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Research Article

Inequalities among Eigenvalues of Second-Order Symmetric Equations on Time Scales

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We consider coupled boundary value problems for second-order symmetric equations on time scales. Existence of eigenvalues of this boundary value problem is proved, numbers of their eigenvalues are calculated, and their relationships are obtained. These results not only unify the existing ones of coupled boundary value problems for second-order symmetric differential equations but also contain more complicated time scales.

1. Introduction

In this paper we consider the following second-order symmetric equation:

$$-\left(p(t)y^{\Delta}(t)\right)^{\Delta}+q(t)y^{\sigma}(t)=\lambda r(t)y^{\sigma}(t),\quad t\in[\rho(0),\rho(1)]\cap\mathbb{T},\ \text{and } 0,1\in\mathbb{T} \tag{1.1}$$

with the coupled boundary conditions:

$$\begin{pmatrix} y(1) \\ y^{\Delta}(1) \end{pmatrix}=e^{i\theta}K\begin{pmatrix} y(\rho(0)) \\ y^{\Delta}(\rho(0)) \end{pmatrix}, \tag{1.2}$$

where $\mathbb{T}$ is a time scale; p^{Δ}, q, and r are real and continuous functions in $[\rho(0),\rho(1)]\cap\mathbb{T}$, $p>0$ over $[\rho(0),1]\cap\mathbb{T}$, $r>0$ over $[\rho(0),\rho(1)]\cap\mathbb{T}$, and $p(\rho(0))=p(1)=1$; $\sigma(t)$ and $\rho(t)$ are the forward and backward jump operators in $\mathbb{T}$, y^{Δ} is the delta derivative, and $y^{\sigma}(t) := y(\sigma(t))$; $\theta \neq 0$, $-\pi < \theta < \pi$, is a constant parameter; $i = \sqrt{-1}$,

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \quad k_{ij} \in \mathbf{R}, \quad i,j = 1,2, \text{ with } \det K = 1. \tag{1.3}$$

The boundary condition (1.2) contains the two special cases: the periodic and antiperiodic conditions. In fact, (1.2) is the periodic boundary condition in the case where $\theta = 0$ and $K = I$, the identity matrix, and (1.2) is the antiperiodic condition in the case where $\theta = \pi$ and $K = I$. Equation (1.1) with (1.2) is called a coupled boundary value problem.

Hence, according to [1, Theorem 3.1], the periodic and antiperiodic boundary value problems have $N_d + 1$ real eigenvalues and they satisfy the following inequality:

$$-\infty < \lambda_0(I) < \lambda_0(-I) \le \lambda_0^D \le \lambda_1(-I) < \lambda_1(I) \le \lambda_1^D \le \lambda_2(I) < \lambda_2(-I) \le \lambda_2^D \le \lambda_3(-I) \le \lambda_3^D \le \cdots, \tag{1.4}$$

where $N_d := |[0,1] \cap \mathbb{T}| - \operatorname{def}(\mu(\rho(0))) - 1$, λ_n^D denote the nth Dirichlet eigenvalues. Denote the number of point of a set $S \subset \mathbf{R}$ by $|S|$ and introduce the following notation for $\alpha \in \mathbf{R}$:

$$\operatorname{def} \alpha = \begin{cases} 0, & \text{if } \alpha \neq 0, \\ 1, & \text{if } \alpha = 0. \end{cases} \tag{1.5}$$

Furthermore, if $N_d < \infty$, then

$$\begin{aligned} &\lambda_0(I) < \lambda_0(-I) \le \lambda_1(-I) < \lambda_1(I) \le \lambda_2(I) < \lambda_2(-I) \le \lambda_3(-I) < \lambda_3(I) \\ &\qquad \le \cdots \le \lambda_{N_d-1}(-I) < \lambda_{N_d-1}(I) \le \lambda_{N_d}(I) < \lambda_{N_d}(-I), \quad \text{if } N_d + 1 \text{ is odd}, \\ &\lambda_0(I) < \lambda_0(-I) \le \lambda_1(-I) < \lambda_1(I) \le \lambda_2(I) < \lambda_2(-I) \le \lambda_3(-I) < \lambda_3(I) \\ &\qquad \le \cdots \le \lambda_{N_d-1}(I) < \lambda_{N_d-1}(-I) \le \lambda_{N_d}(-I) < \lambda_{N_d}(I), \quad \text{if } N_d + 1 \text{ is even}. \end{aligned} \tag{1.6}$$

In [2], Eastham et al. considered the second-order differential equation:

$$-\left(p(t)x'(t)\right)' + q(t)x(t) = \lambda w(t)x(t), \quad t \in [a,b], \tag{1.7}$$

with the coupled boundary condition:

$$\begin{pmatrix} y(b) \\ p(b)y'(b) \end{pmatrix} = e^{i\alpha} K \begin{pmatrix} y(a) \\ p(a)y'(a) \end{pmatrix}, \tag{1.8}$$

where $i = \sqrt{-1}$, $-\pi < \alpha \le \pi$, $-\infty < a < b < \infty$, $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$, $k_{ij} \in \mathbf{R}$, $i,j = 1,2$, $\det K = 1$, and $1/p, q, w \in L^1([a,b], \mathbf{R})$, $p > 0$, $w > 0$ a.e. on $[a,b]$. Here $\mathbf{R}$ denote the set of real number, and $L^1([a,b], \mathbf{R})$ the space of real valued Lebesgue integrable functions on $[a,b]$. They obtained the following results: the coupled boundary value problem (1.7) with (1.8) has an infinite but countable number of only real eigenvalues which can be ordered to form a nondecreasing sequence:

$$\begin{aligned}-\infty < \lambda_0(K) < \lambda_0\left(e^{i\alpha}K\right) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1\left(e^{i\alpha}K\right) < \lambda_1(K)\\ \leq \lambda_2(K) < \lambda_2\left(e^{i\alpha}K\right) < \lambda_2(-K) \leq \lambda_3(-K) < \lambda_3\left(e^{i\alpha}K\right) < \lambda_3(K) \leq \cdots .\end{aligned} \tag{1.9}$$

In the present paper, we try to extend these results on time scales. We shall remark that Eastham et al. employed continuous eigenvalue branch which studied in [2], in their proof. Instead, we will make use of some oscillation results that are extended from the results obtained by Agarwal et al. [4] to prove the existence of eigenvalues of (1.1) with (1.2) and compare the eigenvalues as θ varies.

This paper is organized as follows. Section 2 introduces some basic concepts and fundamental theory about time scales and gives some properties of eigenvalues of a kind of separated boundary value problem for (1.1) which will be used in Section 4. Our main result has been introduced in Section 3. Section 4 pays attention to prove some propositions, by which one can easily obtain the existence and the comparison result of eigenvalues of the coupled boundary value problems (1.1) with (1.2). By using these propositions, we give the proof of our main result in Section 5.

2. Preliminaries

In this section, some basic concepts and some fundamental results on time scales are introduced. Next, the eigenvalues of the kind of separated boundary value problem for (1.1) and the oscillation of their eigenfunction are studied. Finally, the reality of the eigenvalues of the coupled boundary value problems for (1.1) is shown.

Let $\mathbb{T} \subset \mathbf{R}$ be a nonempty closed subset. Define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \tag{2.1}$$

where $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered, and left-dense if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, and $\rho(t) = t$, respectively.

We assume throughout the paper that if 0 is right-scattered, then it is also left-scattered, and if 1 is left-scattered, then it is also right-scattered.

Since $\mathbb{T}$ is a nonempty bounded closed subset of $\mathbf{R}$, we put $\mathbb{T}^k = \mathbb{T} \setminus (\rho(\max \mathbb{T}), \max \mathbb{T}]$. The graininess $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{2.2}$$

Let f be a function defined on $\mathbb{T}$. f is said to be (delta) differentiable at $t \in \mathbb{T}^k$ provided there exists a constant a such that for any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) with

$$\left| f(\sigma(t)) - f(s) - a(\sigma(t) - s) \right| \le \varepsilon \, |\sigma(t) - s|, \quad \forall s \in U. \tag{2.3}$$

In this case, denote $f^{\Delta}(t) := a$. If f is (delta) differentiable for every $t \in \mathbb{T}^k$, then f is said to be (delta) differentiable on $\mathbb{T}$. If f is differentiable at $t \in \mathbb{T}^k$, then

$$f^{\Delta}(t) = \begin{cases} \lim\limits_{\substack{s \to t \\ s \in \mathbb{T}}} \dfrac{f(t) - f(s)}{t - s} & \text{if } \mu(t) = 0 \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \text{if } \mu(t) > 0. \end{cases} \tag{2.4}$$

If $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, then $F(t)$ is called an antiderivative of f on $\mathbb{T}$. In this case, define the delta integral by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \quad \forall s, t \in \mathbb{T}. \tag{2.5}$$

Moreover, a function f defined on $\mathbb{T}$ is said to be rd-continuous if it is continuous at every right-dense point in $\mathbb{T}$ and its left-sided limit exists at every left-dense point in $\mathbb{T}$.

For convenience, we introduce the following results ([5, Chapter 1], [6, Chapter 1], and [7, Lemma 1]), which are useful in the paper.

Lemma 2.1. *Let $f, g : \mathbb{T} \to \mathbf{R}$ and $t \in \mathbb{T}^k$.*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f and g are differentiable at t, then fg is differentiable at t and*

$$(fg)^{\Delta}(t) = f^{\sigma}(t)g^{\Delta}(t) + f^{\Delta}(t)g(t) = f^{\Delta}(t)g^{\sigma}(t) + f(t)g^{\Delta}(t). \tag{2.6}$$

(iii) *If f and g are differentiable at t, and $f(t)f^{\sigma}(t) \ne 0$, then $f^{-1}g$ is differentiable at t and*

$$\left(gf^{-1}\right)^{\Delta}(t) = \left(g^{\Delta}(t)f(t) - g(t)f^{\Delta}(t)\right)\left(f^{\sigma}(t)f(t)\right)^{-1}. \tag{2.7}$$

(iv) *If f is rd-continuous on $\mathbb{T}$, then it has an antiderivative on $\mathbb{T}$.*

Now, we turn to discuss some properties of solutions of (1.1) and eigenvalues of its boundary value problems.

