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Research Article

Boundary Controllability of Nonlinear Fractional Integrodifferential Systems

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Sufficient conditions for boundary controllability of nonlinear fractional integrodifferential systems in Banach space are established. The results are obtained by using fixed point theorems. We also give an application for integropartial differential equations of fractional order.

1. Introduction

Let E and U be a pair of real Banach spaces with norms $\|\cdot\|$ and $|\cdot|$, respectively. Let σ be a linear closed and densely defined operator with $D(\sigma) \subseteq E$ and let $\tau \subseteq X$ be a linear operator with $D(\sigma)$ and $R(\tau) \subseteq X$, a Banach space. In this paper we study the boundary controllability of nonlinear fractional integrodifferential systems in the form

$$\begin{gathered} \frac{d^\alpha x(t)}{dt^\alpha} = \sigma x(t) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \quad t \in J = [0, b], \\ \tau x(t) = B_1 u(t), \\ x(0) = x_0, \end{gathered} \tag{1.1}$$

where $0 < \alpha \leq 1$ and $B_1 : U \to X$ is a linear continuous operator, and the control function u is given in $L^1(J, U)$, a Banach space of admissible control functions. The nonlinear operators $f : J \times E \to E$ and $g : \Delta \times E \to E$ are given and $\Delta : (t, s);\ 0 \leq s \leq t \leq b$.

Let $A : E \to E$ be the linear operator defined by

$$D(A) = \{x \in D(\sigma); \tau x = 0\}, \qquad Ax = \{\sigma x,\ \text{for } x \in D(A)\}. \tag{1.2}$$

The controllability of integrodifferential systems has been studied by many authors (see [1–6]). This work may be regarded as a direct attempt to generalize the work in [7, 8].

2. Main Result

Definition 2.1. System (1.1) is said to be controllable on the interval J if for every $x_0, x_1 \in E$ there exists a control $u \in L^2(J, U)$ such that $x(\cdot)$ of (1.1) satisfies $x(b) = x_1$.

To establish the result, we need the following hypotheses.

(H1) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to the graph norm of $D(\sigma)$.

(H2) The operator A is the infinitesimal generator of a compact semigroup $T(t)$ and there exists a constant $M_1 > 0$ such that $\|T(t)\| \leq M_1$.

(H3) There exists a linear continuous operator $B : U \to E$ such that $\sigma B \in L(U, E)$, $\tau(Bu) = B_1 u$, for all $u \in U$. Also $Bu(t)$ is continuously differentiable and $\|Bu\| \leq C\|B_1 u\|$ for all $u \in U$, where C is a constant.

(H4) For all $t \in (0, b]$ and $u \in U$, $T(t)Bu \in D(A)$. Moreover, there exists a positive constant $K_1 > 0$ such that $\|AT(t)\| \leq K_1$.

(H5) The nonlinear operators $f(t, x(t))$ and $g(t, s, x(s))$, for $t, s \in J$, satisfy

$$\|f(t, x(t))\| \leq L_1, \qquad \|g(t, s, x(s))\| \leq L_2, \tag{2.1}$$

where $L_1 \geq 0$ and $L_2 \geq 0$.

(H6) The linear operator W from $L^2(J, U)$ into E defined by

$$Wu = \alpha \int_0^b \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) \left[T\left((b-s)^\alpha \theta \right) \sigma - AT\left((b-s)^\alpha \theta \right) \right] Bu(s) d\theta \, ds, \tag{2.2}$$

where $\xi_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$ (see [9, 10]) and induces an invertible operator $\widetilde{W}^{-1}$ defined on $L^2(J, U)/\ker W$, and there exists a positive constant $M_2 > 0$ and $M_3 > 0$ such that $\|B\| \leq M_2$ and $\|\widetilde{W}^{-1}\| \leq M_3$. Let $x(t)$ be the solution of (1.1). Then we define a function $z(t) = x(t) - Bu(t)$ and from our assumption we see that $z(t) \in D(A)$. Hence (1.1) can be written in terms of A and B as

$$\begin{gathered} \frac{d^\alpha x(t)}{dt^\alpha} = Az(t) + \sigma Bu(t) + f(t, x(t)) + \int_0^t g(t, s, x(s)) ds, \quad t \in J, \\ x(t) = z(t) + Bu(t), \qquad x(0) = x_0. \end{gathered} \tag{2.3}$$

