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Research Article

Pairs of Function Spaces and Exponential Dichotomy on the Real Line

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We provide a complete diagram of the relation between the admissibility of pairs of Banach function spaces and the exponential dichotomy of evolution families on the real line. We prove that if $W \in \mathscr{H}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$ are two Banach function spaces with the property that either $W \in \mathcal{W}(\mathbb{R})$ or $V \in \mathcal{U}(\mathbb{R})$, then the admissibility of the pair $(W(\mathbb{R}, X), V(\mathbb{R}, X))$ implies the existence of the exponential dichotomy. We study when the converse implication holds and show that the hypotheses on the underlying function spaces cannot be dropped and that the obtained results are the most general in this topic. Finally, our results are applied to the study of exponential dichotomy of C_0-semigroups.

1. Introduction

In the study of the asymptotic behavior of evolution equations the input-output conditions are very efficient tools, with wide applicability area, and give a nice connection between control theory and the qualitative theory of differential equations (see [1–16] and the reference therein). Starting with the pioneering work of Perron (see [8]) these methods were developed and improved in remarkable books (see [1, 4, 6]). A new and interesting perspective on this framework was proposed in [5], where the authors presented a complete study of stability, expansiveness, and dichotomy of evolution families on the half-line in terms of input-output methods. This paper was the starting point for an entire collection of studies dedicated to the input-output techniques and their applications to the qualitative theory of differential and difference equations.

If one analyzes the dichotomous properties of differential equations, then it is easily seen that there are some main technical differences between the case of evolution families

on the half-line (see [5, 9, 10]) and the case of evolution families on the real line (see [11–16]), which require a distinct analysis for each case. For instance, when one determines sufficient conditions for the existence of exponential dichotomy on the half-line, an important hypothesis is that the initial stable subspace is closed and complemented (see, e.g., [5, Theorem 4.3] or [9, Theorem 3.3]). This assumption may be dropped when we study the exponential dichotomy on the real line (see, e.g., [11, Theorem 5.1] or [16, Theorem 5.3]). These facts implicitly generate the differences between the admissibility concepts used on the real line compared with those used on the half-line and also interesting technical approaches in each case.

The aim of the present paper is to provide new and very general conditions for the existence of exponential dichotomy on the real line. We consider the problem of finding connections between the solvability of an integral equation and the existence of exponential dichotomy of evolution families on the real line. The main purpose is to obtain a complete diagram and a classification of the classes of function spaces that may be used in the study of exponential dichotomy via admissibility.

For the beginning we will present the previous results in this topic and the main objectives will be clearly specified in the context of the actual state of knowledge. We denote by $\mathcal{T}(\mathbb{R})$ the class of all Banach sequence spaces B which are invariant under translations, contain the continuous functions with compact support, satisfy an integral property and if $B \setminus L^1(\mathbb{R},\mathbb{R}) \neq \emptyset$, then there is a continuous function $\varphi \in B \setminus L^1(\mathbb{R},\mathbb{R})$. We consider $\mathcal{H}(\mathbb{R})$ the subclass of $\mathcal{T}(\mathbb{R})$ satisfying the ideal property. We associate two subclasses of $\mathcal{H}(\mathbb{R})$: $\mathcal{W}(\mathbb{R})$—the class of all Banach function spaces with unbounded fundamental function and $\mathcal{U}(\mathbb{R})$—the class of all Banach function spaces which contain at least a nonintegrable function. A pair of function spaces $(W(\mathbb{R},X),V(\mathbb{R},X))$ is called admissible for an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ on the Banach space X if for every test function in the input space $V(\mathbb{R},X)$ there exists a unique solution function in the output space $W(\mathbb{R},X)$ for the associated integral equation given by the variation of constants formula (see Definition 3.5 below).

For the first time, we have proposed in [11] a sufficient condition for exponential dichotomy, using certain Banach function spaces which are invariant under translations and we obtained the following theorem.

Theorem 1.1. *If $V \in \mathcal{U}(\mathbb{R})$ and the pair $(C_b(\mathbb{R},X),V(\mathbb{R},X))$ is admissible for an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$, then $\mathcal{U}$ is exponentially dichotomic.*

Our study has been continued and extended in [16], both for uniform dichotomy and exponential dichotomy. According to the proof of Theorem 4.8 in [16] we may give the following sufficient condition for uniform dichotomy.

Theorem 1.2. *If $W \in \mathcal{H}(\mathbb{R})$, $V \in \mathcal{T}(\mathbb{R})$, and the pair $(W(\mathbb{R},X),V(\mathbb{R},X))$ is admissible for an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$, then $\mathcal{U}$ is uniformly dichotomic.*

From the proof of Theorem 5.3(i) in [16] we deduce the following sufficient condition for exponential dichotomy.

Theorem 1.3. *If $W \in \mathcal{W}(\mathbb{R})$, $V \in \mathcal{T}(\mathbb{R})$, and the pair $(W(\mathbb{R},X),V(\mathbb{R},X))$ is admissible for an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$, then $\mathcal{U}$ is exponentially dichotomic.*

Taking into account the above results and their consequences, the natural question arises whether, in the general case, the output space may belong to the class $\mathcal{H}(\mathbb{R})$ and

if so, which is the most general class where the input space should belong to. The aim of the present paper is to answer this question and to provide a complete study of the exponential dichotomy on the real line via integral admissibility. The answer to the above question will establish clearly how should one modify the hypotheses of Theorem 1.2 such that the admissibility of the pair $(W(\mathbb{R},X),V(\mathbb{R},X))$ implies the existence of the exponential dichotomy.