Define the Wronskian by

$$W(x, y) := xy^{\Delta} - yx^{\Delta}, \quad x, y \in C^2_{\mathrm{rd}}(\mathbb{T}), \tag{2.8}$$

where $C^2_{\mathrm{rd}}(\mathbb{T})$ is the set of twice differentiable functions with rd-continuous second derivative. The following result can be derived from the Lagrange Identity [5, Theorem 4.30].

Lemma 2.2. *For any two solutions x and y of* (1.1), *$p(t)W(x,y)(t)$ is a constant on $[\rho(0),1]\cap\mathbb{T}$.*

In [4], Agarwal et al. studied the following second-order symmetric linear equation:

$$y^{\Delta\Delta}+q(t)y^{\sigma}=-\lambda y^{\sigma},\quad t\in[\rho(a),\rho(b)]\cap\mathbb{T},\ \text{and}\ a,b\in\mathbb{T} \tag{2.9}$$

with the boundary conditions:

$$R_a(y):=\alpha y(\rho(a))+\beta y^{\Delta}(\rho(a))=0,\qquad R_b(y):=\gamma y(b)+\delta y^{\Delta}(b)=0, \tag{2.10}$$

where $q:[\rho(a),\rho(b)]\cap\mathbb{T}\to\mathbf{R}$ is continuous; $(\alpha^2+\beta^2)(\gamma^2+\delta^2)\neq0$; $a<b$ satisfy that if a is right-scattered, then it is also left-scattered; and if b is left-scattered, then it is also right-scattered. A solution y of (2.9) is said to have a node at $(t+\sigma(t))/2$ if $y(t)y(\sigma(t))<0$. A generalized zero of y is defined as its zero or its node. Without loss of generality, they assumed that α and β in (2.10) satisfy

(**H**) $\beta>\alpha\mu(\rho(a))$ if $\beta\neq\alpha\mu(\rho(a))$ and $\alpha=-1$ if $\beta=\alpha\mu(\rho(a))$

and obtained the following oscillation result.

Lemma 2.3 (see [4, Theorem 1]). *The eigenvalues of* (2.9) *with* (2.10) *may be arranged as $-\infty<\lambda_0<\lambda_1<\lambda_2<\cdots$ and an eigenfunction corresponding to λ_k has exactly k-generalized zeros in the open interval (a,b).*

In order to study the kind of separated boundary value problem for (1.1), we now extend the above oscillation theorem to the more general equation (1.1) with

$$R_0(y)=R_1(y)=0. \tag{2.11}$$

By $n(\lambda)$ denote the number of generalized zeros of the solution $y(t,\lambda)$ of (1.1) with the initial conditions

$$y(\rho(0),\lambda)=\beta,\qquad y^{\Delta}(\rho(0),\lambda)=-\alpha \tag{2.12}$$

in the open interval (0,1), where α and β satisfy (**H**) with a and b replaced by 0 and 1, respectively. It can be easily verified that

$$\text{either } y(0,\lambda)>0 \text{ or } y(0,\lambda)=0 \text{ and } y^{\Delta}(0,\lambda)=1, \tag{2.13}$$

which is independent of λ.

Lemma 2.4 (see [1, Lemma 2.5]). *Let $y(t,\lambda)$ be the solution of* (1.1) *with* (2.12)*. Then $y^{\Delta}(t,\lambda)/y(t,\lambda)$ is strictly decreasing in $\lambda\in\mathbf{R}$ for each $t\in(0,1]\cap\mathbb{T}$ whenever $y(t,\lambda)\neq0$.*

Lemma 2.5 (see [1, Lemma 2.6]). *If there exists $\lambda_0 \in \mathbf{R}$ such that $n(\lambda_0) = 0$, then $n(\lambda) = 0$ for all $\lambda < \lambda_0$.*

With a similar argument to that used in the proof of [4, Theorem 1], one can show the following result.

Theorem 2.6. *All the eigenvalues of* (1.1) *with* (2.11) *are simple and can be arranged as $-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, and an eigenfunction corresponding to λ_k has exactly k-generalized zeros in the open interval $(0,1)$, where $0, 1 \in \mathbb{T}$ satisfy that if 0 is right-scattered, then it is also left-scattered; if 1 is left-scattered, then it is also right-scattered. Furthermore, the number of its eigenvalues is equal to $|[0,1] \cap \mathbb{T}| - \operatorname{def}(\beta - \alpha\mu(\rho(0))) - \operatorname{def}\delta$.*

Setting $\beta = 0$, $\gamma = k_{22}$, and $\delta = -k_{12}$ in (2.11), where k_{12}, k_{22} are elements of K in (1.2), we get the following separated boundary conditions:

$$y(\rho(0)) = 0, \qquad k_{22}y(1) - k_{12}y^{\Delta}(1) = 0. \tag{2.14}$$

The following result is a direct consequence of Theorem 2.6.

Theorem 2.7. *All the eigenvalues of* (1.1) *with* (2.14) *are simple and can be arranged as*

$$-\infty < \mu_0 < \mu_1 < \mu_2 < \cdots, \tag{2.15}$$

and an eigenfunction corresponding to μ_k has exactly k-generalized zeros in $(0,1)$. Furthermore, the number of its eigenvalues is equal to $N_d := |[0,1] \cap \mathbb{T}| - \operatorname{def}(\mu(\rho(0))) - \operatorname{def} k_{12}$.

For convenience, we shall write $\mu_{k+1} = \infty$ if $N_d = k < \infty$.

Lemma 2.8. *For each $\lambda \in (\mu_k, \mu_{k+1}]$, $n(\lambda) = k + 1$, $k \geq 0$.*

Proof. The proof is similar to that of [4, Theorem 6]. So the details are omitted. □

Lemma 2.9. *All the eigenvalues of the coupled boundary value problem* (1.1) *with* (1.2) *are real.*

Proof. The proof is similar to that of [1, Lemma 2.8]. So the details are omitted. □

3. Main Result

In this section we state our main results: general inequalities among eigenvalues of coupled boundary value problem of (1.1) with (1.2).

Theorem 3.1. *If $k_{11} > 0$ and $k_{12} \leq 0$ or $k_{11} \geq 0$ and $k_{12} < 0$, then, for every fixed $\theta \neq 0$, $-\pi < \theta < \pi$, coupled boundary value problem* (1.1) *with* (1.2) *has $N_d + 1$ eigenvalues and these eigenvalues satisfy the following inequalities:*

$$\begin{aligned} &\lambda_0(K) < \lambda_0\left(e^{i\theta}K\right) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1\left(e^{i\theta}K\right) < \lambda_1(K) \\ &\quad \leq \lambda_2(K) < \lambda_2\left(e^{i\theta}K\right) < \lambda_2(-K) \leq \lambda_3(-K) < \lambda_3\left(e^{i\theta}K\right) < \lambda_3(K) \leq \cdots, \end{aligned} \tag{3.1}$$

where $N_d := |[0,1] \cap \mathbb{T}| - \operatorname{def}(\mu(\rho(0))) - \operatorname{def} k_{12}$. *Furthermore, if* $N_d < \infty$ *then*

$$\begin{aligned}
&\lambda_0(K) < \lambda_0\left(e^{i\theta}K\right) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1\left(e^{i\theta}K\right) < \lambda_1(K) \\
&\quad \leq \lambda_2(K) < \lambda_2\left(e^{i\theta}K\right) < \lambda_2(-K) \leq \lambda_3(-K) < \lambda_3\left(e^{i\theta}K\right) < \lambda_3(K) \\
&\quad \leq \cdots \leq \lambda_{N_d-1}(-K) < \lambda_{N_d-1}\left(e^{i\theta}K\right) < \lambda_{N_d-1}(K) \leq \lambda_{N_d}(K) \\
&\quad < \lambda_{N_d}\left(e^{i\theta}K\right) < \lambda_{N_d}(-K), \quad \textit{if } N \textit{ is odd}, \\
&\lambda_0(K) < \lambda_0\left(e^{i\theta}K\right) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1\left(e^{i\theta}K\right) < \lambda_1(K) \\
&\quad \leq \lambda_2(K) < \lambda_2\left(e^{i\theta}K\right) < \lambda_2(-K) \leq \lambda_3(-K) < \lambda_3\left(e^{i\theta}K\right) < \lambda_3(K) \\
&\quad \leq \cdots \leq \lambda_{N_d-1}(K) < \lambda_{N_d-1}\left(e^{i\theta}K\right) < \lambda_{N_d-1}(-K) \leq \lambda_{N_d}(-K) \\
&\quad < \lambda_{N_d}\left(e^{i\theta}K\right) < \lambda_{N_d}(K), \quad \textit{if } N \textit{ is even}.
\end{aligned} \tag{3.2}$$

Remark 3.2. If $k_{11} \leq 0$ and $k_{12} > 0$ or $k_{11} < 0$ and $k_{12} \geq 0$, a similar result can be obtained by applying Theorem 3.1 to $-K$. In fact, $e^{i\theta}K = e^{i(\pi+\theta)}(-K)$ for $\theta \in (-\pi, 0)$ and $e^{i\theta}K = e^{i(-\pi+\theta)}(-K)$ for $\theta \in (0,\pi)$. Hence, the boundary condition (1.2) in the cases of $k_{11} \leq 0$, $k_{12} > 0$ or $k_{11} < 0$, $k_{12} \geq 0$ and $\theta \neq 0$, $-\pi < \theta < \pi$, can be written as condition (1.2), where θ is replaced by $\pi + \theta$ for $\theta \in (-\pi, 0)$ and $-\pi + \theta$ for $\theta \in (0,\pi)$, and K is replaced by $-K$.

4. The Characteristic Function $D(\lambda)$

Before showing Theorem 3.1, we need to prove the following six propositions.