If u is continuously differentiable on $[0, b]$, then z can be defined as a mild solution to be the Cauchy problem

$$\begin{gathered} \frac{d^\alpha z(t)}{dt^\alpha} = Az(t) + \sigma Bu(t) - B\frac{d^\alpha u(t)}{dt^\alpha} + f(t, x(t)) + \int_0^t g(t, s, x(s)) ds, \quad t \in J, \\ z(0) = x_0 - Bu(0), \end{gathered} \tag{2.4}$$

and the solution of (1.1) is given by

$$\begin{aligned}
x(t) &= \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta)[x_0 - Bu(0)]d\theta + Bu(t) \\
&+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\left((t-s)^\alpha \theta\right) f(s, x(s)) d\theta\, ds \\
&+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\left((t-s)^\alpha \theta\right) \left[\sigma Bu(s) - B\frac{d^\alpha u(s)}{ds^\alpha}\right] d\theta\, ds \\
&+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\left((t-s)^\alpha \theta\right) \left[\int_0^s g(s, \tau, x(\tau)) d\tau\right] d\theta\, ds
\end{aligned} \tag{2.5}$$

(see [11–13]).

Since the differentiability of the control u represents an unrealistic and severe requirement, it is necessary of the solution for the general inputs $u \in L^1(J, U)$. Integrating (2.5) by parts, we get

$$\begin{aligned}
x(t) = &\int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta) x_0 d\theta + \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) \left[T\left((t-s)^\alpha \theta\right)\sigma - AT\left((t-s)^\alpha \theta\right)\right] Bu(s) d\theta\, ds \\
&+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\left((t-s)^\alpha \theta\right) f(s, x(s)) d\theta\, ds \\
&+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\left((t-s)^\alpha \theta\right) \left[\int_0^s g(s, \tau, x(\tau)) d\tau\right] d\theta\, ds.
\end{aligned} \tag{2.6}$$

Thus (2.6) is well defined and it is called a mild solution of system (1.1).

Theorem 2.2. *If hypotheses (H1)–(H6) are satisfied, then the boundary control fractional integrodifferential system* (1.1) *is controllable on* J.

Proof. Using assumption (H6), for an arbitrary function $x(\cdot)$ define the control

$$\begin{aligned}
u(t) = \widetilde{W}^{-1} \Bigg\{ &x_1 - \int_0^\infty \xi_\alpha(\theta) T(b^\alpha \theta) x_0 d\theta - \alpha \int_0^b \int_0^\infty \theta (b-s)^{\alpha-1} \xi_\alpha(\theta) T\left((b-s)^\alpha \theta\right) f(s, x(s)) d\theta\, ds \\
&- \alpha \int_0^b \int_0^\infty \theta (b-s)^{\alpha-1} \xi_\alpha(\theta) T\left((b-s)^\alpha \theta\right) \left[\int_0^s g(s, \tau, x(\tau)) d\tau\right] d\theta\, ds \Bigg\}(t).
\end{aligned} \tag{2.7}$$

We shall now show that, when using this control, the operator defined by

$$\begin{aligned}(\Phi x)(t) &= \int_0^\infty \xi_\alpha(\theta) T(t^\alpha\theta) x_0 d\theta \\ &\quad + \alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)\left[T\left((t-s)^\alpha\theta\right)\sigma - AT\left((t-s)^\alpha\theta\right)\right]Bu(s)d\theta\, ds \\ &\quad + \alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)f(s,x(s))d\theta\, ds \\ &\quad + \alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]d\theta\, ds\end{aligned} \tag{2.8}$$

has a fixed point. This fixed point is then a solution of (1.1). Clearly, $(\Phi x)(b) = x_1$, which means that the control u steers the nonlinear fractional integrodifferential system from the initial state x_0 to x_1 in time T, provided we can obtain a fixed point of the nonlinear operator Φ.