We will prove that if $W \in \mathscr{H}(\mathbb{R})$ and $V \in \mathcal{U}(\mathbb{R})$, then the admissibility of the pair $(W(\mathbb{R},X),V(\mathbb{R},X))$ is a sufficient condition for exponential dichotomy. Consequently, we will deduce a complete diagram of the study of exponential dichotomy on the real line in terms of the admissibility of function spaces (see Theorem 3.11). Specifically, if $W \in \mathscr{H}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$ are two Banach function spaces with the property that either $W \in \mathcal{W}(\mathbb{R})$ or $V \in \mathcal{U}(\mathbb{R})$, then the admissibility of the pair $(W(\mathbb{R},X),V(\mathbb{R},X))$ implies the existence of the exponential dichotomy. Also, in certain conditions, we deduce that the exponential dichotomy of an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is equivalent with the admissibility of the pair $(W(\mathbb{R},X),V(\mathbb{R},X))$.

By an example we motivate our techniques and show that the hypotheses from our main results cannot be removed. Precisely, if $W \in \mathscr{H}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$ are such that $W \notin \mathcal{W}(\mathbb{R})$ and $V \notin \mathcal{U}(\mathbb{R})$, then we prove that the admissibility of the pair $(W(\mathbb{R},X),V(\mathbb{R},X))$ does not imply the exponential dichotomy. Moreover, we show that the obtained results and their consequences are the most general in this topic.

Finally, our results are applied at the study of the exponential dichotomy of C_0-semigroups. Using function spaces which are invariant under translations, we obtain a classification of the classes of input and output spaces which may be used in the study of exponential dichotomy of semigroups in terms of input-output techniques with respect to associated integral equations.

2. Preliminaries: Banach Function Spaces

In this section, for the sake of clarity, we present some definitions and notations and we introduce the main classes of function spaces that will be used in our study. Let $\mathscr{M}(\mathbb{R})$ be the linear space of all Lebesgue measurable functions $u : \mathbb{R} \to \mathbb{R}$, identifying the functions equal almost everywhere.

Definition 2.1. A linear subspace B of $\mathscr{M}(\mathbb{R})$ is called *normed function space* if there is a mapping $|\cdot|_B : B \to \mathbb{R}_+$ such that

(i) $|u|_B = 0$ if and only if $u = 0$ a.e.;

(ii) $|\alpha u|_B = |\alpha||u|_B$, for all $(\alpha,u) \in \mathbb{R}\times B$;

(iii) $|u+v|_B \leq |u|_B + |v|_B$, for all $u,v \in B$;

(iv) if $u,v \in B$ and $|u| \leq |v|$ a.e. then $|u|_B \leq |v|_B$;

(v) if $u \in B$, then $|u| \in B$.

If $(B,|\cdot|_B)$ is complete, then B is called *Banach function space*.

Definition 2.2. A Banach function space $(B,|\cdot|_B)$ is said to be *invariant under translations* if for every $(u,s) \in B\times\mathbb{R}$, the function $u_s : \mathbb{R} \to \mathbb{R}$, $u_s(t) = u(t-s)$ belongs to B and $|u_s|_B = |u|_B$.

Notations 1. Let $C_c(\mathbb{R},\mathbb{R})$ denote the linear space of all continuous functions $v:\mathbb{R}\to\mathbb{R}$ with compact support. Throughout this paper, we denote by $\mathcal{T}(\mathbb{R})$ the class of all Banach function spaces B, which are invariant under translations, $C_c(\mathbb{R},\mathbb{R})\subset B$, and satisfy the following conditions:

(i) for every $t>s$ there is $\alpha(t,s)>0$ such that $\int_s^t|u(\tau)|d\tau\le\alpha(t,s)|u|_B$, for all $u\in B$;

(ii) if $B\setminus L^1(\mathbb{R},\mathbb{R})\neq\emptyset$ then there is a continuous function $\varphi\in B\setminus L^1(\mathbb{R},\mathbb{R})$.

For examples of Banach function spaces from the class $\mathcal{T}(\mathbb{R})$ we refer to [11].

Let $\mathcal{H}(\mathbb{R})$ be the class of all Banach function spaces $B\in\mathcal{T}(\mathbb{R})$ with the property that if $|u|\le|v|$ a.e. and $v\in B$, then $u\in B$.

For every $A\subset\mathbb{R}$ we denote by χ_A the characteristic function of the set A. Then, if $B\in\mathcal{H}(\mathbb{R})$, we have that $\chi_{[a,b)}\in B$, for every $a,b\in\mathbb{R}$ with $a<b$.

Definition 2.3. Let $B\in\mathcal{H}(\mathbb{R})$. The mapping $F_B:(0,\infty)\to\mathbb{R}$, $F_B(t)=|\chi_{[0,t)}|_B$ is called *the fundamental function* of the space B.

For the proof of the next proposition we refer to [16, Proposition 2.8].

Proposition 2.4. *Let $B\in\mathcal{H}(\mathbb{R})$ and $\nu>0$. If $u:\mathbb{R}\to\mathbb{R}_+$ is a function, which belongs to B and with the property that $q_u:\mathbb{R}\to\mathbb{R}_+$, $q_u(t)=\int_t^{t+1}u(s)ds$ belongs to B, then the functions*

$$f_u,g_u:\mathbb{R}\longrightarrow\mathbb{R}_+,\qquad f_u(t)=\int_{-\infty}^{t}e^{-\nu(t-s)}u(s)ds,\qquad g_u(t)=\int_t^{\infty}e^{-\nu(s-t)}u(s)ds \tag{2.1}$$

belong to B.

Example 2.5. Let $M^1(\mathbb{R},\mathbb{R})$ be the linear space of all $u\in\mathcal{M}(\mathbb{R})$ with the property that $\sup_{t\in\mathbb{R}}\int_t^{t+1}|u(s)|ds<\infty$. With respect to the norm

$$\|u\|_{M^1}:=\sup_{t\in\mathbb{R}}\int_t^{t+1}|u(s)|ds, \tag{2.2}$$

this is a Banach function space which belongs to $\mathcal{H}(\mathbb{R})$.