Let $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ be the solutions of (1.1) satisfying the following initial conditions:

$$\varphi(\rho(0),\lambda) = 1, \quad \varphi^{\Delta}(\rho(0),\lambda) = 0; \qquad \psi(\rho(0),\lambda) = 0, \quad \psi^{\Delta}(\rho(0),\lambda) = 1, \tag{4.1}$$

respectively. Obviously, $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are two linearly independent solutions of (1.1). By Lemma 2.2 we have

$$p(t)\left[\varphi(t,\lambda)\psi^{\Delta}(t,\lambda) - \varphi^{\Delta}(t,\lambda)\psi(t,\lambda)\right] = 1, \quad t \in \left[\rho(0), 1\right] \cap \mathbb{T}, \tag{4.2}$$

which, together with the assumption of $p(1) = 1$, implies

$$\varphi(1,\lambda)\psi^{\Delta}(1,\lambda) - \varphi^{\Delta}(1,\lambda)\psi(1,\lambda) = 1. \tag{4.3}$$

For any fixed $K \in \mathrm{SL}(2,R)$, $\det K = 1$, and all $\lambda \in \mathbf{C}$, we define

$$D(\lambda) = k_{11}\psi^{\Delta}(1,\lambda) - k_{21}\psi(1,\lambda) + k_{22}\varphi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda), \tag{4.4}$$

$$A(\lambda) = k_{11}\varphi^{\Delta}(1,\lambda) - k_{21}\varphi(1,\lambda), \tag{4.5}$$

$$B(\lambda) = k_{11}\psi^{\Delta}(1,\lambda) + k_{12}\varphi^{\Delta}(1,\lambda) - k_{22}\varphi(1,\lambda) - k_{21}\psi(1,\lambda), \tag{4.6}$$

$$B_1(\lambda) = k_{11}\psi^{\Delta}(1,\lambda) - k_{21}\psi(1,\lambda), \tag{4.7}$$

$$B_2(\lambda) = k_{22}\varphi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda), \tag{4.8}$$

$$C(\lambda) = k_{22}\psi(1,\lambda) - k_{12}\psi^{\Delta}(1,\lambda). \tag{4.9}$$

Note that

$$D(\lambda) = B_1(\lambda) + B_2(\lambda), \qquad B(\lambda) = B_1(\lambda) - B_2(\lambda). \tag{4.10}$$

Let

$$\phi(t,\lambda) = \begin{pmatrix} \varphi(t,\lambda) & \psi(t,\lambda) \\ p(t)\varphi^{\Delta}(t,\lambda) & p(t)\psi^{\Delta}(t,\lambda) \end{pmatrix}, \quad t \in [\rho(0),1] \cap \mathbb{T}. \tag{4.11}$$

Hence, we have

$$K^{-1}\phi(1,\lambda) = \begin{pmatrix} B_2(\lambda) & C(\lambda) \\ A(\lambda) & B_1(\lambda) \end{pmatrix}, \tag{4.12}$$

and by Lemma 2.2, we get

$$\phi^{-1}(t,\lambda) = \begin{pmatrix} p(t)\psi^{\Delta}(t,\lambda) & -\psi(t,\lambda) \\ -p(t)\varphi^{\Delta}(t,\lambda) & \varphi(t,\lambda) \end{pmatrix}, \quad t \in [\rho(0),1] \cap \mathbb{T}. \tag{4.13}$$

Proposition 4.1. *For $\lambda \in \mathbf{C}$, λ is an eigenvalue of* (1.1) *with* (1.2) *if and only if*

$$D(\lambda) = 2\cos\theta. \tag{4.14}$$

Moreover, λ is a multiple eigenvalue of (1.1) *with* (1.2) *if and only if*

$$\phi(1,\lambda) = e^{i\theta}K. \tag{4.15}$$

Proof. Since $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are linearly independent solutions of (1.1), then λ is an eigenvalue of the problem (1.1) with (1.2) if and only if there exist two constants C_1 and C_2, not both zero, such that $C_1\varphi(t,\lambda) + C_2\psi(t,\lambda)$ satisfies (1.2), which yields

$$\begin{pmatrix} \varphi(1,\lambda) - e^{i\theta}k_{11} & \psi(1,\lambda) - e^{i\theta}k_{12} \\ \varphi^{\Delta}(1,\lambda) - e^{i\theta}k_{21} & \psi^{\Delta}(1,\lambda) - e^{i\theta}k_{22} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0. \tag{4.16}$$

It is evident that (4.15) has a nontrivial solution (C_1, C_2) if and only if

$$\det\begin{pmatrix} \varphi(1,\lambda) - e^{i\theta}k_{11} & \psi(1,\lambda) - e^{i\theta}k_{12} \\ \varphi^{\Delta}(1,\lambda) - e^{i\theta}k_{21} & \psi^{\Delta}(1,\lambda) - e^{i\theta}k_{22} \end{pmatrix} = 0, \tag{4.17}$$

which together with (4.3), (4.4) and $\det K = k_{11}k_{22} - k_{12}k_{21} = 1$ implies that

$$1 + e^{2i\theta} - e^{i\theta}D(\lambda) = 0. \tag{4.18}$$

It follows from the above relation and the fact that $e^{-i\theta} + e^{i\theta} = 2\cos\theta$ that λ is an eigenvalue of (1.1) with (1.2) if and only if λ satisfies

$$D(\lambda) = 2\cos\theta. \tag{4.19}$$

On the other hand, (1.1) has two linearly independent solutions satisfying (1.2) if and only if all the entries of the coefficient matrix of (4.16) are zero. Hence, λ is a multiple eigenvalue of (1.1) with (1.2) if and only if (4.15) holds. This completes the proof. □

The following result is a direct consequence of the first result of Proposition 4.1.

Corollary 4.2. *For any* $\theta \in (-\pi, \pi]$,

$$\lambda_n\left(e^{i\theta}K\right) = \lambda_n\left(e^{-i\theta}K\right), \quad n \geq 0. \tag{4.20}$$

For $K \in \mathrm{SL}(2, \mathbf{R})$ and $\det K = 1$, we consider the separated boundary problem (1.1) with (2.14). Let μ_n, $0 \leq n \leq N_d - 1$, be all the eigenvalues of (1.1) with (2.14) and ordered as that in Theorem 2.7. Since $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are all entire functions in $\lambda \in \mathbf{C}$ for each $t \in [\rho(0), \sigma(1)] \cap \mathbb{T}$, $D(\lambda)$ is an entire functions in $\mathbf{C}$. Denote

$$\frac{d}{d\lambda}D(\lambda) := D'(\lambda), \qquad \frac{d^2}{d\lambda^2}D(\lambda) := D''(\lambda). \tag{4.21}$$

Proposition 4.3. *Assume that* $k_{11} > 0$ *and* $k_{12} \leq 0$ *or* $k_{11} \geq 0$ *and* $k_{12} < 0$. *For each* n, $n \geq 0$, $D(\mu_n) \geq 2$ *if* n *is odd, and* $D(\mu_n) \leq -2$ *if* n *is even.*

Proof. It is noted that λ is eigenvalue of (1.1) with (2.14) if and only if $k_{22}\psi(1,\lambda) - k_{12}\psi^{\Delta}(1,\lambda) = 0$. Hence, $\psi(t,\mu_n)$ is an eigenfunction with respect to μ_n. By Theorem 2.7 and the last two relations in (4.1), we have that $\psi(t,\mu_n)$ has exactly n generalized zeros in $(0,1)$ and

$$\operatorname{sgn}\psi(1,\mu_n) = (-1)^n. \tag{4.22}$$

(i) If $k_{12} < 0$, then it follows from $k_{22}\psi(1,\mu_n) - k_{12}\psi^{\Delta}(1,\mu_n) = 0$ that

$$\frac{\psi(1,\mu_n)}{k_{12}} = \frac{\psi^{\Delta}(1,\mu_n)}{k_{22}}, \qquad k_{11}k_{22}\psi(1,\mu_n) = k_{11}k_{12}\psi^{\Delta}(1,\mu_n). \tag{4.23}$$

By (4.3) and the first relation in (4.23) we have

$$\begin{aligned}1 &= \varphi(1,\mu_n)\psi^{\Delta}(1,\mu_n) - \varphi^{\Delta}(1,\mu_n)\psi(1,\mu_n)\\ &= \varphi(1,\mu_n)\frac{k_{22}}{k_{12}}\psi(1,\mu_n) - \varphi^{\Delta}(1,\mu_n)\psi(1,\mu_n)\\ &= \left(k_{22}\varphi(1,\mu_n) - k_{12}\varphi^{\Delta}(1,\mu_n)\right)\frac{\psi(1,\mu_n)}{k_{12}}.\end{aligned} \tag{4.24}$$

By the definition of $D(\lambda)$, the second relation in (4.23), and $\det K = 1$, we get

$$\begin{aligned}k_{12}D(\mu_n) &= k_{11}k_{12}\psi^{\Delta}(1,\mu_n) - k_{12}k_{21}\psi(1,\mu_n) + k_{12}k_{22}\varphi(1,\mu_n) - k_{12}^2\varphi^{\Delta}(1,\mu_n)\\ &= k_{11}k_{22}\psi(1,\mu_n) - k_{12}k_{21}\psi(1,\mu_n) + k_{12}k_{22}\varphi(1,\mu_n) - k_{12}^2\varphi^{\Delta}(1,\mu_n)\\ &= \psi(1,\mu_n) + k_{12}k_{22}\varphi(1,\mu_n) - k_{12}^2\varphi^{\Delta}(1,\mu_n).\end{aligned} \tag{4.25}$$

Hence,

$$D(\mu_n) = \left(k_{22}\varphi(1,\mu_n) - k_{12}\varphi^{\Delta}(1,\mu_n)\right) + \frac{\psi(1,\mu_n)}{k_{12}}. \tag{4.26}$$

Noting $(k_{22}\varphi(1,\mu_n) - k_{12}\varphi^{\Delta}(1,\mu_n))(\psi(1,\mu_n)/k_{12}) = 1$, $k_{12} < 0$, and (4.22), we have that if n is odd, then

$$D(\mu_n) = \left(\sqrt{\frac{\psi(1,\mu_n)}{k_{12}}} - \sqrt{k_{22}\varphi(1,\mu_n) - k_{12}\varphi^{\Delta}(1,\mu_n)}\right)^2 + 2 \geq 2, \tag{4.27}$$

and if n is even, then

$$D(\mu_n) = -\left(\sqrt{-\frac{\psi(1,\mu_n)}{k_{12}}} - \sqrt{-(k_{22}\varphi(1,\mu_n) - k_{12}\varphi^{\Delta}(1,\mu_n))}\right)^2 - 2 \leq -2. \tag{4.28}$$

(ii) If $k_{12} = 0$, then it is noted that λ is eigenvalue of (1.1) with (2.14) if and only if $\psi(1,\lambda) = 0$. Hence, $\psi(t,\mu_n)$ is an eigenfunction with respect to μ_n. By Theorem 2.7, $\psi(t,\mu_n)$ has exactly n generalized zeros in $(0,1)$ and

$$\psi(\rho(0),\mu_n) = \psi(1,\mu_n) = 0, \qquad \psi^{\Delta}(\rho(0),\mu_n) = 1, \quad n \geq 0. \tag{4.29}$$

Hence $\psi(t,\mu_0) > 0$ for all $t \in (0,1) \cap \mathbb{T}$.