Let $Y = C(J,X)$ and $Y_0 = \{x \in Y : \|x(t)\| \le r,\ \text{for } t \in J\}$, where the positive constant r is given by

$$\begin{aligned}r &= M_1\|x_0\| + b^\alpha[M_1\|\sigma\| + K_1]M_2M_3\left[\|x_1\| + M_1\|x_0\| + M_1L_1b^\alpha + M_1L_2b^{\alpha+1}\right] \\ &\quad + M_1L_1b^\alpha + M_1L_2b^{\alpha+1}.\end{aligned} \tag{2.9}$$

Then Y_0 is clearly a bounded, closed, and convex subset of Y. We define a mapping $\Phi : Y \to Y_0$ by

$$\begin{aligned}(\Phi x)(t) &= \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)x_0 d\theta + \alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)\left[T\left((t-s)^\alpha\theta\right)\sigma - AT\left((t-s)^\alpha\theta\right)\right]B\widetilde{W}^{-1} \\ &\qquad \times\left\{x_1 - \int_0^\infty \xi_\alpha(\theta)T(b^\alpha\theta)x_0 d\theta - \alpha\int_0^b\int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta)T\left((b-s)^\alpha\theta\right)f(s,x(s))d\theta\, ds\right. \\ &\qquad\quad \left. - \alpha\int_0^b\int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta)T\left((b-s)^\alpha\theta\right)\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]d\theta\, ds\right\}(s)d\theta\, ds \\ &\quad + \alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)f(s,x(s))d\theta\, ds \\ &\quad + \alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]d\theta\, ds.\end{aligned} \tag{2.10}$$

Consider

$$
\begin{aligned}
&\|(\Phi x)(t)\| \\
&\quad\le \left\|\int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)x_0 d\theta\right\| + \alpha\int_0^t \left\|\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)\left[T\left((t-s)^\alpha\theta\right)\sigma - AT\left((t-s)^\alpha\theta\right)\right]d\theta\right\|\|B\| \\
&\qquad\times \left\|\widetilde{W}^{-1}\right\|\left\{\|x_1\| - \left\|\int_0^\infty \xi_\alpha(\theta)T(b^\alpha\theta)x_0 d\theta\right\| - \alpha\int_0^b \left\|\int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta)T\left((b-s)^\alpha\theta\right)d\theta\right\|\right. \\
&\qquad\qquad\times \|f(s,x(s))\|ds - \alpha\int_0^b \left\|\int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta)T\left((b-s)^\alpha\theta\right)d\theta\right\| \\
&\qquad\qquad\left.\times \left\|\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]\right\|ds\right\}(s)ds \\
&\qquad + \alpha\int_0^t \left\|\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)d\theta\right\|\|f(s,x(s))\|ds \\
&\qquad + \alpha\int_0^t \left\|\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)d\theta\right\|\left\|\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]\right\|ds \\
&\quad\le M_1\|x_0\| + b^\alpha[M_1\|\sigma\| + K_1]M_2M_3\left[\|x_1\| + M_1\|x_0\| + M_1L_1b^\alpha + M_1L_2b^{\alpha+1}\right] \\
&\qquad + M_1L_1b^\alpha + M_1L_2b^{\alpha+1} \le r.
\end{aligned}
\tag{2.11}
$$

