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Research Article

Oscillation of Second-Order Sublinear Dynamic Equations with Damping on Isolated Time Scales

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This paper concerns the oscillation of solutions to the second sublinear dynamic equation with damping $x^{\Delta\Delta}(t) + q(t)x^{\Delta^\sigma}(t) + p(t)x^\alpha(\sigma(t)) = 0$, on an isolated time scale $\mathbb{T}$ which is unbounded above. In $0 < \alpha < 1$, α is the quotient of odd positive integers. As an application, we get the difference equation $\Delta^2 x(n) + n^{-\gamma}\Delta x(n+1) + [(1/n(\ln n)^\beta) + b((-1)^n/(\ln n)^\beta)]x^\alpha(n+1) = 0$, where $\gamma > 0$, $\beta > 0$, and b is any real number, is oscillatory.

1. Introduction

During the past years, there has been an increasing interest in studying the oscillation of solution of second-order damped dynamic equations on time scale which attempts to harmonize the oscillation theory for continuousness and discreteness, to include them in one comprehensive theory, and to eliminate obscurity from both. We refer the readers to the papers [1–4] and the references cited therein.

In [5], Bohner et al. consider the second-order nonlinear dynamic equation with damping

$$x^{\Delta\Delta}(t) + q(t)x^{\Delta^\sigma}(t) + p(t)\left(f \circ x^\sigma(t)\right) = 0, \tag{1.1}$$

where p and q are real-valued, right-dense continuous functions on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup \mathbb{T} = \infty$. $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies $f'(x) > 0$ and $xf(x) > 0$ for

$x \neq 0$. When $f(x) = x^\alpha$, where $0 < \alpha < 1$, $\alpha > 0$ is the quotient of odd positive integers, (1.1) is the second-order sublinear dynamic equation with damping

$$x^{\Delta\Delta}(t) + q(t)x^{\Delta^\sigma}(t) + p(t)x^\alpha(\sigma(t)) = 0. \tag{1.2}$$

When $q(t) = 0$, (1.2) is the second-order sublinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^\alpha(\sigma(t)) = 0. \tag{1.3}$$

When $\mathbb{T} = \mathbb{N}_0$, (1.3) is the second-order sublinear difference equation

$$\Delta^2 x(n) + p(t)x^\alpha(n+1) = 0. \tag{1.4}$$

In [6], under the assumption of $\mathbb{T}$ being an isolated time scale, we prove that, when $p(t)$ is allowed to take on negative values, $\int_{t_1}^{\infty} t^\alpha p(t)\Delta t = \infty$ is sufficient for the oscillation of the dynamic equation (1.3). As an application, we get that, when $p(n)$ is allowed to take on negative values, $\sum_{n=1}^{\infty} n^\alpha p(n) = \infty$ is sufficient for the oscillation of the dynamic equation (1.4), which improves a result of Hooker and Patula [7, Theorem 4.1] and Mingarelli [8].

In this paper, we extend the result of [6] to dynamic equation (1.1). As an application, we get that the difference equation with damping

$$\Delta^2 x(n) + n^{-\gamma}\Delta x(n+1) + \left[\frac{1}{n(\ln n)^\beta} + b\frac{(-1)^n}{(\ln n)^\beta}\right]x^\alpha(n+1) = 0, \tag{1.5}$$

where $0 < \alpha < 1$, $\gamma > 0$, $\beta > 0$, and b is any real number, is oscillatory.

For completeness (see [9, 10] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \tag{1.6}$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \tag{1.7}$$

where $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say t is right scattered, while, if $\rho(t) < t$, we say t is left scattered. If $\sigma(t) = t$, we say t is right dense, while, if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, we say t is left-dense. Given a time scale interval $[c,d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c,d]^{\kappa}{}_{\mathbb{T}}$ denotes the interval $[c,d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c,d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^{\Delta}(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s} \tag{1.8}$$

exists when $\sigma(t) = t$ (here, by $s \to t$, it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}. \tag{1.9}$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence, our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales (e.g., the time scale $q^{\mathbb{N}_0} := \{1, q, q^2, \ldots\}$ which is very important in quantum theory [11]).

2. Lemmas

We will need the following second mean value theorem (see [10, Theorem 5.45]).