Next we will show $\psi^{\Delta}(1,\mu_0) < 0$. In the case that $\rho(1) < 1$, $\psi(\rho(1),\mu_0) > 0$ and $\psi^{\Delta}(\rho(1),\mu_0) < 0$. It follows from (1.1) and (2.4) that

$$\left(p(\rho(1))\psi^{\Delta}(\rho(1),\mu_0)\right)^{\Delta} = \frac{p(1)\psi^{\Delta}(1,\mu_0) - p(\rho(1))\psi^{\Delta}(\rho(1),\mu_0)}{1-\rho(1)} = 0, \tag{4.30}$$

which implies

$$\psi^{\Delta}(1,\mu_0) = p(\rho(1))\psi^{\Delta}(\rho(1),\mu_0) < 0. \tag{4.31}$$

In the other case that $\rho(1) = 1$, then

$$\psi^{\Delta}(1,\mu_0) = \lim_{t\to 1^-}\frac{\psi(t,\mu_0)-\psi(1,\mu_0)}{t-1} = -\lim_{t\to 1^-}\frac{\psi(t,\mu_0)}{1-t} \le 0. \tag{4.32}$$

Further, by the existence and uniqueness theorem of solutions of initial value problems for (1.1) [5, Theorem 4.5], we obtain that $\psi^{\Delta}(1,\mu_0) < 0$.

With a similar argument from above, we get $\operatorname{sgn}\psi^{\Delta}(1,\mu_n) = (-1)^{n+1}$, $n \ge 0$.

By referring to $\psi(1,\mu_n) = 0$ and from (4.3), it follows that

$$\varphi(1,\mu_n)\psi^{\Delta}(1,\mu_n) = 1. \tag{4.33}$$

Hence, noting $\det K = k_{11}k_{22} = 1$ and $k_{22} > 0$, if n is odd, then

$$D(\mu_n) = \frac{k_{22}}{\psi^{\Delta}(1,\mu_n)} + k_{11}\psi^{\Delta}(1,\mu_n) \ge 2, \tag{4.34}$$

and if n is even, then

$$D(\mu_n) \le -2. \tag{4.35}$$

This completes the proof. $\square$

Proposition 4.4. *Assume that $k_{11} > 0$ and $k_{12} \le 0$ or $k_{11} \ge 0$ and $k_{12} < 0$. There exists a constant ν_0 such that $\nu_0 < \mu_0$ and $D(\nu_0) \ge 2$.*

Proof. Since $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are solutions of (1.1), we have

$$\begin{aligned} -\left(p(t)\varphi^{\Delta}(t,\lambda)\right)^{\Delta} + q(t)\varphi^{\sigma}(t,\lambda) &= \lambda r(t)\varphi^{\sigma}(t,\lambda), \\ -\left(p(t)\psi^{\Delta}(t,\lambda)\right)^{\Delta} + q(t)\psi^{\sigma}(t,\lambda) &= \lambda r(t)\psi^{\sigma}(t,\lambda). \end{aligned} \tag{4.36}$$

By integration, it follows from (4.1) and (4.36) that

$$\begin{aligned} \varphi^{\Delta}(1,\lambda) &= \int_{\rho(0)}^{1} \left(q(s)-\lambda r(s)\right)\varphi(\sigma(s),\lambda)\Delta s, \\ \psi^{\Delta}(1,\lambda) &= 1+\int_{\rho(0)}^{1} \left(q(s)-\lambda r(s)\right)\psi(\sigma(s),\lambda)\Delta s, \end{aligned} \tag{4.37}$$

where $p(\rho(0)) = p(1) = 1$ is used. In addition, from (4.36), we obtain

$$\begin{aligned} \left(\varphi(t,\lambda)\left(p(t)\varphi^{\Delta}(t,\lambda)\right)\right)^{\Delta} &= p(t)\left(\varphi^{\Delta}(t,\lambda)\right)^{2} + \left(q(t)-\lambda r(t)\right)\left(\varphi^{\sigma}(t,\lambda)\right)^{2}, \\ \left(\psi(t,\lambda)\left(p(t)\psi^{\Delta}(t,\lambda)\right)\right)^{\Delta} &= p(t)\left(\psi^{\Delta}(t,\lambda)\right)^{2} + \left(q(t)-\lambda r(t)\right)\left(\psi^{\sigma}(t,\lambda)\right)^{2}, \end{aligned} \tag{4.38}$$

which, similarly together with (4.1) and by integration, imply that

$$\begin{aligned} \varphi(1,\lambda)\varphi^{\Delta}(1,\lambda) &= \int_{\rho(0)}^{1} \left(p(s)\left(\varphi^{\Delta}(s,\lambda)\right)^{2} + \left(q(s)-\lambda r(s)\right)\varphi^{2}(\sigma(s),\lambda)\right)\Delta s \\ &\geq \int_{\rho(0)}^{1} \left(q(s)-\lambda r(s)\right)\varphi^{2}(\sigma(s),\lambda)\Delta s, \\ \psi(1,\lambda)\psi^{\Delta}(1,\lambda) &= \int_{\rho(0)}^{1} \left(p(s)\left(\psi^{\Delta}(s,\lambda)\right)^{2} + \left(q(s)-\lambda r(s)\right)\psi^{2}(\sigma(s),\lambda)\right)\Delta s \\ &\geq \int_{\rho(0)}^{1} \left(q(s)-\lambda r(s)\right)\psi^{2}(\sigma(s),\lambda)\Delta s. \end{aligned} \tag{4.39}$$

On the other hand, it follows from Lemma 2.5 and (4.1) that for all sufficiently large $-\lambda$, $\varphi^{\sigma}(t) > 0$, $\psi^{\sigma}(t) > 0$, for all $t \in (\rho(0),1) \cap \mathbb{T}$, where α and β in (2.11) are taken as $\alpha = 0$, $\beta = 1$, and $\alpha = -1$, $\beta = 0$, respectively, which satisfy (**H**). So, from (4.37) and (4.39), we obtain that

$$\lim_{\lambda\to-\infty}\psi^{\Delta}(1,\lambda) = \lim_{\lambda\to-\infty}\varphi^{\Delta}(1,\lambda) = \lim_{\lambda\to-\infty}\left(\psi(1,\lambda)\psi^{\Delta}(1,\lambda)\right) = \lim_{\lambda\to-\infty}\left(\varphi(1,\lambda)\varphi^{\Delta}(1,\lambda)\right) = \infty, \tag{4.40}$$

and by Lemma 2.4, it implies

$$\begin{aligned} \lim_{\lambda\to-\infty} D(\lambda) &= \lim_{\lambda\to-\infty}\left(k_{22}\varphi(1,\lambda) + k_{11}\psi^{\Delta}(1,\lambda) - k_{21}\psi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda)\right) \\ &= \lim_{\lambda\to-\infty}\left(k_{11}\psi(1,\lambda)\left(\frac{\psi^{\Delta}(1,\lambda)}{\psi(1,\lambda)} - \frac{k_{21}}{k_{11}}\right) - k_{12}\varphi(1,\lambda)\left(\frac{\varphi^{\Delta}(1,\lambda)}{\varphi(1,\lambda)} - \frac{k_{22}}{k_{12}}\right)\right) \\ &= \infty. \end{aligned} \tag{4.41}$$

By Proposition 4.3, $D(\mu_0) \leq -2$. Therefore, there exists a $\nu_0 < \mu_0$ such that $D(\nu_0) \geq 2$. This completes the proof. $\square$

Lemma 4.5. *For any $\lambda \in \mathbf{C}$ one has*

$$D'(\lambda) = \int_{\rho(0)}^{1} \Big(A(\lambda)\psi^2(\sigma(s),\lambda) - B(\lambda)\psi(\sigma(s),\lambda)\varphi(\sigma(s),\lambda) - C(\lambda)\varphi^2(\sigma(s),\lambda)\Big) r(s)\Delta s, \tag{4.42}$$

$$\begin{aligned} 4C(\lambda)D'(\lambda) &= -\int_{\rho(0)}^{1} (2C(\lambda)\varphi(\sigma(s),\lambda) + B(\lambda)\psi(\sigma(s),\lambda))^2 r(s)\Delta s \\ &\quad -(4 - D^2(\lambda))\int_{\rho(0)}^{1} \psi^2(\sigma(s),\lambda) r(s)\Delta s, \end{aligned} \tag{4.43}$$

$$\begin{aligned} 4A(\lambda)D'(\lambda) &= \int_{\rho(0)}^{1} (2A(\lambda)\psi(\sigma(s),\lambda) - B(\lambda)\varphi(\sigma(s),\lambda))^2 r(s)\Delta s \\ &\quad +(4 - D^2(\lambda))\int_{\rho(0)}^{1} \varphi^2(\sigma(s),\lambda) r(s)\Delta s. \end{aligned} \tag{4.44}$$

Proof. Since $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are solutions of (1.1) with (4.1), then they satisfy (4.36). Differentiating (4.36) with respect to λ, we have

$$\begin{gathered} \Big(p(t)\varphi_\lambda^\Delta(t,\lambda)\Big)^\Delta + (q(t) + \lambda r(t))\varphi_\lambda^\sigma(t,\lambda) = -r(t)\varphi^\sigma(t,\lambda) \\ \varphi_\lambda(\rho(0),\lambda) = \varphi_\lambda^\Delta(\rho(0),\lambda) = 0, \\ \Big(p(t)\psi_\lambda^\Delta(t,\lambda)\Big)^\Delta + (q(t) + \lambda r(t))\psi_\lambda^\sigma(t,\lambda) = -r(t)\psi^\sigma(t,\lambda) \\ \psi_\lambda(\rho(0),\lambda) = \psi_\lambda^\Delta(\rho(0),\lambda) = 0. \end{gathered} \tag{4.45}$$