Lemma 2.6. *If $B\in\mathcal{T}(\mathbb{R})$, then $B\subset M^1(\mathbb{R},\mathbb{R})$.*

Proof. Let $\alpha>0$ be such that $\int_0^1|u(\tau)|d\tau\le\alpha|u|_B$, for all $u\in B$. Then, we have that

$$\int_t^{t+1}|v(\tau)|d\tau=\int_0^1|v_t(\tau)|d\tau\le\alpha|v_t|_B=\alpha|v|_B,\quad\forall t\in\mathbb{R},\ \forall v\in B. \tag{2.3}$$

□

Notations 2. In what follows we denote by

(i) $\mathcal{W}(\mathbb{R})$ the class of all Banach function spaces $B \in \mathcal{L}(\mathbb{R})$ with $\sup_{t>0} F_B(t) = \infty$;

(ii) $\mathcal{U}(\mathbb{R})$ the class of all Banach function spaces $B \in \mathcal{T}(\mathbb{R})$ with the property that $B \setminus L^1(\mathbb{R},\mathbb{R}) \neq \emptyset$;

(iii) $\mathcal{O}(\mathbb{R})$ the class of all Banach function spaces $B \in \mathcal{L}(\mathbb{R})$ with the property that for every $u : \mathbb{R} \to \mathbb{R}_+$ in B, the function $q_u : \mathbb{R} \to \mathbb{R}_+$, $q_u(t) = \int_t^{t+1} u(s)ds$ belongs to B.

Remark 2.7. (i) For examples of Banach function spaces from the class $\mathcal{W}(\mathbb{R})$ we refer to [16, Proposition 2.9].

(ii) If $B \in \mathcal{U}(\mathbb{R})$ then there is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}_+$ with $\varphi \in B \setminus L^1(\mathbb{R},\mathbb{R})$.

Notation 1. Let $C_0(\mathbb{R},\mathbb{R})$ be the space of all continuous functions $v : \mathbb{R} \to \mathbb{R}$ with $\lim_{t\to\pm\infty} v(t) = 0$, which is Banach space with respect to the norm $|||v||| := \sup_{t\in\mathbb{R}} |v(t)|$.

Lemma 2.8. *Let B be a Banach function space with $B \in \mathcal{L}(\mathbb{R}) \setminus \mathcal{W}(\mathbb{R})$. Then $C_0(\mathbb{R},\mathbb{R}) \subset B$.*

Proof. Let $L := \sup_{t>0} F_B(t)$. Let $v \in C_0(\mathbb{R},\mathbb{R})$. Then there is an unbounded increasing sequence $(t_n) \subset (0,\infty)$ such that $|v(t)| \leq 1/(n+1)$, for all $|t| \geq t_n$ and all $n \in \mathbb{N}$. Setting $v_n = v\chi_{[-t_n,t_n]}$ we have that

$$\left|v_{n+p} - v_n\right|_B \leq \frac{1}{n+1}\left(\left|\chi_{[-t_{n+p},-t_n)}\right|_B + \left|\chi_{(t_n,t_{n+p}]}\right|_B\right) \leq \frac{2L}{n+1}, \quad \forall p \in \mathbb{N}^*,\ \forall n \in \mathbb{N}. \tag{2.4}$$

From the above inequality we deduce that (v_n) is fundamental in the Banach space B, so there is $w \in B$ such that $v_n \to w$ in B. According to [16, Lemma 2.4] there is a subsequence (v_{k_n}) such that $v_{k_n} \to w$ a.e. This implies that $v \equiv w$ a.e., so $v = w$ in B. Thus $v \in B$ and the proof is complete. □

Notation 2. Let X be a real or complex Banach space. For every $B \in \mathcal{T}(\mathbb{R})$ we denote by $B(\mathbb{R},X)$ the linear space of all Bochner measurable functions $v : \mathbb{R} \to X$ with the property that the mapping $N_v : \mathbb{R} \to \mathbb{R}_+$, $N_v(t) = \|v(t)\|$ lies in B. With respect to the norm $\|v\|_{B(\mathbb{R},X)} := |N_v|_B$, $B(\mathbb{R},X)$ is a Banach space.

3. Exponential Dichotomy for Evolution Families on the Real Line

Let X be a real or complex Banach space. The norm on X and on $\mathcal{B}(X)$, the Banach algebra of all bounded linear operators on X, will be denoted by $\|\cdot\|$. Denote by I_d the identity operator on X. First, we remind some basic definitions.

Definition 3.1. A family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ of bounded linear operators on X is called an *evolution family* if the following properties hold:

(i) $U(t_0,t_0) = I_d$ and $U(t,s)U(s,t_0) = U(t,t_0)$, for all $t \geq s \geq t_0$;

(ii) for every $x \in X$ and every $t_0 \in \mathbb{R}$ the mapping $t \mapsto U(t,t_0)x$ is continuous on $[t_0,\infty)$ and the mapping $s \mapsto U(t_0,s)x$ is continuous on $(-\infty,t_0]$;

(iii) there are $M \geq 1$ and $\omega > 0$ such that $\|U(t,t_0)\| \leq Me^{\omega(t-t_0)}$, for all $t \geq t_0$.

Definition 3.2. An evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is said to be *uniformly dichotomic* if there are a family of projections $\{P(t)\}_{t\in\mathbb{R}}$ and a constant $K \geq 1$ such that

(i) $U(t,t_0)P(t_0) = P(t)U(t,t_0)$, for all $t \geq t_0$;

(ii) the restriction $U(t,t_0)_| : \ker P(t_0) \to \ker P(t)$ is an isomorphism, for all $t \geq t_0$;

(iii) $\|U(t,t_0)x\| \leq K\|x\|$, for all $x \in \operatorname{Im} P(t_0)$ and all $t \geq t_0$;

(iv) $\|U(t,t_0)y\| \geq (1/K)\|y\|$, for all $y \in \ker P(t_0)$ and all $t \geq t_0$.