Lemma 2.1. *Let h be a bounded function that is integrable on $[a, b]_{\mathbb{T}}$. Let m_H and M_H be the infimum and supremum, respectively, of the function $H(t) := \int_a^t h(s)\Delta s$ on $[a, b]_{\mathbb{T}}$. Suppose that g is nonincreasing with $g(t) \geq 0$ on $[a, b]_{\mathbb{T}}$. Then, there is some number Λ with $m_H \leq \Lambda \leq M_H$ such that*

$$\int_a^b h(t)g(t)\Delta t = g(a)\Lambda. \tag{2.1}$$

Lemmas 2.2 and 2.4 give two lower bounds of definite integrals on time scale, respectively.

Lemma 2.2. *Assume that $\mathbb{T} = \{t_i\}_{i=0}^{\infty}$, where $1 = t_0 < t_1 < \cdots < t_i \cdots$. If there exists a real number $K \geq 1$ such that $(t_{i+1} - t_i)/(t_i - t_{i-1}) = K \geq 1$, for all $i \geq 1$, then, for $u(t) > 0$, $t \geq T$, one has*

$$\int_T^t \frac{u^{\Delta^{\sigma}}(s)}{u^{\alpha}(\sigma(s))}\Delta s \geq -\frac{u^{1-\alpha}(\sigma(T))}{K(1-\alpha)}. \tag{2.2}$$

Remark 2.3. It is easy to know that, when $\mathbb{T} = \mathbb{N}$, $K = 1$ and, when $\mathbb{T} = q^{\mathbb{N}_0}$, $K = q$.

Proof. For $i \geq 1$, using Theorem 1.75 of [9], we have

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} \frac{u^{\Delta^\sigma}(s)}{u^\alpha(\sigma(s))}\Delta s &= \frac{(t_i - t_{i-1})u^{\Delta^\sigma}(t_{i-1})}{u^\alpha(\sigma(t_{i-1}))} \\
&= \frac{(t_i - t_{i-1})u^{\Delta}(t_i)}{u^\alpha(t_i)} \\
&= \frac{(t_i - t_{i-1})[u(t_{i+1}) - u(t_i)]}{(t_{i+1} - t_i)u^\alpha(t_i)} \\
&= \frac{u(t_{i+1}) - u(t_i)}{Ku^\alpha(t_i)}.
\end{aligned} \tag{2.3}$$

We consider the two cases $u(t_i) \leq u(t_{i+1})$ and $u(t_i) > u(t_{i+1})$. First, if $u(t_i) \leq u(t_{i+1})$, then we have that

$$\frac{u(t_{i+1}) - u(t_i)}{u^\alpha(t_i)} \geq \int_{u(t_i)}^{u(t_{i+1})} \frac{1}{s^\alpha} ds = \frac{1}{1-\alpha}\left[u^{1-\alpha}(t_{i+1}) - u^{1-\alpha}(t_i)\right]. \tag{2.4}$$

On the other hand, if $u(t_i) > u(t_{i+1})$, then

$$\frac{u(t_i) - u(t_{i+1})}{u^\alpha(t_i)} \leq \int_{u(t_{i+1})}^{u(t_i)} \frac{1}{s^\alpha} ds = \frac{1}{1-\alpha}\left[u^{1-\alpha}(t_i) - u^{1-\alpha}(t_{i+1})\right], \tag{2.5}$$

which implies that

$$\frac{u(t_{i+1}) - u(t_i)}{u^\alpha(t_i)} \geq \frac{1}{1-\alpha}\left[u^{1-\alpha}(t_{i+1}) - u^{1-\alpha}(t_i)\right]. \tag{2.6}$$

From (2.3)–(2.6) and the additivity of the integral, we have

$$\begin{aligned}
\int_T^t \frac{u^{\Delta^\sigma}(s)}{u^\alpha(\sigma(s))}\Delta s &\geq \frac{1}{K(1-\alpha)}\left[u^{1-\alpha}(\sigma(t)) - u^{1-\alpha}(\sigma(T))\right] \\
&\geq -\frac{u^{1-\alpha}(\sigma(T))}{K(1-\alpha)}.
\end{aligned} \tag{2.7}$$

