) - Pf(y) + Pf(1)\| \leq{}& \|-f((xy)^p z^q) + Pf(xy) + Qf(z)\| \\ &+ \left\|f((xy)^p z^q) - Pf(x) - Qf\left(y^{p/q}z\right)\right\| \\ &+ \|f(y^p z^q) - Pf(y) - Qf(z)\| \\ &+ \left\|-f(y^p z^q) + Pf(1) + Qf\left(y^{p/q}z\right)\right\| \\ \leq{}& 4\epsilon.\end{aligned} \tag{2.23}$$

Dividing (2.23) by $|P|$ and using Theorem A, we obtain that there exists a unique logarithmic function $L : \mathbb{R}_+ \to B$ such that

$$\|f(x) - L(x) - f(1)\| \leq \frac{4\epsilon}{|P|} \tag{2.24}$$

for all $x \in \mathbb{R}_+$. Assume that $k \neq 0$. For given $x, y \in \mathbb{R}_+$, choose a $z > 0$ such that $x^s y^s z^k \geq d$, $x^{qk/p} y^s z^k \geq d$, $x^s z^k \geq d$ and $x^{qk/p} z^k \geq d$. Replacing y by xy, x by z; y by y, x by $x^{q/p}z$; y by x, x by z; y by 1, x by $x^{q/p}z$ in (2.20) we have

$$\begin{aligned}\|Qf(xy) - Qf(x) - Qf(y) + Qf(1)\| \leq{}& \|-f((xy)^q z^p) + Pf(z) + Qf(xy)\| \\ &+ \left\|f((xy)^q z^p) - Pf\left(x^{q/p}z\right) - Qf(y)\right\| \\ &+ \|f(x^q z^p) - Pf(z) - Qf(x)\| \\ &+ \left\|-f(x^q z^p) + Pf\left(x^{q/p}z\right) + Qf(1)\right\| \\ \leq{}& 4\epsilon.\end{aligned} \tag{2.25}$$

Dividing (2.25) by $|Q|$ and using Theorem A, we obtain that there exists a unique logarithmic function $L : \mathbb{R}_+ \to B$ such that

$$\|f(x) - L(x) - f(1)\| \leq \frac{4\epsilon}{|Q|} \tag{2.26}$$

for all $x \in \mathbb{R}_+$. This completes the proof. □

From Theorem 2.4, using the same approach as in the proof of Theorem 2.3 we have the following.

Theorem 2.5. *Let $\epsilon, d > 0$, $k, s \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$. Suppose that $p \neq P$ and p or P is a rational number, or $q \neq Q$ and q or Q is a rational number, and $f : \mathbb{R}_+ \to B$ satisfies*

$$\| f(x^p y^q) - Pf(x) - Qf(y) \| \leq \epsilon \tag{2.27}$$

for all $x, y > 0$, with $x^k y^s \geq d$. Then one has

$$\| f(x) - f(1) \| \leq \frac{4\epsilon}{|P|} \tag{2.28}$$

for all $x \in \mathbb{R}_+$ if $s \neq 0$, and

$$\| f(x) - f(1) \| \leq \frac{4\epsilon}{|Q|} \tag{2.29}$$

for all $x \in \mathbb{R}_+$ if $k \neq 0$.

We call $A : \mathbb{R} \to B$ *an additive function* provided that

$$A(x + y) = A(x) + A(y) \tag{2.30}$$

for all $x, y \in \mathbb{R}$. Using Theorem 2.2 we have the following.

Corollary 2.6 (see [22]). *Let $\epsilon > 0$, $d, k, s \in \mathbb{R}$ with $k/p \neq s/q$. Suppose that $g : \mathbb{R} \to B$ satisfies*

$$\| g(px + qy) - Pg(x) - Qg(y) \| \leq \epsilon \tag{2.31}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then there exists a unique additive function $A : \mathbb{R} \to B$ such that

$$\| g(x) - A(x) - g(0) \| \leq 4\epsilon \tag{2.32}$$

for all $x \in \mathbb{R}$.