Definition 3.3. An evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is said to be *exponentially dichotomic* if there exist a family of projections $\{P(t)\}_{t\in\mathbb{R}}$ and two constants $K \geq 1$ and $\nu > 0$ such that

(i) $U(t,t_0)P(t_0) = P(t)U(t,t_0)$, for all $t \geq t_0$;

(ii) the restriction $U(t,t_0)_| : \ker P(t_0) \to \ker P(t)$ is an isomorphism, for all $t \geq t_0$;

(iii) $\|U(t,t_0)x\| \leq Ke^{-\nu(t-t_0)}\|x\|$, for all $x \in \operatorname{Im} P(t_0)$ and all $t \geq t_0$;

(iv) $\|U(t,t_0)y\| \geq (1/K)e^{\nu(t-t_0)}\|y\|$, for all $y \in \ker P(t_0)$ and all $t \geq t_0$.

Remark 3.4. It is obvious that if an evolution family is exponentially dichotomic, then it is uniformly dichotomic.

One of the most efficient tool in the study of the dichotomic behavior of an evolution family is represented by the so-called input-output techniques. The input-output method considered in this paper is the admissibility of a pair of function spaces. Indeed, let W, V be two Banach function spaces such that $W \in \mathcal{H}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$.

Definition 3.5. The pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is said to be *admissible* for $\mathcal{U}$ if for every $v \in V(\mathbb{R},X)$ there exists a unique $f \in W(\mathbb{R},X)$ such that the pair (f,v) satisfies the equation

$$f(t) = U(t,s)f(s) + \int_s^t U(t,\tau)v(\tau)d\tau, \quad \forall t \geq s. \tag{$E_{\mathcal{U}}$}$$

Remark 3.6. If the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for $\mathcal{U}$, then it makes sense to define the operator $Q : V(\mathbb{R},X) \to W(\mathbb{R},X)$, $Q(v) = f$, where $f \in W(\mathbb{R},X)$ is such that the pair (f,v) satisfies $(E_{\mathcal{U}})$. Then Q is a bounded linear operator (see [16, Proposition 4.4]).

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on X and $W \in \mathcal{H}(\mathbb{R})$. For every $t \in \mathbb{R}$, we consider *the stable subspace* $X_s(t)$ as the space of all $x \in X$ with the property that the function

$$\delta_x : \mathbb{R} \longrightarrow X, \quad \delta_x(\tau) = \begin{cases} U(\tau,t)x, & \tau \geq t, \\ 0, & \tau < t \end{cases} \tag{3.1}$$

belongs to $W(\mathbb{R},X)$ and we define *the unstable subspace* $X_u(t)$ as the space of all $x \in X$ with the property that there is a function $\varphi_x \in W(\mathbb{R},X)$ such that $\varphi_x(t) = x$ and $\varphi_x(\tau) = U(\tau,s)\varphi_x(s)$, for all $s \leq \tau \leq t$.

An important information concerning the structure of the projection family associated with a uniformly dichotomic evolution family was obtained in [16, Theorem 4.8] and this is given by the following.

Theorem 3.7. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on X and let W, V be two Banach function spaces with $W \in \mathscr{M}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$. If the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for the evolution family $\mathcal{U}$, then $\mathcal{U}$ is uniformly dichotomic with respect to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$, where*

$$\operatorname{Im} P(t) = X_s(t), \qquad \ker P(t) = X_u(t), \quad \forall t \in \mathbb{R}. \tag{3.2}$$

Taking into account the results obtained in [11, 16], an interesting open question is whether in the study of exponential dichotomy, the output space may belong to the general class $\mathscr{M}(\mathbb{R})$. To answer this question, the first purpose of this paper is to prove the following theorem.

Theorem 3.8. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on the Banach space X and let W, V be two Banach function spaces with $W \in \mathscr{M}(\mathbb{R})$ and $V \in \mathcal{U}(\mathbb{R})$. If the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for $\mathcal{U}$, then $\mathcal{U}$ is uniformly exponentially dichotomic.*

The proof will be constructive and therefore, we will present several intermediate results.

Theorem 3.9. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on the Banach space X and let W, V be two Banach function spaces with $W \in \mathscr{M}(\mathbb{R})$ and $V \in \mathcal{U}(\mathbb{R})$. If the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for $\mathcal{U}$, then there are $K, \nu > 0$ such that*

$$\|U(t,t_0)x\| \leq Ke^{-\nu(t-t_0)}\|x\|, \quad \forall t \geq t_0,\ \forall x \in X_s(t_0). \tag{3.3}$$

Proof. According to Theorem 3.7 and Definition 3.2(iii) we have that there is $\lambda > 0$ such that

$$\|U(t,t_0)x\| \leq \lambda\|x\|, \quad \forall t \geq t_0,\ \forall x \in X_s(t_0). \tag{3.4}$$

Since $V \in \mathcal{U}(\mathbb{R}_+)$, from Remark 2.7(ii) we have that there is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}_+$ with $\varphi \in V \setminus L^1(\mathbb{R},\mathbb{R})$. Using the invariance under translations of the space V, we may assume that there is $r > 1$ such that

$$\int_0^r \varphi(s)ds \geq 2\frac{e\lambda^2\|Q\|\,|\varphi|_V}{F_W(1)}. \tag{3.5}$$

Since $\int_{1/n}^{r-1/n}\varphi(s)ds \underset{n\to\infty}{\to} \int_0^r\varphi(s)ds$ there is $n_0 \in \mathbb{N}^*$ such that

$$\int_{1/n_0}^{r-1/n_0} \varphi(s)ds \geq \frac{1}{2}\int_0^r \varphi(s)ds. \tag{3.6}$$