□

Lemma 2.4. *Assume that* $\mathbb{T} = \{t_i\}_{i=0}^\infty$, *where* $1 = t_0 < t_1 < \cdots < t_i \cdots$, *with* $t_k \to \infty$. *Then, for* $u(t) > 0$, $t \geq T$, *one has*

$$\int_T^t \frac{(u^\alpha(s))^\Delta u^\sigma(s)}{u^\alpha(s)u^\alpha(\sigma(s))}\Delta s \geq -\frac{\alpha u^{1-\alpha}(T)}{1-\alpha}. \tag{2.8}$$

Proof. For $i \geq 1$, using Theorem 1.75 of [9], we have

$$L_i(u) := \int_{t_{i-1}}^{t_i} \frac{(u^\alpha(s))^\Delta u^\sigma(s)}{u^\alpha(s)u^\alpha(\sigma(s))}\Delta s = \left[\frac{1}{u^\alpha(t_{i-1})} - \frac{1}{u^\alpha(t_i)}\right]u(t_i). \tag{2.9}$$

Setting $v(t_i) := 1/u^\alpha(t_i)$, we have that

$$L_i(u) := \frac{v(t_{i-1}) - v(t_i)}{v^{1/\alpha}(t_i)}. \tag{2.10}$$

We consider the two possible cases $v(t_{i-1}) \geq v(t_i)$ and $v(t_{i-1}) < v(t_i)$. First, if $v(t_{i-1}) \geq v(t_i)$, we have that

$$\begin{aligned} \frac{v(t_{i-1}) - v(t_i)}{v^{1/\alpha}(t_i)} &\geq \int_{v(t_i)}^{v(t_{i-1})} \frac{1}{s^{1/\alpha}}ds \\ &= \frac{\alpha}{1-\alpha}\left[v^{1-1/\alpha}(t_i) - v^{1-1/\alpha}(t_{i-1})\right]. \end{aligned} \tag{2.11}$$

On the other hand, if $v(t_{i-1}) < v(t_i)$, then

$$\begin{aligned} \frac{v(t_i) - v(t_{i-1})}{v^{1/\alpha}(t_i)} &\leq \int_{v(t_{i-1})}^{v(t_i)} \frac{1}{s^{1/\alpha}}ds \\ &= \frac{\alpha}{1-\alpha}\left[v^{1-1/\alpha}(t_{i-1}) - v^{1-1/\alpha}(t_i)\right], \end{aligned} \tag{2.12}$$

which implies that

$$\frac{v(t_{i-1}) - v(t_i)}{v^{1/\alpha}(t_i)} \geq \frac{\alpha}{1-\alpha}\left[v^{1-1/\alpha}(t_i) - v^{1-1/\alpha}(t_{i-1})\right]. \tag{2.13}$$

Hence, from (2.10)–(2.13), we have that

$$L_i(u) \geq \frac{\alpha}{1-\alpha}\left[v^{1-1/\alpha}(t_i) - v^{1-1/\alpha}(t_{i-1})\right]. \tag{2.14}$$

From (2.9), (2.10), (2.14), and the additivity of the integral, we have

$$\begin{aligned} \int_T^t \frac{(u^\alpha(s))^\Delta u^\sigma(s)}{u^\alpha(s)u^\alpha(\sigma(s))}\Delta s &\geq \frac{\alpha}{1-\alpha}\left[v^{1-1/\alpha}(t) - v^{1-1/\alpha}(T)\right] \\ &= \frac{\alpha}{1-\alpha}\left[u^{1-\alpha}(t) - u^{1-\alpha}(T)\right] \geq -\frac{\alpha u^{1-\alpha}(T)}{1-\alpha}. \end{aligned} \tag{2.15}$$

□

3. Main Theorem

Theorem 3.1. *Assume that* $\mathbb{T} = \{t_0, t_1, \ldots, t_k \ldots\}$, *where* $1 = t_0 < t_1 < \cdots < t_i \cdots$. *Suppose that*

(i) *there exists a real number* $K \geq 1$ *such that* $(t_{i+1} - t_i)/(t_i - t_{i-1}) = K \geq 1$, *for all* $i \geq 1$;

(ii) *there exists a* C^1_{rd} *function* $a(t)$ *such that for* $t \in [T, \infty)$,

$$a(t) > 0, \quad a^{\Delta}(t) \geq 0, \quad a^{\Delta\Delta}(t) \leq 0, \quad q(t) \geq 0, \quad \left(q(t)a^{\sigma}(t)\right)^{\Delta} \leq 0,$$
$$\int_T^{\infty} a^{\alpha}(\sigma(s))p(s)\Delta s = \int_T^{\infty} \frac{1}{a(s)}\Delta s = \infty. \tag{3.1}$$