By the variation of constants formula [5, Theorem 4.24], we get

$$\begin{aligned} \varphi_\lambda(t,\lambda) &= \int_{\rho(0)}^{t} r(s)(\psi(\sigma(s),\lambda)\varphi(t,\lambda) - \varphi(\sigma(s),\lambda)\psi(t,\lambda))\varphi(\sigma(s),\lambda)\Delta s, \\ \psi_\lambda(t,\lambda) &= \int_{\rho(0)}^{t} r(s)(\psi(\sigma(s),\lambda)\varphi(t,\lambda) - \varphi(\sigma(s),\lambda)\psi(t,\lambda))\psi(\sigma(s),\lambda)\Delta s. \end{aligned} \tag{4.46}$$

Further, it follows from [5, Theorem 1.117] that

$$\begin{aligned}\varphi_\lambda^\Delta(t,\lambda) &= \int_{\rho(0)}^{t} r(s)\Big(\psi(\sigma(s),\lambda)\varphi^\Delta(t,\lambda)-\varphi(\sigma(s),\lambda)\psi^\Delta(t,\lambda)\Big)\varphi(\sigma(s),\lambda)\Delta s,\\ \psi_\lambda^\Delta(t,\lambda) &= \int_{\rho(0)}^{t} r(s)\Big(\psi(\sigma(s),\lambda)\varphi^\Delta(t,\lambda)-\varphi(\sigma(s),\lambda)\psi^\Delta(t,\lambda)\Big)\psi(\sigma(s),\lambda)\Delta s.\end{aligned} \tag{4.47}$$

From (4.46), (4.47), and (4.11), we have

$$\phi_\lambda(t,\lambda) = \begin{pmatrix}\varphi_\lambda(t,\lambda) & \psi_\lambda(t,\lambda)\\ p(t)\varphi_\lambda^\Delta(t,\lambda) & p(t)\psi_\lambda^\Delta(t,\lambda)\end{pmatrix} = \int_{\rho(0)}^{t} r(s)\begin{pmatrix}A_{11}(s,t,\lambda) & A_{12}(s,t,\lambda)\\ A_{21}(s,t,\lambda) & A_{22}(s,t,\lambda)\end{pmatrix}\Delta s, \tag{4.48}$$

where

$$\begin{aligned}A_{11}(s,t,\lambda) &= \left(\psi^\sigma(s,\lambda)\varphi(t,\lambda)-\varphi^\sigma(s,\lambda)\psi(t,\lambda)\right)\varphi^\sigma(s,\lambda),\\ A_{12}(s,t,\lambda) &= \left(\psi^\sigma(s,\lambda)\varphi(t,\lambda)-\varphi^\sigma(s,\lambda)\psi(t,\lambda)\right)\psi^\sigma(s,\lambda),\\ A_{21}(s,t,\lambda) &= p(t)\Big(\psi^\sigma(s,\lambda)\varphi^\Delta(t,\lambda)-\varphi^\sigma(s,\lambda)\psi^\Delta(t,\lambda)\Big)\varphi^\sigma(s,\lambda),\\ A_{22}(s,t,\lambda) &= p(t)\Big(\psi^\sigma(s,\lambda)\varphi^\Delta(t,\lambda)-\varphi^\sigma(s,\lambda)\psi^\Delta(t,\lambda)\Big)\psi^\sigma(s,\lambda).\end{aligned} \tag{4.49}$$

It follows from (4.11) and (4.13) that

$$\phi(t,\lambda)\phi^{-1}(\sigma(s),\lambda)R(s)\phi(\sigma(s),\lambda) = -\begin{pmatrix}A_{11}(s,t,\lambda) & A_{12}(s,t,\lambda)\\ A_{21}(s,t,\lambda) & A_{22}(s,t,\lambda)\end{pmatrix}, \tag{4.50}$$

where $R(t) = \begin{pmatrix}0 & 0\\ r(t) & 0\end{pmatrix}$, $t \in [\rho(0),\rho(1)]\cap\mathbb{T}$.

Hence,

$$\phi_\lambda(t,\lambda) = -\int_{\rho(0)}^{1} \phi(t,\lambda)\phi^{-1}(\sigma(s),\lambda)R(s)\phi(\sigma(s),\lambda)\Delta s. \tag{4.51}$$

By (4.10) and (4.12), we have

$$D(\lambda) = B_1(\lambda) + B_2(\lambda) = \operatorname{trace} K^{-1}\phi(1,\lambda). \tag{4.52}$$

Differentiating above relation with respect to λ, and with (4.12), we have

$$\begin{aligned}
&D'(\lambda)\\
&= \operatorname{trace} K^{-1}\phi_\lambda(1,\lambda)\\
&= -\operatorname{trace}\int_{\rho(0)}^{1} K^{-1}\phi(1,\lambda)\phi^{-1}(\sigma(s),\lambda)R(s)\phi(\sigma(s),\lambda)\Delta s\\
&= -\operatorname{trace}\int_{\rho(0)}^{1}\begin{pmatrix} B_2(\lambda) & C(\lambda)\\ A(\lambda) & B_1(\lambda)\end{pmatrix}\begin{pmatrix}\psi^{\Delta}(\sigma(s),\lambda) & -\psi(\sigma(s),\lambda)\\ -\varphi^{\Delta}(\sigma(s),\lambda) & \varphi(\sigma(s),\lambda)\end{pmatrix}\\
&\quad\times\begin{pmatrix}0 & 0\\ r(s) & 0\end{pmatrix}\begin{pmatrix}\varphi(\sigma(s),\lambda) & \psi(\sigma(s),\lambda)\\ \varphi^{\Delta}(\sigma(s),\lambda) & \psi^{\Delta}(\sigma(s),\lambda)\end{pmatrix}\Delta s\\
&= \int_{\rho(0)}^{1}\Big(A(\lambda)\psi^2(\sigma(s),\lambda) - (B_1(\lambda)-B_2(\lambda))\psi(\sigma(s),\lambda)\varphi(\sigma(s),\lambda) - C(\lambda)\varphi^2(\sigma(s),\lambda)\Big)r(s)\Delta s,
\end{aligned} \tag{4.53}$$

which together with (4.10) confirm (4.42).

To establish (4.43), from (4.12) and (4.10), we obtain

$$\begin{aligned}
B_1(\lambda)B_2(\lambda) - A(\lambda)C(\lambda) &= \det\Big(K^{-1}\phi(1,\lambda)\Big) = 1,\\
4 - D^2(\lambda) &= 4 - (B_1(\lambda)+B_2(\lambda))^2 = 4 - (B_1(\lambda)-B_2(\lambda))^2 - 4B_1(\lambda)B_2(\lambda)\\
&= 4(1 - B_1(\lambda)B_2(\lambda)) - B^2(\lambda) = -\big(4A(\lambda)C(\lambda) + B^2(\lambda)\big).
\end{aligned} \tag{4.54}$$

Thus

$$\begin{aligned}
&4C(\lambda)D'(\lambda)\\
&\quad= \int_{\rho(0)}^{1}\Big(4A(\lambda)C(\lambda)\psi^{\sigma^2}(s,\lambda) - 4B(\lambda)C(\lambda)\psi^{\sigma}(s,\lambda)\varphi^{\sigma}(s,\lambda) - 4C^2(\lambda)\varphi^{\sigma^2}(s,\lambda)\Big)r(s)\Delta s\\
&\quad= \int_{\rho(0)}^{1}\Big(-(2C(\lambda)\varphi^{\sigma}(s,\lambda) + B(\lambda)\psi^{\sigma}(s,\lambda))^2 + \Big(4A(\lambda)C(\lambda) + B^2(\lambda)\Big)\psi^{\sigma^2}(s,\lambda)\Big)r(s)\Delta s\\
&\quad= -\int_{\rho(0)}^{1}(2C(\lambda)\varphi^{\sigma}(s,\lambda) + B(\lambda)\psi^{\sigma}(s,\lambda))^2 r(s)\Delta s - \Big(4 - D^2(\lambda)\Big)\int_{\rho(0)}^{1}\psi^{\sigma^2}(s,\lambda)r(s)\Delta s.
\end{aligned} \tag{4.55}$$

That is, (4.43) holds. The identity (4.44) can be verified similarly. This completes the proof. □

Corollary 4.6. *If $\lambda \in \mathbf{R}$ satisfies $|D(\lambda)| < 2$, then $A(\lambda) \neq 0$, $C(\lambda) \neq 0$, and $D'(\lambda) \neq 0$.*

Proof. These are direct consequences of (4.43) and (4.44). □

Lemma 4.7. *$C(\lambda) = 0$ if and only if $\lambda = \mu_n$ for some $n \in \{0, 1, \ldots, N_d - 1\}$ and $\psi(\cdot, \mu_n)$ is an eigenfunction of μ_n.*

Proof. It is directly follows from the definition of $C(\lambda)$ and the initial conditions (4.1). □

Lemma 4.8. *Assum that $\theta = 0$ or $\theta = \pi$ and λ is a multiple eigenvalue of (1.1) with (1.2) if and only if $D'(\lambda) = 0$.*

Proof. Assume that $\theta = 0$. By (4.15) λ is a multiple eigenvalue if and only if

$$\phi(1, \lambda) = K; \tag{4.56}$$

hence, it follows from (4.12) that

$$\begin{pmatrix} B_2(\lambda) & C(\lambda) \\ A(\lambda) & B_1(\lambda) \end{pmatrix} = K^{-1}\phi(1, \lambda) = K^{-1}K = I. \tag{4.57}$$

Therefore $A(\lambda) = C(\lambda) = 0$ and $B_1(\lambda) = B_2(\lambda) = 1$.

(i) Suppose that λ is a multiple eigenvalue of (1.1) with (1.2). Then $A(\lambda) = C(\lambda) = 0$ and $B(\lambda) = B_1(\lambda) - B_2(\lambda) = 0$. By (4.42), $D'(\lambda) = 0$.