Let $\alpha : \mathbb{R} \to [0,1]$ be a continuous function with $\operatorname{supp}\alpha \subset (0,r)$ and $\alpha(t) = 1$, for $t \in [1/n_0, r-1/n_0]$. Then, the function $\psi : \mathbb{R} \to \mathbb{R}_+$, $\psi(t) = \alpha(t)\varphi(t)$ is continuous and from (3.5) and (3.6) we have that

$$\int_0^r \psi(t)dt \geq \int_{1/n_0}^{r-1/n_0} \varphi(t)dt \geq \frac{e\lambda^2\|Q\|\,|\varphi|_V}{F_W(1)}. \tag{3.7}$$

Let $t_0 \in \mathbb{R}$ and let $x \in X_s(t_0)$. We consider the functions

$$v : \mathbb{R} \to X, \quad v(t) = \varphi(t - t_0)U(t, t_0)x, \tag{3.8}$$

$$f : \mathbb{R} \to X, \quad f(t) = \begin{cases} \displaystyle\int_{t_0}^{t} \varphi(\tau - t_0)d\tau U(t, t_0)x, & t \geq t_0, \\ 0, & t < t_0. \end{cases} \tag{3.9}$$

Since $v \in C_c(\mathbb{R}, X)$ it follows that $v \in V(\mathbb{R}, X)$. Setting $q = \int_0^r \varphi(\tau)d\tau$ we observe that $f(t) = qU(t, t_0)x$, for all $t \geq t_0 + r$. Since $x \in X_s(t_0)$ we deduce that $f \in W(\mathbb{R}, X)$. A simple computation shows that the pair (f, v) satisfies $(E_{\mathcal{U}})$, so $f = Q(v)$. This implies that

$$\|f\|_{W(\mathbb{R},X)} \leq \|Q\|\|v\|_{V(\mathbb{R},X)}. \tag{3.10}$$

According to relation (3.4) we observe that

$$\|v(t)\| = \varphi(t - t_0)\|U(t, t_0)x\| \leq \lambda\|x\|\varphi_{t_0}(t), \quad \forall t \in \mathbb{R}, \tag{3.11}$$

and using the invariance under translations of the space V we deduce that $\|v\|_{V(\mathbb{R},X)} \leq \lambda\|x\||\varphi|_V$. From $\varphi(t) \leq \varphi(t)$, for all $t \in \mathbb{R}$, we have that $|\varphi|_V \leq |\varphi|_V$. Thus we obtain that

$$\|v\|_{V(\mathbb{R},X)} \leq \lambda\|x\|\left|\varphi\right|_V. \tag{3.12}$$

From $\|U(t_0 + r + 1, t_0)x\| \leq \lambda\|U(\tau, t_0)x\| = (\lambda/q)\|f(\tau)\|$, for all $\tau \in [t_0 + r, t_0 + r + 1]$, we have that $\|U(t_0 + r + 1, t_0)x\|\chi_{[t_0+r,t_0+r+1)}(\tau) \leq (\lambda/q)\|f(\tau)\|$, for all $\tau \in \mathbb{R}$, which implies that

$$F_W(1)\|U(t_0 + r + 1, t_0)x\| \leq \frac{\lambda}{q}\|f\|_{W(\mathbb{R},X)}. \tag{3.13}$$

Setting $l = r + 1$, from relations (3.10)–(3.13) it follows that

$$qF_W(1)\|U(t_0 + l, t_0)x\| \leq \lambda^2\|Q\|\left|\varphi\right|_V\|x\|. \tag{3.14}$$

Using relations (3.7) and (3.14) we deduce that $\|U(t_0+l, t_0)x\| \leq (1/e)\|x\|$. Taking into account that l does not depend on t_0 or x, we have that

$$\|U(t_0 + l, t_0)x\| \leq \frac{1}{e}\|x\|, \quad \forall t_0 \in \mathbb{R}, \ \forall x \in X_s(t_0). \tag{3.15}$$

Let $\nu = 1/l$ and $K = \lambda e$. Let $t \geq t_0$ and $x \in X_s(t_0)$. Then, there are $j \in \mathbb{N}$ and $s \in [0, l)$ such that $t = t_0 + jl + s$. Using relations (3.4) and (3.15) we obtain that $\|U(t, t_0)x\| \leq \lambda\|U(t_0 + jl, t_0)x\| \leq \lambda e^{-j}\|x\| \leq Ke^{-\nu(t-t_0)}\|x\|$, which completes the proof. □

Theorem 3.10. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on the Banach space X and let W, V be two Banach function spaces with $W \in \mathcal{M}(\mathbb{R})$ and $V \in \mathcal{U}(\mathbb{R})$. If the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for $\mathcal{U}$, then there are $K, \nu > 0$ such that*

$$\|U(t,t_0)x\| \geq \frac{1}{K}e^{\nu(t-t_0)}\|x\|, \quad \forall t \geq t_0,\ \forall x \in X_u(t_0). \tag{3.16}$$

Proof. Let $M \geq 1$ and $\omega > 0$ be given by Definition 3.1. According to Theorem 3.7 and Definition 3.2(iv) we have that there is $\lambda > 0$ such that

$$\|U(t,t_0)x\| \geq \frac{1}{\lambda}\|x\|, \quad \forall t \geq t_0,\ \forall x \in X_u(t_0). \tag{3.17}$$

Since $V \in \mathcal{U}(\mathbb{R}_+)$, from Remark 2.7(ii) we have that there is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}_+$ with $\varphi \in V \setminus L^1(\mathbb{R},\mathbb{R})$. Using the invariance under translations of the space V we may assume that there is $r > 1$ such that

$$\int_0^r \varphi(s)ds \geq 2\frac{\lambda M e^{\omega+1}\|Q\|\|\varphi\|_V}{F_W(1)}. \tag{3.18}$$

Using similar arguments with those in the proof of Theorem 3.9 we obtain that there is a continuous function $\psi : \mathbb{R} \to \mathbb{R}_+$ with $\operatorname{supp}\psi \subset (0,r)$, $\psi(t) \leq \varphi(t)$, for all $t \in \mathbb{R}$ and

$$\int_0^r \psi(\tau)d\tau \geq \frac{\lambda M e^{\omega+1}\|Q\|\|\varphi\|_V}{F_W(1)}. \tag{3.19}$$