Proof. Replacing x by $\ln u$, y by $\ln v$ in (2.31) and setting $f(x) = g(\ln x)$ we have

$$\| f(u^p v^q) - Pf(u) - Qf(v) \| \leq \epsilon \tag{2.33}$$

for all $u, v \in \mathbb{R}$, with $u^k v^s \geq e^d$. Using Theorem 2.2, we have

$$\| f(x) - L(x) - f(1) \| \leq 4\epsilon \tag{2.34}$$

for all $x \in \mathbb{R}_+$, which implies

$$\| g(x) - L(e^x) - g(0) \| \leq 4\epsilon \tag{2.35}$$

for all $x \in \mathbb{R}$. Letting $A(x) = L(e^x)$ we get the result. □

Using Theorem 2.3, we have the following.

Corollary 2.7. *Let $\epsilon > 0$, $d, k, s \in \mathbb{R}$ with $k/p \neq s/q$. Suppose that $p \neq P$ and p or P is a rational number, or $q \neq Q$ and q or Q is a rational number, and $g : \mathbb{R} \to B$ satisfies*

$$\|g(px + qy) - Pg(x) - Qg(y)\| \leq \epsilon \tag{2.36}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then one has

$$\|g(x) - g(0)\| \leq 4\epsilon \tag{2.37}$$

for all $x \in \mathbb{R}$.

Using Theorem 2.4, we have the following.

Corollary 2.8. *Let $\epsilon > 0$, $d, k, s \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$. Suppose that $g : \mathbb{R} \to B$ satisfies*

$$\|g(px + qy) - Pg(x) - Qg(y)\| \leq \epsilon \tag{2.38}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then there exists a unique additive function $A : \mathbb{R} \to B$ such that

$$\|g(x) - A(x) - g(0)\| \leq \frac{4\epsilon}{|P|} \tag{2.39}$$

for all $x \in \mathbb{R}$ if $s \neq 0$, and

$$\|g(x) - A(x) - g(0)\| \leq \frac{4\epsilon}{|Q|} \tag{2.40}$$

for all $x \in \mathbb{R}$ if $k \neq 0$.

Using Theorem 2.5, we have the following.

Corollary 2.9. *Let $\epsilon > 0$, $d, k, s \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$. Suppose that $p \neq P$ and p or P is a rational number, or $q \neq Q$ and q or Q is a rational number, and $g : \mathbb{R} \to B$ satisfies*

$$\|g(px + qy) - Pg(x) - Qg(y)\| \leq \epsilon \tag{2.41}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then one has

$$\|g(x) - g(0)\| \leq \frac{4\epsilon}{|P|} \tag{2.42}$$

for all $x \in \mathbb{R}$ if $s \neq 0$, and

$$\|g(x) - g(0)\| \leq \frac{4\epsilon}{|Q|} \tag{2.43}$$

for all $x \in \mathbb{R}$ if $k \neq 0$.

3. Asymptotic Behavior of the Inequality

In this section, we consider asymptotic behaviors of the inequalities (1.4) and (2.1).

Theorem 3.1. *Let $k, s \in \mathbb{R}$ satisfy one of the conditions; $k \neq 0$, $s \neq 0$. Suppose that $f : \mathbb{R}_+ \to B$ satisfies the asymptotic condition*

$$\|f(xy) - f(x) - f(y)\| \longrightarrow 0 \tag{3.1}$$

as $x^k y^s \to \infty$. Then f is a logarithmic function.

Proof. By the condition (3.1), for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\|f(xy) - f(x) - f(y)\| \leq \frac{1}{n} \tag{3.2}$$

for all $x, y > 0$, with $x^k y^s \geq d_n$. By Theorem 2.1, there exists a unique logarithmic function $L_n : \mathbb{R}_+ \to B$ such that

$$\|f(x) - L_n(x)\| \leq \frac{3}{n} \tag{3.3}$$

for all $x \in \mathbb{R}_+$. From (3.4) we have

$$\|L_n(x) - L_m(x)\| \leq \frac{3}{n} + \frac{3}{m} \leq 6 \tag{3.4}$$

for all $x \in \mathbb{R}_+$ and all positive integers n, m. Now, the inequality (3.4) implies $L_n = L_m$. Indeed, for all $x > 0$ and rational numbers $r > 0$ we have

$$\|L_n(x) - L_m(x)\| = \frac{1}{r}\|L_n(x^r) - L_m(x^r)\| \leq \frac{6}{r}. \tag{3.5}$$