Then, (1.1) *is oscillatory.*

Proof. For the sake of contradiction, assume that (1.1) is nonoscillatory. Then, without loss of generality, there is a solution $x(t)$ of (1.1) and a $T \in \mathbb{T}$ with $x(t) > 0$, for all $t \in [T, \infty)_{\mathbb{T}}$. Making the substitution $x(t) = a(t)u(t)$ in (1.1) and noticing that

$$x^{\Delta}(t) = a^{\Delta}(t)u^{\sigma}(t) + a(t)u^{\Delta}(t),$$
$$x^{\Delta^{\sigma}}(t) = a^{\Delta^{\sigma}}(t)u\left(\sigma^2(t)\right) + a^{\sigma}(t)u^{\Delta^{\sigma}}(t), \tag{3.2}$$
$$x^{\Delta\Delta}(t) = a^{\Delta\Delta}(t)u^{\sigma}(t) + a^{\Delta^{\sigma}}(t)(u^{\sigma}(t))^{\Delta} + a^{\Delta}(t)u^{\Delta}(t) + a^{\sigma}(t)u^{\Delta\Delta}(t),$$

we get that

$$a^{\Delta\Delta}(t)u^{\sigma}(t) + a^{\Delta^{\sigma}}(t)(u^{\sigma}(t))^{\Delta} + a^{\Delta}(t)u^{\Delta}(t) + a^{\sigma}(t)u^{\Delta\Delta}(t)$$
$$+ q(t)a^{\Delta^{\sigma}}(t)u\left(\sigma^2(t)\right) + q(t)a^{\sigma}(t)u^{\Delta^{\sigma}}(t) + a^{\alpha}(\sigma(t))p(t)u^{\alpha}(\sigma(t)) = 0. \tag{3.3}$$

Multiplying both sides of (3.3) by $1/u^{\alpha}(\sigma(t))$, integrating from T to t, and using an integration by parts formula, we get

$$\int_T^t \frac{a^{\Delta\Delta}(s)u^{\sigma}(s)}{u^{\alpha}(\sigma(s))}\Delta s + \int_T^t \frac{a^{\Delta^{\sigma}}(s)(s)(u^{\sigma}(s))^{\Delta}}{u^{\alpha}(\sigma(s))}\Delta s$$
$$+ \int_T^t \frac{a^{\Delta}(s)u^{\Delta}(s)}{u^{\alpha}(\sigma(s))}\Delta s + \left[\frac{a(s)u^{\Delta}(s)}{u^{\alpha}(s)}\right]_T^t - \int_T^t \left[\frac{a(s)}{u^{\alpha}(s)}\right]^{\Delta} u^{\Delta}(s)\Delta s$$
$$+ \int_T^t \frac{q(s)a^{\Delta^{\sigma}}(s)u(\sigma^2(s))}{u^{\alpha}(\sigma(s))}\Delta s + \int_T^t \frac{q(s)a^{\sigma}(s)u^{\Delta^{\sigma}}(s)}{u^{\alpha}(\sigma(s))}\Delta s \tag{3.4}$$
$$+ \int_T^t a^{\alpha}(\sigma(s))p(s)\Delta s = 0.$$

Next, using the quotient rule and then Pötzsche's chain rule [9, Theorem 1.90] gives

$$\begin{aligned}
\int_T^t \left[\frac{a(s)}{u^\alpha(s)}\right]^\Delta u^\Delta(s)\Delta s &= \int_T^t \frac{a^\Delta(s)u^\Delta(s)}{u^\alpha(\sigma(s))}\Delta s - \int_T^t \frac{a(s)[u^\alpha(s)]^\Delta u^\Delta(s)}{u^\alpha(s)u^\alpha(\sigma(s))}\Delta s \\
&= \int_T^t \frac{a^\Delta(s)u^\Delta(s)}{u^\alpha(\sigma(s))}\Delta s - \alpha\int_T^t \frac{a(s)\left[u^\Delta(s)\right]^2}{u^\alpha(s)u^\alpha(\sigma(s))}\int_0^1 [u_h(s)]^{\alpha-1}dh\Delta s \\
&\leq \int_T^t \frac{a^\Delta(s)u^\Delta(s)}{u^\alpha(\sigma(s))}\Delta s,
\end{aligned} \tag{3.5}$$