Let $t_0 \in \mathbb{R}$ and $x \in X_u(t_0)$. Then, there is $\varphi_x \in W(\mathbb{R},X)$ such that $\varphi_x(t_0) = x$ and $\varphi_x(\tau) = U(\tau,s)\varphi_x(s)$, for all $s \leq \tau \leq t_0$. We consider the functions

$$v : \mathbb{R} \longrightarrow X, \quad v(t) = -\psi(t-t_0)U(t,t_0)x, \tag{3.20}$$

$$f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} \displaystyle\int_t^\infty \psi(\tau-t_0)d\tau U(t,t_0)x, & t \geq t_0, \\ q\varphi_x(t), & t < t_0, \end{cases} \tag{3.21}$$

where $q = \int_0^r \psi(\tau)d\tau$. We have that $v \in C_c(\mathbb{R},X)$, so $v \in V(\mathbb{R},X)$. Using relation (3.17) we have that

$$\|f(t)\| \leq q\|\varphi_x(t)\| + q\lambda\|U(t_0+r,t_0)x\|\chi_{[t_0,t_0+r)}(t), \quad \forall t \in \mathbb{R}. \tag{3.22}$$

From this inequality, since $W \in \mathcal{M}(\mathbb{R})$ we deduce that $f \in W(\mathbb{R},X)$. An easy computation shows that the pair (f,v) satisfies $(E_{\mathcal{U}})$, so $f = Q(v)$. Then, we have that

$$\|f\|_{W(\mathbb{R},X)} \leq \|Q\|\|v\|_{V(\mathbb{R},X)}. \tag{3.23}$$

Using relation (3.17) we have that

$$\|v(t)\| = \varphi(t-t_0)\|U(t,t_0)x\| \le \varphi(t-t_0)\lambda\|U(t_0+r,t_0)x\|, \quad \forall t \in \mathbb{R} \tag{3.24}$$

which implies that

$$\|v\|_{V(\mathbb{R},X)} \le |\varphi|_V \lambda\|U(t_0+r,t_0)x\| \le |\varphi|_V \lambda\|U(t_0+r,t_0)x\|. \tag{3.25}$$

Since $x = \varphi_x(t_0) = U(t_0,t)\varphi_x(t)$, for all $t \in [t_0-1,t_0]$, we deduce that

$$\|x\|\chi_{[t_0-1,t_0)}(t) \le Me^{\omega}\|\varphi_x(t)\|\chi_{[t_0-1,t_0)}(t) \le \frac{Me^{\omega}}{q}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{3.26}$$

This shows that

$$q\|x\|F_W(1) \le Me^{\omega}\|f\|_{W(\mathbb{R},X)}. \tag{3.27}$$

From relations (3.19)–(3.27) it follows that $\|U(t_0+r,t_0)x\| \ge e\|x\|$. Taking into account that r does not depend on t_0 or x, we have that

$$\|U(t_0+r,t_0)x\| \ge e\|x\|, \quad \forall t_0 \in \mathbb{R},\ \forall x \in X_u(t_0). \tag{3.28}$$

We set $\nu = 1/r$ and $K = \lambda e$. Let $t \ge t_0$ and $x \in X_u(t_0)$. Then, there are $j \in \mathbb{N}$ and $s \in [0,r)$ such that $t = t_0 + jr + s$. Using relations (3.17) and (3.28) we obtain that $\|U(t,t_0)x\| \ge (1/\lambda)\|U(t_0 + jr,t_0)x\| \ge (1/\lambda)e^j\|x\| \ge (1/K)e^{\nu(t-t_0)}\|x\|$, which completes the proof. □

Now, we may give the proof of Theorem 3.8.

Proof of Theorem 3.8. This immediately follows from Theorems 3.7, 3.9, and 3.10. □

Now, we may give the main result of the paper, which establishes a complete diagram concerning the study of exponential dichotomy on the real line in terms of integral admissibility.

Theorem 3.11. *Let $\mathcal{U} = \{U(t,s)\}_{t\ge s}$ be an evolution family on the Banach space X and let W, V be two Banach function spaces with $W \in \mathcal{A}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$. If $W \in \mathcal{W}(\mathbb{R})$ or $V \in \mathcal{V}(\mathbb{R})$, then the following assertions hold:*

(i) *if the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for $\mathcal{U}$, then $\mathcal{U}$ is uniformly exponentially dichotomic;*

(ii) *if $V \subset W$ and one of the spaces V, W belongs to the class $\mathcal{O}(\mathbb{R})$, then $\mathcal{U}$ is exponentially dichotomic if and only if the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for $\mathcal{U}$.*

Proof. (i) This follows from Theorems 1.3 and 3.8.

(ii) *Necessity.* Suppose that $\mathcal{U}$ is exponentially dichotomic with respect to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$ and the constants $K, \nu > 0$. Then, we have that $L := \sup_{t\in\mathbb{R}}\|P(t)\| < \infty$ (see, e.g., [13]).

Let $v \in V(\mathbb{R}, X)$. We consider the function

$$f : \mathbb{R} \longrightarrow X, \quad f(t) = \int_{-\infty}^{t} U(t,s)P(s)v(s)ds - \int_{t}^{\infty} U(s,t)_{|}^{-1}(I - P(s))v(s)ds, \tag{3.29}$$

where for every $s > t$, $U(s,t)_{|}^{-1}$ denotes the inverse of the operator $U(s,t)_{|} : \ker P(t) \to \ker P(s)$.

If $V \in \mathcal{O}(\mathbb{R})$, then using Proposition 2.4 we obtain that $f \in V(\mathbb{R}, X)$. Since $V \subset W$ we deduce that $f \in W(\mathbb{R}, X)$.