(ii) Suppose that λ is an eigenvalue of (1.1) with (1.2) and $D'(\lambda) = 0$. Then by (4.14), $D(\lambda) = 0$. From (4.43) and (4.44) we get

$$\begin{aligned} &2C(\lambda)\varphi(\sigma(s), \lambda) + B(\lambda)\psi(\sigma(s), \lambda) = 0, \\ &2A(\lambda)\psi(\sigma(s), \lambda) - B(\lambda)\varphi(\sigma(s), \lambda) = 0. \end{aligned} \tag{4.58}$$

Since $\varphi(t, \lambda)$ and $\psi(t, \lambda)$ are linearly independence solutions of (1.1), we have

$$A(\lambda) = B(\lambda) = C(\lambda) = 0. \tag{4.59}$$

It follows from $B(\lambda) = B_1(\lambda) - B_2(\lambda) = 0$ and $D(\lambda) = B_1(\lambda) + B_2(\lambda) = 2$ that $B_1(\lambda) = B_2(\lambda) = 1$. Thus, λ is a multiple eigenvalue of (1.1) with (1.2).

The case $\theta = \pi$ can be established by replacing K by $-K$ in the above argument. This completes the proof. □

Lemma 4.9. *Assume $\theta = 0$ or $\theta = \pi$. If λ is a multiple eigenvalue of (1.1) with (1.2), then there exists $n \in \{0, 1, \ldots, N_d - 1\}$ such that $\lambda = \mu_n$.*

Proof. Assume that λ is a multiple eigenvalue of (1.1) with (1.2). From the proof of Lemma 4.8 we see that $C(\lambda) = 0$. From (4.9) we have

$$k_{22}\psi(1, \lambda) - k_{12}\psi^{\Delta}(1, \lambda) = 0. \tag{4.60}$$

This means that $\lambda = \mu_n$ for some $n \in \{0, 1, \ldots, N_d - 1\}$. This completes the proof. □

Proposition 4.10. *Assume that $k_{11} > 0$ and $k_{12} \le 0$ or $k_{11} \ge 0$ and $k_{12} < 0$.*

(i) *Equations $D'(\lambda) = 0$ and $D(\lambda) = 2$ or -2 hold if and only if λ is a multiple eigenvalue of* (1.1) *with* (1.2) *with $\theta = 0$ or $\theta = \pi$.*

(ii) *If $D(\lambda) = 2$ or -2 and λ is a multiple eigenvalue of* (1.1) *with* (1.2)*, then $\lambda = \mu_n$, $n \in \{0, 1, \dots, N_d - 1\}$.*

(iii) *If $D(\lambda) = 2$ or -2 for some $\lambda \neq \mu_n$, $n \in \{0, 1, \dots, N_d - 1\}$, then λ is a simple eigenvalue of* (1.1) *with* (1.2) *with $\theta = 0$ or $\theta = \pi$.*

(iv) *Moreover, for every $\lambda \neq \mu_n$, $n \in \{0, 1, \dots, N_d - 1\}$, with $-2 \le D(\lambda) \le 2$ one has*

$$\begin{gathered} D'(\lambda) < 0, \quad \nu_0 < \lambda < \mu_0; \\ (-1)^n D'(\lambda) > 0, \quad \mu_n < \lambda < \mu_{n+1}, \quad n \ge 0, \end{gathered} \tag{4.61}$$

and in the case of $N_d < \infty$,

$$(-1)^{N_d - 1} D'(\lambda) > 0, \quad \lambda > \mu_{N_d - 1}. \tag{4.62}$$

Proof. Parts (i), (ii), and (iii) follow from Lemmas 4.8 and 4.9. It follows from Propositions 4.3 and 4.4 and Corollary 4.6 that $D(\mu_0) \le -2$, $D(\nu_0) \ge 2$ with $\nu_0 < \mu_0$ and $D'(\lambda) \neq 0$ when $|D(\lambda)| < 2$. Hence, $D'(\lambda) < 0$ with $\nu_0 < \lambda < \mu_0$, $-2 \le D(\lambda) \le 2$. Similarly, by Proposition 4.3 and Corollary 4.6, we have

$$D(\mu_n) D(\mu_{n+1}) \le -4, \quad D'(\lambda) \neq 0 \quad \text{when } |D(\lambda)| < 2, \tag{4.63}$$

which implies

$$(-1)^n D'(\lambda) > 0 \quad \text{with } \mu_n < \lambda < \mu_{n+1}, \quad -2 \le D(\lambda) \le 2. \tag{4.64}$$

If $N_d < \infty$, then all the points of $[0,1] \cap \mathbb{T}$ are isolated. In this case, (1.1) can be rewritten as

$$p(\sigma(t)) y\left(\sigma^2(t)\right) = (a(t) + \lambda b(t)) y(\sigma(t)) + c(t) y(t), \quad t \in \left[\rho(0), \rho(1)\right] \cap \mathbb{T}, \tag{4.65}$$

where

$$\begin{gathered} a(t) = p(\sigma(t)) + p(t) \frac{\mu(\sigma(t))}{\mu(t)} - q(t)\mu(t)\mu(\sigma(t)), \\ b(t) = -r(t)\mu(t)\mu(\sigma(t)), \qquad c(t) = -p(t)\frac{\mu(\sigma(t))}{\mu(t)}. \end{gathered} \tag{4.66}$$

By Theorem 2.7, (1.1) with (2.14) has N_d eigenvalues:

$$\mu_0 < \mu_1 < \cdots < \mu_{N_d - 1}. \tag{4.67}$$

It follows from (4.65) that $\psi^{\Delta}(1,\lambda) = (\psi(\sigma(1),\lambda) - \psi(1,\lambda))/\mu(1)$ and $\varphi^{\Delta}(1,\lambda) = (\varphi(\sigma(1),\lambda) - \varphi(1,\lambda))/\mu(1)$ are two polynomials of degree $N_d + 1$ in λ and $\psi(1,\lambda)$ and $\varphi(1,\lambda)$ are two polynomials of degree N_d in λ. Then $D(\lambda)$ can be written as

$$D(\lambda) = k_{11}\psi^{\Delta}(1,\lambda) - k_{21}\psi(1,\lambda) + k_{22}\varphi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda) = (-1)^{N_d+1}A_{N_d+1}\lambda^{N_d+1} + h(\lambda), \tag{4.68}$$

where $A_{N_d+1} > 0$ and $h(\lambda)$ is a polynomial in λ whose order is not larger than N_d. By Proposition 4.3, if N_d+1 is odd, then $D(\mu_{N_d-1}) \geq 2$, and if N_d+1 is even, then $D(\mu_{N_d-1}) \leq -2$. It follows that if $N_d + 1$ is odd, then $D(\lambda) \to -\infty$ as $\lambda \to \infty$, and if $N_d + 1$ is even, then $D(\lambda) \to \infty$ as $\lambda \to \infty$. Hence, if $N_d + 1$ is odd, then there exists a constant $\xi_0 > \mu_{N_d-1}$ such that $D(\xi_0) \leq -2$. Similarly, in the other case that $N_d + 1$ is even, there exists a constant $\eta_0 > \mu_{N_d-1}$ such that $D(\eta_0) \geq 2$, and by using Corollary 4.6, we have

$$\begin{gathered} -D'(\lambda) > 0 \quad \text{with } \mu_{N_d-1} < \lambda < \xi_0 \text{ if } N_d + 1 \text{ is odd}, \\ D'(\lambda) > 0 \quad \text{with } \mu_{N_d-1} < \lambda < \eta_0 \text{ if } N_{\mathrm{d}} + 1 \text{ is even}. \end{gathered} \tag{4.69}$$

Hence,

$$(-1)^{N_d-1}D'(\lambda) > 0, \quad \lambda > \mu_{N_d-1}. \tag{4.70}$$

This completes the proof. □

Proposition 4.11. *For any fixed $\theta \neq 0$, $-\pi < \theta < \pi$, each eigenvalue of* (1.1) *with* (1.2) *is simple.*

Proof. It follows from (4.46) and (4.47) that

$$D'(\lambda) = k_{11}\psi_{\lambda}^{\Delta}(1,\lambda) - k_{21}\psi_{\lambda}(1,\lambda) + k_{22}\varphi_{\lambda}(1,\lambda) - k_{12}\varphi_{\lambda}^{\Delta}(1,\lambda) = \int_{\rho(0)}^{1} r(s)\delta(s)\Delta s, \tag{4.71}$$

where

$$\begin{aligned} \delta(s) &:= \left(k_{11}\varphi^{\Delta}(1,\lambda) - k_{21}\varphi(1,\lambda)\right)\psi^2(\sigma(s),\lambda) \\ &\quad + \left(k_{22}\varphi(1,\lambda) - k_{11}\psi^{\Delta}(1,\lambda) + k_{21}\psi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda)\right)\varphi(\sigma(s),\lambda)\psi(\sigma(s),\lambda) \\ &\quad - \left(k_{22}\psi(1,\lambda) - k_{12}\psi^{\Delta}(1,\lambda)\right)\varphi^2(\sigma(s),\lambda) \\ &= \left(\psi(\sigma(s),\lambda), \varphi(\sigma(s),\lambda)\right)H(\lambda)\begin{pmatrix} \psi(\sigma(s),\lambda) \\ \varphi(\sigma(s),\lambda) \end{pmatrix}, \\ H(\lambda) &:= \begin{pmatrix} k_{11}\varphi^{\Delta}(1,\lambda) - k_{21}\varphi(1,\lambda) & \begin{array}{l} \frac{1}{2}\big(k_{22}\varphi(1,\lambda) - k_{11}\psi^{\Delta}(1,\lambda) \\ \quad + k_{21}\psi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda)\big) \end{array} \\ \begin{array}{l} \frac{1}{2}\big(k_{22}\varphi(1,\lambda) - k_{11}\psi^{\Delta}(1,\lambda) \\ \quad + k_{21}\psi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda)\big) \end{array} & k_{12}\psi^{\Delta}(1,\lambda) - k_{22}\psi(1,\lambda) \end{pmatrix}. \end{aligned} \tag{4.72}$$

Then from (4.1), $\det K = 1$, and the definition of $D(\lambda)$, we have

$$\begin{aligned}
\det H(\lambda) &= \left(k_{11}\varphi^{\Delta}(1,\lambda) - k_{21}\varphi(1,\lambda)\right)\left(k_{12}\psi^{\Delta}(1,\lambda) - k_{22}\psi(1,\lambda)\right) \\
&\quad - \frac{\left(k_{22}\varphi(1,\lambda) - k_{11}\psi^{\Delta}(1,\lambda) + k_{21}\psi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda)\right)^2}{4} \\
&= \left(k_{11}\varphi^{\Delta}(1,\lambda) - k_{21}\varphi(1,\lambda)\right)\left(k_{12}\psi^{\Delta}(1,\lambda) - k_{22}\psi(1,\lambda)\right) \\
&\quad - \frac{1}{4}D^2(\lambda) + \left(k_{22}\varphi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda)\right)\left(k_{11}\psi^{\Delta}(1,\lambda) - k_{21}\psi(1,\lambda)\right) \\
&= -\frac{1}{4}D^2(\lambda) + 1.
\end{aligned} \tag{4.73}$$