Letting $r \to \infty$ in (3.5), we have $L_n = L_m$. Thus, letting $n \to \infty$ in (3.3), we get the result. □

Theorem 3.2. *Let $k, s \in \mathbb{R}$ satisfy one of the conditions; $k \neq 0$, $s \neq 0$, $k/p \neq s/q$. Suppose that $f : \mathbb{R}_+ \to B$ satisfies the asymptotic condition*

$$\|f(x^p y^q) - Pf(x) - Qf(y)\| \longrightarrow 0 \tag{3.6}$$

as $x^k y^s \to \infty$. Then there exists a unique logarithmic function $L : \mathbb{R}_+ \to B$ such that

$$f(x) = L(x) + f(1) \tag{3.7}$$

for all $x \in \mathbb{R}_+$.

Proof. By the condition (3.6), for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\|f(x^p y^q) - Pf(x) - Qf(y)\| \le \frac{1}{n} \tag{3.8}$$

for all $x, y > 0$, with $x^k y^s \ge d_n$. By Theorems 2.2 and 2.4, there exists a unique logarithmic function $L_n : \mathbb{R}_+ \to B$ such that

$$\|f(x) - L_n(x) - f(1)\| \le \frac{4}{n} \tag{3.9}$$

if $k/p \ne s/q$,

$$\|f(x) - L_n(x) - f(1)\| \le \frac{4}{n|P|} \tag{3.10}$$

if $s \ne 0$, and

$$\|f(x) - L_n(x) - f(1)\| \le \frac{4}{n|Q|} \tag{3.11}$$

if $k \ne 0$. For all cases (3.9), (3.10), and (3.11), there exists $M > 0$ such that

$$\|L_n(x) - L_m(x)\| \le M \tag{3.12}$$

for all $x \in \mathbb{R}_+$ and all positive integers n, m. Now as in the proof of Theorem 3.1, it follows from (3.12) that $L_n = L_m$ for all $n, m \in \mathbb{N}$. Letting $n \to \infty$ in (3.9), (3.10), and (3.11) we get the result. □

Similarly using Theorems 2.3 and 2.5, we have the following.

Theorem 3.3. *Let $k, s \in \mathbb{R}$ satisfy one of the conditions; $k \ne 0$, $s \ne 0$, $k/p \ne s/q$. Suppose that $p \ne P$ and p or P is a rational number, or $q \ne Q$ and q or Q is a rational number, and $f : \mathbb{R}_+ \to B$ satisfies the asymptotic condition*

$$\|f(x^p y^q) - Pf(x) - Qf(y)\| \longrightarrow 0 \tag{3.13}$$

as $x^k y^s \to \infty$. Then f is a constant function.

Using Corollaries 2.6 and 2.8 we have the following.

Corollary 3.4. *Let* $\epsilon > 0$, $k, s \in \mathbb{R}$ *satisfy one of the conditions* $k \neq 0$, $s \neq 0$, *or* $k/p \neq s/q$. *Suppose that* $g : \mathbb{R} \to B$ *satisfies*

$$\|g(px + qy) - Pg(x) - Qg(y)\| \longrightarrow 0 \tag{3.14}$$

as $kx + sy \to \infty$. *Then there exists a unique additive function* $A : \mathbb{R} \to B$ *such that*

$$g(x) = A(x) + g(0) \tag{3.15}$$

for all $x \in \mathbb{R}$.

Using Corollaries 2.7 and 2.9 we have the following.

Corollary 3.5. *Let* $\epsilon > 0$, $k, s \in \mathbb{R}$ *satisfy one of the conditions* $k \neq 0$, $s \neq 0$, *or* $k/p \neq s/q$. *Suppose that* $p \neq P$ *and* p *or* P *is a rational number, or* $q \neq Q$ *and* q *or* Q *is a rational number, and* $g : \mathbb{R} \to B$ *satisfies*

$$\|g(px + qy) - Pg(x) - Qg(y)\| \longrightarrow 0 \tag{3.16}$$

as $kx + sy \to \infty$. *Then* g *is a constant function.*

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