where we used the fact that $u_h(s) := u(s) + h\mu(s)u^\Delta(s) = (1-h)u(s) + hu^\sigma(s) \geq 0$. Using this last inequality in (3.4), we get

$$\begin{aligned}
&\int_T^t \frac{a^{\Delta\Delta}(s)u^\sigma(s)}{u^\alpha(\sigma(s))}\Delta s + \int_T^t \frac{a^{\Delta^\sigma}(s)(s)(u^\sigma(s))^\Delta}{u^\alpha(\sigma(s))}\Delta s \\
&\quad + \frac{a(t)u^\Delta(t)}{u^\alpha(t)} - \frac{a(T)u^\Delta(T)}{u^\alpha(T)} + \int_T^t \frac{q(s)a^{\Delta^\sigma}(s)u\left(\sigma^2(s)\right)}{u^\alpha(\sigma(s))}\Delta s \\
&\quad + \int_T^t \frac{q(s)a^\sigma(s)u^{\Delta^\sigma}(s)}{u^\alpha(\sigma(s))}\Delta s + \int_T^t a^\alpha(\sigma(s))p(s)\Delta s \leq 0.
\end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned}
\int_T^t \frac{a^{\Delta^\sigma}(s)(u^\sigma(s))^\Delta}{u^\alpha(\sigma(s))}\Delta s &= \left[\frac{a^\Delta(s)u^\sigma(s)}{u^\alpha(s)}\right]_{s=T}^t - \int_T^t \left[\frac{a^\Delta(s)}{u^\alpha(s)}\right]^\Delta u^\sigma(s)\Delta s \\
&= \frac{a^\Delta(t)u^\sigma(t)}{u^\alpha(t)} - \frac{a^\Delta(T)u^\sigma(T)}{u^\alpha(T)} \\
&\quad - \int_T^t \frac{a^{\Delta\Delta}(s)u^\sigma(s)}{u^\alpha(\sigma(s))}\Delta s + \int_T^t \frac{(u^\alpha(s))^\Delta a^\Delta(s)u^\sigma(s)}{u^\alpha(s)u^\alpha(\sigma(s))}\Delta s.
\end{aligned} \tag{3.7}$$

Let us define $A := a(T)u^\Delta(T)/u^\alpha(T)$, $B := a^\Delta(T)u^\sigma(T)/u^\alpha(T)$. Then, we get from (3.6) and (3.7) that

$$\begin{aligned}
&\frac{a^\Delta(t)u^\sigma(t)}{u^\alpha(t)} - B + \int_T^t \frac{(u^\alpha(s))^\Delta a^\Delta(s)u^\sigma(s)}{u^\alpha(s)u^\alpha(\sigma(s))}\Delta s \\
&\quad + \frac{a(t)u^\Delta(t)}{u^\alpha(t)} - A + \int_T^t \frac{q(s)a^{\Delta^\sigma}(s)u\left(\sigma^2(s)\right)}{u^\alpha(\sigma(s))}\Delta s \\
&\quad + \int_T^t \frac{q(s)a^\sigma(s)u^{\Delta^\sigma}(s)}{u^\alpha(\sigma(s))}\Delta s + \int_T^t a^\alpha(\sigma(s))p(s)\Delta s \leq 0,
\end{aligned} \tag{3.8}$$

since $a^{\Delta}(t) \geq 0$, for $t > T$. So the first term of (3.8) is nonnegative. From (3.8), we get that

$$-B + \int_T^t \frac{(u^{\alpha}(s))^{\Delta} a^{\Delta}(s) u^{\sigma}(s)}{u^{\alpha}(s) u^{\alpha}(\sigma(s))} \Delta s + \frac{a(t) u^{\Delta}(t)}{u^{\alpha}(t)} - A$$
$$+ \int_T^t \frac{q(s) a^{\Delta^{\sigma}}(s) u(\sigma^2(s))}{u^{\alpha}(\sigma(s))} \Delta s + \int_T^t \frac{q(s) a^{\sigma}(s) u^{\Delta^{\sigma}}(s)}{u^{\alpha}(\sigma(s))} \Delta s + \int_T^t a^{\alpha}(\sigma(s)) p(s) \Delta s \leq 0. \tag{3.9}$$