Thus, if $|D(\lambda)| \leq 2$, then $\det H(\lambda) \geq 0$. $H(\lambda)$ is always positive semidefinite or negative semidefinite. Consequently, $\delta(s)$ is not change sign in $[\rho(0), 1] \cap \mathbb{T}$. In this case, $D'(\lambda)$ cannot vanish unless $\delta(s) \equiv 0$. Because $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are linearly independent, $\delta(s) \equiv 0$ if and only if all the entries of the matrix $H(\lambda)$ vanish, namely,

$$\begin{gathered}
k_{11}\varphi^{\Delta}(1,\lambda) - k_{21}\varphi(1,\lambda) = 0, \\
k_{12}\psi^{\Delta}(1,\lambda) - k_{22}\psi(1,\lambda) = 0, \\
k_{22}\varphi(1,\lambda) - k_{11}\psi^{\Delta}(1,\lambda) + k_{21}\psi(1,\lambda) - k_{12}\varphi^{\Delta}(1,\lambda) = 0.
\end{gathered} \tag{4.74}$$

Suppose that λ is an eigenvalue of the problem (1.1) with (1.2) and fix $\theta, -\pi < \theta < \pi$ with $\theta \neq 0$. By Proposition 4.1, we have $D^2(\lambda) = 4\cos^2\theta < 4$, then $\det H(\lambda) > 0$, and the matrix $H(\lambda)$ is positive definite or negative definite. Hence, $\delta(s) > 0$ or $\delta(s) < 0$ for $s \in [\rho(0), 1] \cap \mathbb{T}$, since $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are linearly independent.

If λ is a multiple eigenvalue of problem (1.1) with (1.2), then (4.15) holds by Proposition 4.1. By using (4.15), it can be easily verified that (4.74) holds; that is, all the entries of the matrix $H(\lambda)$ are zeros. Then $\delta(s) = 0$, which is contrary to $\delta(s) \neq 0$. Hence, λ is a simple eigenvalue of (1.1) and (1.2). This completes the proof. □

Proposition 4.12. *If n is odd, $D(\mu_n) = 2$, and $D'(\mu_n) = 0$, then $D''(\mu_n) < 0$; if n is even, $D(\mu_n) = -2$, and $D'(\mu_n) = 0$, then $D''(\mu_n) > 0$.*

Proof. Assume $D(\mu_n) = 2$ and $D'(\mu_n) = 0$ with n being odd. It follows from Proposition 4.1 that

$$\phi(1,\mu_n) = K. \tag{4.75}$$

As in the proof of Lemma 4.5 and by (4.11) and (4.13),

$$\begin{aligned}
D'(\lambda) &= -\operatorname{trace}\int_{\rho(0)}^{1} K^{-1}\phi(1,\lambda)\phi^{-1}(\sigma(s),\lambda)R(s)\phi(\sigma(s),\lambda)\Delta s \\
&= -\operatorname{trace}\int_{\rho(0)}^{1} \phi^{-1}(1,\mu_n)\phi(1,\lambda)\begin{pmatrix} p(s)\psi^{\Delta}(\sigma(s),\lambda) & -\psi(\sigma(s),\lambda) \\ -p(s)\varphi^{\Delta}(\sigma(s),\lambda) & \varphi(\sigma(s),\lambda)\end{pmatrix} \\
&\quad\times\begin{pmatrix} 0 & 0 \\ r(s) & 0\end{pmatrix}\begin{pmatrix} \varphi(\sigma(s),\lambda) & \psi(\sigma(s),\lambda) \\ p(s)\varphi^{\Delta}(\sigma(s),\lambda) & p(s)\psi^{\Delta}(\sigma(s),\lambda)\end{pmatrix}\Delta s \\
&= \operatorname{trace}\left(\phi^{-1}(1,\mu_n)\phi(1,\lambda)\int_{\rho(0)}^{1}\begin{pmatrix} \psi(\sigma(s),\lambda)\varphi(\sigma(s),\lambda) & \psi^2(\sigma(s),\lambda) \\ -\varphi^2(\sigma(s),\lambda) & -\psi(\sigma(s),\lambda)\varphi(\sigma(s),\lambda)\end{pmatrix} r(s)\Delta s\right).
\end{aligned} \tag{4.76}$$

Hence,

$$\begin{aligned}
&D''(\lambda) \\
&= \operatorname{trace}\left(\phi^{-1}(1,\mu_n)\phi_{\lambda}(1,\lambda)\int_{\rho(0)}^{1}\begin{pmatrix} \psi(\sigma(s),\lambda)\varphi(\sigma(s),\lambda) & \psi^2(\sigma(s),\lambda) \\ -\varphi^2(\sigma(s),\lambda) & -\psi(\sigma(s),\lambda)\varphi(\sigma(s),\lambda)\end{pmatrix} r(s)\Delta s\right) \\
&\quad+ \operatorname{trace}\left(\phi^{-1}(1,\mu_n)\phi(1,\lambda)\frac{\partial}{\partial\lambda}\int_{\rho(0)}^{1}\begin{pmatrix} \psi(\sigma(s),\lambda)\varphi(\sigma(s),\lambda) & \psi^2(\sigma(s),\lambda) \\ -\varphi^2(\sigma(s),\lambda) & -\psi(\sigma(s),\lambda)\varphi(\sigma(s),\lambda)\end{pmatrix} r(s)\Delta s\right) \\
&= \operatorname{trace}\left(\phi^{-1}(1,\mu_n)\phi(1,\lambda)\int_{\rho(0)}^{1}\begin{pmatrix} \psi^{\sigma}(s,\lambda)\varphi^{\sigma}(s,\lambda) & \psi^{\sigma^2}(s,\lambda) \\ -\varphi^{\sigma^2}(s,\lambda) & -\psi^{\sigma}(s,\lambda)\varphi^{\sigma}(s,\lambda)\end{pmatrix}^2 r^2(s)\Delta s\right) \\
&\quad+ \operatorname{trace}\left(\phi^{-1}(1,\mu_n)\phi(1,\lambda)\frac{\partial}{\partial\lambda}\int_{\rho(0)}^{1}\begin{pmatrix} \psi^{\sigma}(s,\lambda)\varphi^{\sigma}(s,\lambda) & \psi^{\sigma^2}(s,\lambda) \\ -\varphi^{\sigma^2}(s,\lambda) & -\psi^{\sigma}(s,\lambda)\varphi^{\sigma}(s,\lambda)\end{pmatrix} r(s)\Delta s\right),
\end{aligned} \tag{4.77}$$

where (4.51) is used and

$$\begin{aligned}
D''(\mu_n) &= \operatorname{trace}\int_{\rho(0)}^{1}\begin{pmatrix} \psi(\sigma(s),\mu_n)\varphi(\sigma(s),\mu_n) & \psi^2(\sigma(s),\mu_n) \\ -\varphi^2(\sigma(s),\mu_n) & -\psi(\sigma(s),\mu_n)\varphi(\sigma(s),\mu_n)\end{pmatrix}^2 r^2(s)\Delta s \\
&= 2\left(\int_{\rho(0)}^{1}\psi(\sigma(s),\mu_n)\varphi(\sigma(s),\mu_n)r(s)\Delta s\right)^2 \\
&\quad - 2\int_{\rho(0)}^{1}\psi^2(\sigma(s),\mu_n)r(s)\Delta s\int_{\rho(0)}^{1}\varphi^2(\sigma(s),\mu_n)r(s)\Delta s \\
&\le 0
\end{aligned} \tag{4.78}$$

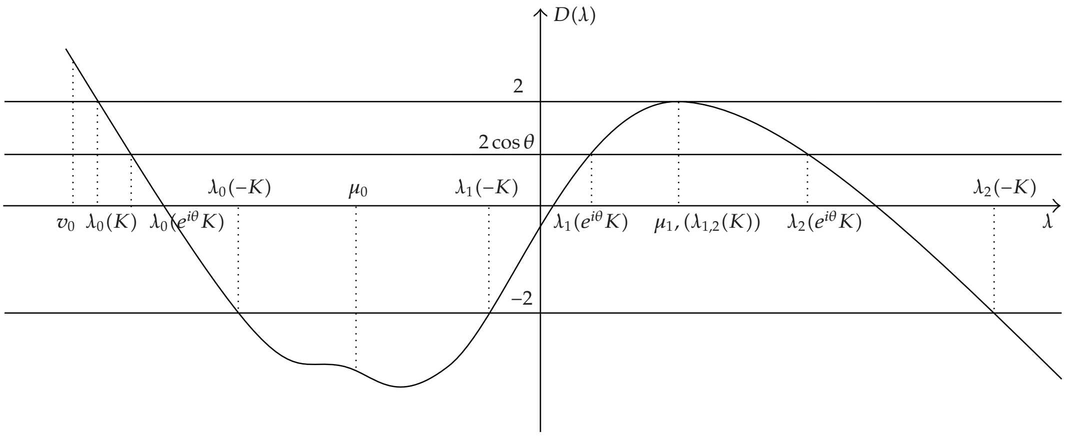

Figure 1: The graph of $D(\lambda)$.