Since f and g are continuous and $\|(\Phi x)(t)\| \le r$, it follows that Φ is also continuous and maps Y_0 into itself. Moreover, Φ maps Y_0 into precompact subset of Y_0. To prove this, we first show that, for every fixed $t \in J$, the set $Y_0(t) = \{(\Phi x)(t) : x \in Y_0\}$ is precompact in X. This is clear for $t = 0$, since $Y_0(0) = \{x_0\}$. Let $t > 0$ be fixed and for $0 < \epsilon < t$ define

$$
\begin{aligned}
(\Phi_\epsilon x)(t) &= \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)x_0 d\theta + \alpha\int_0^{t-\epsilon}\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)\left[T\left((t-s)^\alpha\theta\right)\sigma - AT\left((t-s)^\alpha\theta\right)\right]B\widetilde{W}^{-1} \\
&\quad\times \left\{x_1 - \int_0^\infty \xi_\alpha(\theta)T(b^\alpha\theta)x_0 d\theta - \alpha\int_0^b\int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta)T\left((b-s)^\alpha\theta\right)f(s,x(s))d\theta\, ds\right. \\
&\qquad\left. - \alpha\int_0^b\int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta)T\left((b-s)^\alpha\theta\right)\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]d\theta\, ds\right\}(s)d\theta\, ds \\
&\quad + \alpha\int_0^{t-\epsilon}\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)f(s,x(s))d\theta\, ds \\
&\quad + \alpha\int_0^{t-\epsilon}\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]d\theta\, ds.
\end{aligned}
\tag{2.12}
$$

Since $T(t)$ is compact for every $t > 0$, the set $Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in Y_0\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Furthermore, for $x \in Y_0$, we have

$$\begin{aligned}
&\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| \\
&\quad\leq \left\| \alpha \int_{t-\epsilon}^{t} \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) \left[T\left((t-s)^\alpha \theta\right)\sigma - AT\left((t-s)^\alpha\theta\right)\right] B\widetilde{W}^{-1} \right.\\
&\qquad\quad \times \left\{ x_1 - \int_0^\infty \xi_\alpha(\theta) T(b^\alpha\theta) x_0 d\theta - \alpha \int_0^b \int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta) T\left((b-s)^\alpha\theta\right) f(s,x(s)) d\theta\, ds \right.\\
&\qquad\qquad \left.\left. - \alpha \int_0^b \int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta) T\left((b-s)^\alpha\theta\right) \left[\int_0^s g(s,\tau,x(\tau))d\tau\right] d\theta\, ds \right\}(s) d\theta\, ds \right\| \\
&\qquad + \left\| \alpha \int_{t-\epsilon}^{t} \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\left((t-s)^\alpha\theta\right) f(s,x(s)) d\theta\, ds \right\| \\
&\qquad + \left\| \alpha \int_{t-\epsilon}^{t} \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\left((t-s)^\alpha\theta\right) \left[\int_0^s g(s,\tau,x(\tau))d\tau\right] d\theta\, ds \right\| \\
&\quad\leq \epsilon^\alpha [M_1\|\sigma\| + K_1] M_2 M_3 \left[\|x_1\| + M_1\|x_0\| + M_1 L_1 b^\alpha + M_1 L_2 b^{\alpha+1}\right] + M_1 L_1 \epsilon^\alpha + M_1 L_2 \epsilon^\alpha b,
\end{aligned} \tag{2.13}$$

which implies that $Y_0(t)$ is totally bounded, that is, precompact in X. We want to show that $\Phi(Y_0) = \{\Phi x : x \in Y_0\}$ is an equicontinuous family of functions. For that, let $t_2 > t_1 > 0$. Then we have