From $a^{\Delta\Delta}(t) \leq 0$ and $(q(t)a^{\sigma}(t))^{\Delta} \leq 0$, using the second mean value theorem [10, Theorem 5.45] and Lemmas 2.2 and 2.4, we get that

$$\int_T^t \frac{(u^{\alpha}(s))^{\Delta} a^{\Delta}(s) u^{\sigma}(s)}{u^{\alpha}(s) u^{\alpha}(\sigma(s))} \Delta s \geq -\frac{\alpha a^{\Delta}(T) u^{1-\alpha}(T)}{1-\alpha},$$
$$\int_T^t \frac{q(s) a^{\sigma}(s) u^{\Delta^{\sigma}}(s)}{u^{\alpha}(\sigma(s))} \Delta s \geq -\frac{q(T) a^{\sigma}(T) u^{1-\alpha}(\sigma(T))}{K(1-\alpha)}. \tag{3.10}$$

From $a^{\Delta}(t) \geq 0$ and $q(t) \geq 0$, the fifth term of (3.9) is nonnegative. From (3.9), and (3.10), we get that

$$-B - \frac{\alpha a^{\Delta}(T) u^{1-\alpha}(T)}{1-\alpha} + \frac{a(t) u^{\Delta}(t)}{u^{\alpha}(t)} - A$$
$$- \frac{q(T) a^{\sigma}(T) u^{1-\alpha}(\sigma(T))}{K(1-\alpha)} + \int_T^t a^{\alpha}(\sigma(s)) p(s) \Delta s \leq 0. \tag{3.11}$$

Since $\int_T^{\infty} a^{\alpha}(\sigma(s)) p(s) \Delta s = \infty$, from (3.11), there exists $T_1 \geq T$ such that, for $t \geq T_1$, we have

$$\frac{a(t) u^{\Delta}(t)}{u^{\alpha}(t)} \leq -1. \tag{3.12}$$

Dividing both sides of this last inequality by $a(t)$ and integrating from T_1 to t, we get, using inequality (2.11) in [12], that

$$\frac{1}{1-\alpha}\left[u^{1-\alpha}(t) - u^{1-\alpha}(T_1)\right] \leq \int_{T_1}^t \frac{u^{\Delta}(s)}{u^{\alpha}(s)} \Delta s \leq -\int_{T_1}^t \frac{1}{a(s)} \Delta s. \tag{3.13}$$

Since $\int_{T_1}^{\infty} (1/a(s)) \Delta s = \infty$, we get $u^{1-\alpha}(t) < 0$, for large t, which is a contradiction. Thus, (1.1) is oscillatory. □

When $\mathbb{T} = \mathbb{N}$, $q(t) = 0$, and $a(t) = \rho(t)$, it is easy to get that $a^{\Delta}(t) = 1$, $a^{\Delta\Delta}(t) = 0$. So we have the following corollary (see Corollary 2.4 of [6]). Corollary 3.2 shows that, with no sign assumption on $p(n)$, the condition

$$\sum^{\infty} n^{\alpha} p(n) = \infty \tag{3.14}$$

is sufficient for the oscillation of the difference equation (1.4).

Corollary 3.2. *Assume that* $\mathbb{T} = \mathbb{N}$. *If*

$$\sum_{n=1}^{\infty} n^{\alpha} p(n) = \infty, \tag{3.15}$$

then (1.4) *is oscillatory.*

By using the idea in Theorem 3.1, we can also consider the differential equation

$$x''(t) + q(t)x'(t) + p(t)x^{\alpha}(t) = 0, \quad 0 < \alpha < 1, \tag{3.16}$$

where $p(t), q(t) \in C([1,\infty), \mathbb{R})$. It is easy to get the following.

Theorem 3.3. *Suppose that there exists a function* $a(t) \in C^2([1,\infty), R)$ *such that*

$$\begin{gathered} a(t) > 0, \qquad q'(t) \le 0, \qquad a'(t) + q(t)a(t) \ge 0, \qquad \left(a'(t) + q(t)a(t)\right)' \le 0, \\ \int_1^{\infty} \frac{1}{a(t)} dt = \int_1^{\infty} a^{\alpha}(t) p(t) dt = \infty. \end{gathered} \tag{3.17}$$

Then, the differential equation (3.16) *is oscillatory.*

Example 3.4. Consider the sublinear difference equation

$$\Delta^2 x(n) + n^{-\gamma} \Delta x(n+1) + \left[\frac{1}{n(\ln n)^{\beta}} + b\frac{(-1)^n}{(\ln n)^{\beta}}\right] x^{\alpha}(n+1) = 0, \tag{3.18}$$

where $0 < \alpha < 1$, $\beta > 0$, $\gamma > 0$, and b is any real number.