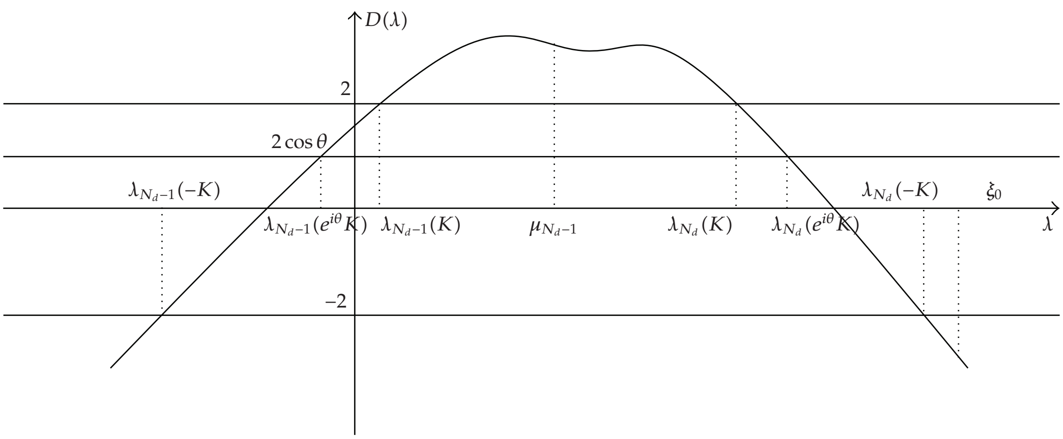

Figure 2: The graph of $D(\lambda)$ in the case that $N_d + 1$ is odd.

by the Holder inequality [8, Lemma 2.2(iv)]. Therefore $D''(\mu_n) < 0$. Since $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are linearly independent, which proves the first conclusion, the second conclusion can be shown similarly. This completes the proof. □

5. Proofs of the Main Results

Proof of Theorem 3.1. By Propositions 4.1–4.12 and the intermediate value theorem, the inequalities in (3.1)–(3.2) can been illustrated with the graph of $D(\lambda)$ (see Figures 1–3). We now give the detail proof of Theorem 3.1.

By Lemma 2.9, all the eigenvalues of the coupled boundary value problem (1.1) with (1.2) are real. By Propositions 4.3–4.10, $D(\mu_0) \leq -2$, $D'(\lambda) < 0$ for all $\lambda < \mu_0$ with $-2 \leq D(\lambda) \leq 2$, and there exists $\nu_0 < \mu_0$ such that $D(\nu_0) \geq 2$. Therefore, by the continuity of $D(\lambda)$ and the intermediate value theorem, (1.1) and (1.2) with $\theta = 0$ have only one eigenvalue $\lambda_0(K) < \mu_0$,

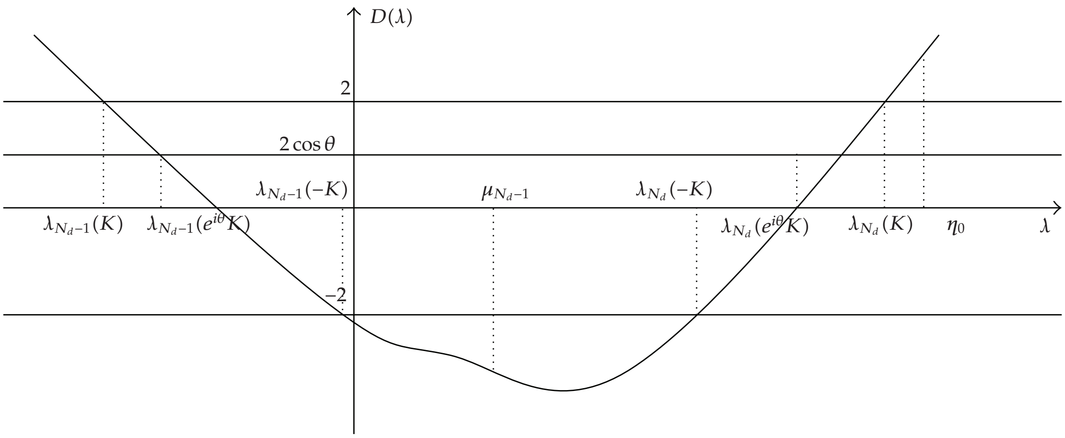

Figure 3: The graph of $D(\lambda)$ in the case that $N_d + 1$ is even.

(1.1) and (1.2) with $\theta = \pi$ hve only one eigenvalue $\lambda_0(-K) \le \mu_0$, and (1.1) and (1.2) with $\theta \neq 0$, $-\pi < \theta < \pi$ have only one eigenvalue $\lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K)$, and they satisfy

$$\nu_0 \le \lambda_0(K) < \lambda_0\left(e^{i\theta}K\right) < \lambda_0(-K) \le \mu_0. \tag{5.1}$$

Similarly, by Propositions 4.1, 4.3, and 4.10, the continuity of $D(\lambda)$, and the intermediate value theorem, $D(\lambda)$ reaches -2, $2\cos\theta$ ($\theta \neq 0$, $-\pi < \theta < \pi$), and 2 exactly one time between any two consecutive eigenvalues of the separated boundary value problem (1.1) with (2.14). Hence, (1.1) and (1.2) with $\theta = 0$, $\theta \neq 0$, $-\pi < \theta < \pi$, and $\theta = \pi$ have only one eigenvalue between any two consecutive eigenvalues of (1.1) with (2.14), respectively. In addition, by Propositions 4.10 and 4.12, if $D(\mu_n) = 2$ or -2 and $D'(\mu_n) = 0$, then μ_n is not only an eigenvalue of (1.1) with (2.14) but also a multiple eigenvalue of (1.1) and (1.2) with $\theta = 0$ and $\theta = \pi$.

If $N_d = \infty$, then it follows from the above discussion that (1.1) and (1.2) with $\theta \neq 0$, $-\pi < \theta < \pi$ have infinitely many eigenvalues, and they are real and satisfy (3.1).

If $N_d < \infty$, then all points of $[0, 1] \cap \mathbb{T}$ are isolated. In this case (1.1) and $D(\lambda)$ can be rewritten as (4.65) and (4.68). By the same method in the proof of Proposition 4.10, that if $N_d + 1$ is even, then there exists a constant $\xi_0 > \mu_{N_d-1}$ such that $D(\xi_0) \le -2$, which together with (4.62), implies that (1.1) and (1.2) with $\theta = 0$, $\theta \neq 0$, $-\pi < \theta < \pi$, and $\theta = \pi$ have only one eigenvalue $\lambda_{N_d}(K)$, $\lambda_{N_d}(e^{i\theta}K)$, and $\lambda_{N_d}(-K)$, satisfying

$$\mu_{N_d-1} \le \lambda_{N_d}(K) < \lambda_{N_d}\left(e^{i\theta}K\right) < \lambda_{N_d}(-K) \le \xi_0 \tag{5.2}$$

(see Figure 2). Similarly, in the other case that $N_d + 1$ is even, there exists a constant $\eta_0 > \mu_{N_d-1}$ such that $D(\eta_0) \ge 2$, which, together with (4.62) implies that (1.1) and (1.2) with $\theta = 0$, $\theta \neq 0$, $-\pi < \theta < \pi$, and $\theta = \pi$ have only one eigenvalue $\lambda_{N_d}(K)$, $\lambda_{N_d}(e^{i\theta}K)$, and $\lambda_{N_d}(-K)$, satisfying

$$\mu_{N_d-1} \le \lambda_{N_d}(-K) < \lambda_{N_d}\left(e^{i\theta}K\right) < \lambda_{N_d}(K) \le \eta_0 \tag{5.3}$$

(see Figure 3). Therefore, we get that (1.1) and (1.2) with $\theta \neq 0$, $-\pi < \theta < \pi$ have $N_d + 1$ eigenvalues and they are real and satisfy

$$\begin{aligned} \nu_0 &\le \lambda_0(K) < \lambda_0\left(e^{i\theta}K\right) < \lambda_0(-K) \le \mu_0 \le \lambda_1(-K) < \lambda_1\left(e^{i\theta}K\right) < \lambda_1(K) \le \mu_1 \le \lambda_2(K) \\ &< \cdots < \lambda_{N_d-1}(K) \le \mu_{N_d-1} \le \lambda_{N_d}(K) < \lambda_{N_d}\left(e^{i\theta}K\right) < \lambda_{N_d}(-K) \le \xi_0 \end{aligned} \tag{5.4}$$

if $N_d + 1$ is odd; and

$$\begin{aligned} \nu_0 &\le \lambda_0(K) < \lambda_0\left(e^{i\theta}K\right) < \lambda_0(-K) \le \mu_0 \le \lambda_1(-K) < \lambda_1\left(e^{i\theta}K\right) < \lambda_1(K) \le \mu_1 \le \lambda_2(K) \\ &< \cdots < \lambda_{N_d-1}(-K) \le \mu_{N_d-1} \le \lambda_{N_d}(-K) < \lambda_{N_d}\left(e^{i\theta}K\right) < \lambda_{N_d}(K) \le \eta_0 \end{aligned} \tag{5.5}$$

if $N_d + 1$ is even. This completes the proof. □

Remark 5.1. In the continuous case: $\mu(t) = 0$, $N_d = \infty$, by Theorem 3.1, the coupled boundary value problems (1.1) and (1.2) have infinitely many eigenvalues: $\{\lambda_n(e^{i\theta}K)\}_{n=0}^{\infty}$ for $\theta \neq 0, -\pi < \theta < \pi$; $\{\lambda_n(K)\}_{n=0}^{\infty}$ for $\theta = 0$; $\{\lambda_n(-K)\}_{n=0}^{\infty}$ for $\theta = \pi$, and they satisfy inequality (3.1). This result is the same as that obtained by Eastham et al. for second-order differential equations [2, Theorem 3.2].

Example 5.2. Consider the following three specific cases:

$$\begin{gathered} [\rho(0), 1] \cap \mathbb{T} = \left[0, \frac{1}{2}\right] \cup \left[\frac{2}{3}, 1\right]; \\ [\rho(0), 1] \cap \mathbb{T} = \left[0, \frac{1}{2}\right] \cup \left\{\frac{1}{2(N-1)}, \frac{1}{(N-1)}, \frac{3}{2(N-1)}, \ldots, 1\right\}, \quad N > 2; \\ [\rho(0), 1] \cap \mathbb{T} = \left\{q^k \mid k \ge 0, k \in \mathbf{Z}\right\} \cup \{0\}, \quad \text{where } 0 < q < 1. \end{gathered} \tag{5.6}$$

It is evident that $|[\rho(0), 1] \cap \mathbb{T}| = \infty$ and then $N_d = \infty$ in these three cases. By Theorem 3.1, the coupled boundary value problems (1.1) and (1.2) have infinitely many real eigenvalues and they satisfy the inequality (3.1). Obviously, the above three cases are not continuous and not discrete. So the existing results are not available now.

By Remark 5.1 and Example 5.2, our result in Theorem 3.1 not only extends the results in the discrete cases but also contains more complicated time scales.

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