$$\begin{aligned}
&\|(\Phi x)(t_1) - (\Phi x)(t_2)\| \\
&\quad\leq \left\| \alpha \int_0^{t_1} \int_0^\infty \theta \xi_\alpha(\theta) \left[(t_1-s)^{\alpha-1} \left[T\left((t_1-s)^\alpha\theta\right)\sigma - AT\left((t_1-s)^\alpha\theta\right)\right] \right.\right.\\
&\qquad\qquad\qquad\qquad \left. - (t_2-s)^{\alpha-1} \left[T\left((t_2-s)^\alpha\theta\right)\sigma - AT\left((t_2-s)^\alpha\theta\right)\right] \right.\\
&\qquad \times B\widetilde{W}^{-1} \left\{ x_1 - \int_0^\infty \xi_\alpha(\theta) T(b^\alpha\theta) x_0 d\theta - \alpha \int_0^b \int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta) T\left((b-s)^\alpha\theta\right) f(s,x(s)) d\theta\, ds \right.\\
&\qquad\qquad \left.\left. - \alpha \int_0^b \int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta) T\left((b-s)^\alpha\theta\right) \left[\int_0^s g(s,\tau,x(\tau))d\tau\right] d\theta\, ds \right\}(s) d\theta\, ds \right\| \\
&\qquad + \left\| \alpha \int_{t_1}^{t_2} \int_0^\infty \theta \xi_\alpha(\theta) (t_2-s)^{\alpha-1} \left[T\left((t_2-s)^\alpha\theta\right)\sigma - AT\left((t_2-s)^\alpha\theta\right)\right] B\widetilde{W}^{-1} \right.\\
&\qquad \times \left\{ x_1 - \int_0^\infty \xi_\alpha(\theta) T(b^\alpha\theta) x_0 d\theta - \alpha \int_0^b \int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta) T\left((b-s)^\alpha\theta\right) f(s,x(s)) d\theta\, ds \right.\\
&\qquad\qquad \left.\left. - \alpha \int_0^b \int_0^\infty \theta(b-s)^{\alpha-1}\xi_\alpha(\theta) T\left((b-s)^\alpha\theta\right) \left[\int_0^s g(s,\tau,x(\tau))d\tau\right]\ d\theta\, ds \right\}(s) d\theta\, ds \right\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\alpha\int_0^{t_1}\int_0^{\infty}\theta\xi_\alpha(\theta)\left[(t_1-s)^{\alpha-1}T\left((t_1-s)^\alpha\theta\right)-(t_2-s)^{\alpha-1}T\left((t_2-s)^\alpha\theta\right)\right]f(s,x(s))d\theta\, ds\right\|\\
&+\left\|\alpha\int_0^{t}\int_0^{\infty}\theta\xi_\alpha(\theta)\left[(t_1-s)^{\alpha-1}\quad T\left((t_1-s)^\alpha\theta\right)-(t_2-s)^{\alpha-1}T\left((t_2-s)^\alpha\theta\right)\right]\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]\right.\\
&\qquad\left.\times\, d\theta\, ds\right\|\\
&+\left\|\alpha\int_{t_1}^{t_2}\int_0^{\infty}\theta\xi_\alpha(\theta)(t_2-s)^{\alpha-1}T\left((t_2-s)^\alpha\theta\right)f(s,x(s))d\theta\, ds\right\|\\
&+\left\|\alpha\int_{t_1}^{t_2}\int_0^{\infty}\theta\xi_\alpha(\theta)(t_2-s)^{\alpha-1}\quad T\left((t_2-s)^\alpha\theta\right)\left[\int_0^s g(s,\tau,x(\tau))d\tau\right]d\theta\, ds\right\|.
\end{aligned}
\tag{2.14}
$$