If $W \in \mathcal{O}(\mathbb{R})$ then, since $V \subset W$ we have that $v \in W(\mathbb{R}, X)$. Using Proposition 2.4 it follows that $f \in W(\mathbb{R}, X)$.

An easy computation shows that the pair (f, v) satisfies $(E_{\mathcal{U}})$. The uniqueness of f is immediate (see, e.g., [16, the Necessity part of Theorem 5.3]). In conclusion, the pair $(W(\mathbb{R}, X), V(\mathbb{R}, X))$ is admissible for the evolution family $\mathcal{U}$. □

The natural question arises whether the hypotheses from Theorem 3.11 can be dropped and also if the conditions given by this theorem are the most general in this topic. The answers are given by the following example.

Example 3.12. Let $W \in \mathcal{M}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$ be such that $W \notin \mathcal{W}(\mathbb{R})$ and $V \notin \mathcal{U}(\mathbb{R})$. Then $V \subset L^1(\mathbb{R}, \mathbb{R})$ and according to Lemma 2.8 we have that $C_0(\mathbb{R}, \mathbb{R}) \subset W$.

We consider the function

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R}, \quad \varphi(t) = \begin{cases} \dfrac{1}{t+1}, & t \geq 0, \\ 1 - t, & t < 0. \end{cases} \tag{3.30}$$

Then φ is a decreasing function.

Let $X = \mathbb{R}^2$ endowed with the norm $\|(x_1, x_2)\| := |x_1| + |x_2|$, for all $(x_1, x_2) \in \mathbb{R}^2$. For every $t \geq s$ we consider the operator

$$U(t,s) : X \longrightarrow X, \quad U(t,s)(x_1, x_2) = \left(\frac{\varphi(t)}{\varphi(s)}x_1, e^{t-s}x_2\right). \tag{3.31}$$

Then $\mathcal{U} = \{U(t,s)\}_{t \geq s}$ is an evolution family on X.

We prove that the pair $(W(\mathbb{R}, X), V(\mathbb{R}, X))$ is admissible for $\mathcal{U}$. Let $v = (v_1, v_2) \in V(\mathbb{R}, X)$. Then $v_1, v_2 \in L^1(\mathbb{R}, \mathbb{R})$. We consider the function

$$f : \mathbb{R} \longrightarrow X, \quad f(t) = \left(\int_{-\infty}^{t} \frac{\varphi(t)}{\varphi(\tau)}v_1(\tau)d\tau, -\int_{t}^{\infty} e^{-(\tau - t)}v_2(\tau)d\tau\right). \tag{3.32}$$

We have that f is correctly defined and an easy computation shows that the pair (f, v) satisfies $(E_{\mathcal{U}})$. We set

$$f_1(t) = \int_{-\infty}^{t} \frac{\varphi(t)}{\varphi(s)} v_1(s)d\tau, \qquad f_2(t) = -\int_{t}^{\infty} e^{-(\tau-t)} v_2(\tau)d\tau, \quad \forall t \in \mathbb{R}. \tag{3.33}$$

We prove that $f_1 \in C_0(\mathbb{R}, \mathbb{R})$. Since $v_1 \in L^1(\mathbb{R}, \mathbb{R})$, from

$$\left|f_1(t)\right| \leq \int_{-\infty}^{t} |v_1(\tau)|d\tau, \quad \forall t \in \mathbb{R}, \tag{3.34}$$

we have that $\lim_{t\to-\infty} f_1(t) = 0$. Let $\varepsilon > 0$. Then, there is $\delta > 0$ such that $\int_{\delta}^{\infty} |v_1(\tau)|d\tau < \varepsilon$. It follows that

$$\left|f_1(t)\right| \leq \frac{\varphi(t)}{\varphi(\delta)} \int_{-\infty}^{\delta} \frac{\varphi(\delta)}{\varphi(\tau)} |v_1(\tau)|d\tau + \int_{\delta}^{t} |v_1(\tau)|d\tau < \frac{\delta+1}{t+1} \|v_1\|_1 + \varepsilon, \quad \forall t \geq \delta. \tag{3.35}$$

The above inequality implies that $\overline{\lim_{t\to\infty}} \ |f_1(t)| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary we obtain that there exists $\lim_{t\to\infty} f_1(t) = 0$. Using similar arguments with those in (3.34) we deduce that $\lim_{t\to\infty} f_2(t) = 0$.

Let $\varepsilon > 0$. Then there is $\gamma < 0$ such that $\int_{-\infty}^{\gamma} |v_2(\tau)|d\tau < \varepsilon$. It follows that

$$\left|f_2(t)\right| \leq e^{t-\gamma} \int_{\gamma}^{\infty} e^{\gamma-\tau} |v_2(\tau)|d\tau + \int_{t}^{\gamma} |v_2(\tau)|d\tau < e^{t-\gamma} \|v_2\|_1 + \varepsilon, \quad \forall t \leq \gamma. \tag{3.36}$$

From this inequality we have that $\overline{\lim_{t\to-\infty}} \ |f_2(t)| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that there exists $\lim_{t\to-\infty} f_2(t) = 0$. Thus, we deduce that $f \in C_0(\mathbb{R}, X)$, so $f \in W(\mathbb{R}, X)$.

To prove the uniqueness of f, let $\widetilde{f} \in W(\mathbb{R}, X)$ be such that the pair $(\widetilde{f}, v)$ satisfies $(E_{\mathcal{U}})$. Setting $g = \widetilde{f} - f$ we have that $g \in W(\mathbb{R}, X)$ and $g(t) = U(t, s)g(s)$, for all $t \geq s$. If $g = (g_1, g_2)$, then we deduce that

$$g_1(t) = \frac{\varphi(t)}{\varphi(s)} g_1(s), \quad \forall t \geq s, \tag{3.37}$$

$$g_2(t) = e^{t-s} g_2(s), \quad \forall t \geq s. \tag{3.38}$$

Let $t \in \mathbb{R}$. Using Lemma 2.6 and integrating in (3.37) we have that

$$\left|g_1(t)\right| = \int_{s}^{s+1} \frac{\varphi(t)}{\varphi(\tau)} \left|g_1(\tau)\right| d\tau \leq \frac{\varphi(t)}{\varphi(s+1)} \left\|g\right\|_{M_1(\mathbb{R},X)} \quad \forall s \leq t-1. \tag{3.39}$$

For $s \to -\infty$ in (3.39) we obtain that $g_1(t) = 0$.