Take $a(n) = (\ln(n-1))^{\beta/\alpha}$, $n > 2$. We have

$$\begin{gathered} a(n) > 0, \qquad \Delta a(n) = (\ln n)^{\beta/\alpha} - (\ln(n-1))^{\beta/\alpha} > 0, \quad \text{for } n > 2, \\ \Delta^2 a(n) = (\ln(n+1))^{\beta/\alpha} - 2(\ln n)^{\beta/\alpha} + (\ln(n-1))^{\beta/\alpha}. \end{gathered} \tag{3.19}$$

Let $h(t) = (\ln t)^{\beta/\alpha}$. Then, we have $h''(t) < 0$, for large t. So $h(t)$ is concave for large t. Therefore, we have

$$\frac{h(n+1) + h(n-1)}{2} \leq h(n), \tag{3.20}$$

for large n. That means $\Delta^2 a(n) \leq 0$. It is easy to get that $\sum_{n=1}^{\infty} a^{\alpha}(n+1)p(n) = \sum_{n=1}^{\infty}(1/n + b(-1)^n) = \infty$ and $q(n)a(n+1) = n^{-\gamma}(\ln n)^{\beta/\alpha}$ is nonincreasing for large n. So from Theorem 3.1, (3.18) is oscillatory.

Example 3.5. Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, and consider the q-difference equation

$$x^{\Delta\Delta}(t) + t^{-\gamma} x^{\Delta^{\sigma}}(t) + \frac{1 + b(-1)^n}{t^{1+\beta}} x^{\alpha}(qt) = 0, \tag{3.21}$$

where $0 < \alpha < 1$, $0 < \beta \leq \alpha$, $\gamma \geq \beta/\alpha$, $b > 1$ is any real number. Take $a(t) = (t/q)^{\beta/\alpha}$. We have

$$a^{\Delta}(t) = \frac{(q^{\beta/\alpha} - 1)t^{\beta/\alpha - 1}}{(q-1)q^{\beta/\alpha}} > 0, \qquad a^{\Delta\Delta}(t) = \frac{(q^{\beta/\alpha} - 1)(q^{\beta/\alpha - 1} - 1)}{(q-1)^2 q^{\beta/\alpha}} t^{\beta/\alpha - 2} \leq 0,$$

$$\int_1^{\infty} \frac{1}{a(t)} \Delta t = q^{2\beta/\alpha}\left(1 - q^{-1}\right) \sum_{n=1}^{\infty} q^{(1-\beta/\alpha)n} = \infty, \tag{3.22}$$

$$\int_1^{\infty} a^{\alpha}(qt)p(t)\Delta t = \sum_{n=1}^{\infty}\left(1 + b(-1)^n\right)(q-1) = \infty,$$

and $q(t)a^{\sigma}(t) = t^{\beta/\alpha - \gamma}$ is nonincreasing. So from Theorem 3.1, (3.21) is oscillatory.

Example 3.6. Let $\mathbb{T} = [1, \infty)$, and consider the differential equation

$$x''(t) + t^{-\gamma} x'(t) + \left(\frac{1}{t(\ln t)^{\beta}} + \frac{b \sin t}{(\ln t)^{\beta}}\right) x^{\alpha}(t) = 0, \tag{3.23}$$

where $0 < \alpha < 1$, $\gamma \geq 0$, $\beta \geq 0$, and b is any real number.

Take $a(t) = (\ln t)^{\beta/\alpha}$. It is easy to know that

$$a(t) > 0, \quad q'(t) \leq 0, \quad a'(t) + q(t)a(t) > 0, \quad \left(a'(t) + q(t)a(t)\right)' < 0, \quad \text{for large } t,$$
$$\int_1^{\infty} \frac{1}{a(t)} dt = \infty, \qquad \int_1^{\infty} a^{\alpha}(t)p(t)dt = \int_1^{\infty}\left(\frac{1}{t} + b \sin t\right) dt = \infty. \tag{3.24}$$

So from Theorem 3.3, (3.23) is oscillatory.

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