By using conditions (H2)–(H6), we get

$$
\begin{aligned}
&\|(\Phi x)(t_1)-(\Phi x)(t_2)\|\\
&\quad\le\alpha\int_0^{t_1}\left\|(t_1-s)^{\alpha-1}\left[T\left((t_1-s)^\alpha\theta\right)\sigma-AT\left((t_1-s)^\alpha\theta\right)\right]\right.\\
&\qquad\qquad\left.-\ (t_2-s)^{\alpha-1}\left[T\left((t_2-s)^\alpha\theta\right)\sigma-AT\left((t_2-s)^\alpha\theta\right)\right]\right\|\\
&\qquad\times\left[M_2M_3\left\{\|x_1\|+M_1\|x_0\|+M_1\left(L_1b^\alpha+L_2b^{\alpha+1}\right)\right\}\right]ds\\
&\qquad+(t_2-t_1)^\alpha\left[M_1M_2M_3\|\sigma\|+K_1M_2M_3\right]\left\{\|x_1\|+M_1\|x_0\|+M_1\left(L_1b^\alpha+L_2b^{\alpha+1}\right)\right\}\\
&\qquad+\alpha\int_0^{t_1}\left\|\left[(t_1-s)^{\alpha-1}T\left((t_1-s)^\alpha\theta\right)-(t_2-s)^{\alpha-1}T\left((t_2-s)^\alpha\theta\right)\right]\right\|\\
&\qquad\times\left[L_1+L_2b^\alpha\right]ds+(t_2-t_1)^\alpha M_1\left[L_1+L_2b^\alpha\right].
\end{aligned}
\tag{2.15}
$$

The compactness of $T(t), t > 0$, implies that $T(t)$ is continuous in the uniform operator topology for $t > 0$. Thus, the right hand side of (2.15) tends to zero as $t_2 \to t_1$. So, $\Phi(Y_0)$ is an equicontinuous family of functions. Also, $\Phi(Y_0)$ is bounded in Y, and so by the Arzela-Ascoli theorem, $\Phi(Y_0)$ is precompact. Hence, from the Schauder fixed point in Y_0, any fixed point of Φ is a mild solution of (1.1) on J satisfying

$$(\Phi x)(t) = x(t) \in \mathrm{X}. \tag{2.16}$$

Thus, system (1.1) is controllable on J. □

3. Application

Let $\Omega \subset R^n$ be bounded with smooth boundary Γ.

Consider the boundary control fractional integropartial differential system

$$\frac{\partial^\alpha y(t,x)}{\partial t^\alpha} - \Delta y(t,x) = F\big(t, y(t,x)\big) + \int_0^t G\big(t,s,y(s,x)\big)ds, \quad \text{in } Y = (0,b)\times\Omega,$$
$$y(t,0) = u(t,0) \quad \text{on } \Sigma = (0,b)\times\Gamma, \quad t \in [0,b], \tag{3.1}$$
$$y(0,x) = y_0(x), \quad \text{for } x \in \Omega.$$

The above problem can be formulated as a boundary control problem of the form of (1.1) by suitably taking the spaces E, X, U and the operators B_1, σ, and τ as follows.

Let $E = L^2(\Omega)$, $X = H^{-1/2}(\Gamma)$, $U = L^2(\Gamma)$, $B_1 = I$, the identity operator and $D(\sigma) = \{y \in L^2(\Omega) : \Delta y \in L^2(\Omega)\}$, $\sigma = \Delta$. The operator τ is the trace operator such that $\tau y = y|_\Gamma$ is well defined and belongs to $H^{-1/2}(\Gamma)$ for each $y \in D(\sigma)$ and the operator A is given by $A = \Delta$, $D(A) = H_0^1(\Omega) \cup H^2(\Omega)$ where $H^k(\Omega)$, $H^\beta(\Omega)$, and $H_0^1(\Omega)$ are usual Sobolev spaces on Ω, Γ. We define the linear operator $B : L^2(\Gamma) \to L^2(\Omega)$ by $Bu = w_u$ where w_u is the unique solution to the Dirichlet boundary value problem

$$\Delta w_u = 0 \quad \text{in } \Omega,$$
$$w_u = u \quad \text{in } \Gamma. \tag{3.2}$$

We also introduce the nonlinear operator defined by

$$f(t,x(t)) = F\big(t,y(t,x)\big), \qquad g(t,s,x(s)) = G\big(t,s,y(s,x)\big). \tag{3.3}$$

Choose b and other constants such that conditions (H1)–(H6) are satisfied. Consequently Theorem 2.2 can be applied for (3.1), so (3.1) is controllable on $[0,b]$.

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