From relation (3.38) we have that

$$g_2(t) = e^{-(\tau-t)}g_2(\tau), \quad \forall \tau \geq t. \tag{3.40}$$

Integrating in relation (3.40) on $[\tau, \tau+1]$ and using Lemma 2.6, we deduce that

$$\left|g_2(t)\right| = \int_{\tau}^{\tau+1} e^{-(\xi-t)}\left|g_2(\xi)\right|d\xi \leq e^{-(\tau-t)}\left\|g\right\|_{M_1(\mathbb{R},X)}, \quad \forall \tau \geq t. \tag{3.41}$$

For $\tau \to \infty$ in (3.41) it follows that $g_2(t) = 0$. Since $t \in \mathbb{R}$ was arbitrary we have that $g = 0$, so $\widetilde{f} = f$. Thus, the pair $(W(\mathbb{R}, X), V(\mathbb{R}, X))$ is admissible for the evolution family $\mathcal{U}$.

Suppose that $\mathcal{U}$ is exponentially dichotomic with respect to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$ and the constants $K, \nu > 0$. According to [13, Proposition 3.1] we have that $\operatorname{Im} P(t) = \{x \in X : \sup_{\tau\geq t}\|U(\tau,t)x\| < \infty\}$, which implies that $\operatorname{Im} P(t) = \mathbb{R} \times \{0\}$, for all $t \in \mathbb{R}$. Then, we obtain that

$$\left|\frac{\varphi(t)}{\varphi(s)}x_1\right| \leq Ke^{-\nu(t-s)}|x_1|, \quad \forall t \geq s, \ \forall x_1 \in \mathbb{R} \tag{3.42}$$

or equivalently

$$\frac{\varphi(t)}{\varphi(s)} \leq Ke^{-\nu(t-s)}, \quad \forall t \geq s \tag{3.43}$$

which is absurd. In conclusion, the pair $(W(\mathbb{R}, X), V(\mathbb{R}, X))$ is admissible for $\mathcal{U}$, but, for all that, the evolution family $\mathcal{U}$ is not exponentially dichotomic.

4. Applications to the Case of C_0-Semigroups

In this section, by applying the central theorems from the Section 3 we deduce several consequences of the main results for the study of exponential dichotomy of C_0-semigroups. Let X be a real or complex Banach space.

Definition 4.1. A family $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators on X is said to be a C_0-*semigroup* if the following properties are satisfied:

(i) $T(0) = I_d$, the identity operator on X;

(ii) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$;

(iii) $\lim_{t\searrow 0} T(t)x = x$, for every $x \in X$.

Definition 4.2. A C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be *exponentially dichotomic* if there exist a projection $P \in \mathcal{B}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that

(i) $PT(t) = T(t)P$, for all $t \geq 0$;

(ii) $\|T(t)x\| \leq Ke^{-\nu t}\|x\|$, for all $x \in \operatorname{Im} P$ and all $t \geq 0$;

(iii) $\|T(t)x\| \geq (1/K)e^{\nu t}\|x\|$, for all $x \in \ker P$ and all $t \geq 0$;

(iv) the restriction $T(t)_{|} : \ker P \to \ker P$ is an isomorphism, for every $t \geq 0$.

Remark 4.3. (i) If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a C_0-semigroup, we can associate to $\mathbf{T}$ an evolution family $\mathcal{U}_T = \{U_T(t,s)\}_{t\geq s}$, by $U_T(t,s) = T(t-s)$, for every $t \geq s$.

(ii) A C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is exponentially dichotomic if and only if the associated evolution family $\mathcal{U}_T = \{U_T(t,s)\}_{t\geq s}$ is exponentially dichotomic (see [12, Proposition 4.4]).

Let W, V be two Banach function spaces such that $W \in \mathcal{M}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$.

Definition 4.4. The pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is said to be *admissible* for the C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ if for every $v \in V(\mathbb{R},X)$ there is a unique $f \in W(\mathbb{R},X)$ such that

$$f(t) = T(t-s)f(s) + \int_s^t T(t-\tau)v(\tau)d\tau, \quad \forall t \geq s. \tag{E_T}$$

Theorem 4.5. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on the Banach space X and let W, V be two Banach function spaces with $W \in \mathcal{M}(\mathbb{R})$ and $V \in \mathcal{T}(\mathbb{R})$. If $W \in \mathcal{W}(\mathbb{R})$ or $V \in \mathcal{U}(\mathbb{R})$, then the following assertions hold:*

(i) *if the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for $\mathbf{T}$, then $\mathbf{T}$ is uniformly exponentially dichotomic;*

(ii) *if $V \subset W$ and one of the spaces V, W belongs to the class $\mathcal{O}(\mathbb{R})$, then $\mathbf{T}$ is exponentially dichotomic if and only if the pair $(W(\mathbb{R},X), V(\mathbb{R},X))$ is admissible for $\mathbf{T}$.*

Proof. This follows from Theorem 3.11 and Remark 4.3(ii). □

Remark 4.6. According to the example given in the previous section we deduce that the hypothesis $W \in \mathcal{W}(\mathbb{R})$ or $V \in \mathcal{U}(\mathbb{R})$ cannot be removed. Moreover, Theorem 4.5 provides a complete answer concerning the study of exponential dichotomy of semigroups using input-output techniques with respect to the associated integral equation.

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