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Research Article

Meromorphic Solutions of Some Complex Difference Equations

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The main purpose of this paper is to present the properties of the meromorphic solutions of complex difference equations of the form $\sum_{\{J\}} \alpha_J(z)(\prod_{j\in J} f(z+c_j)) = R(z, f(z))$, where $\{J\}$ is a collection of all subsets of $\{1, 2, \dots, n\}$, c_j $(j \in J)$ are distinct, nonzero complex numbers, $f(z)$ is a transcendental meromorphic function, $\alpha_J(z)$'s are small functions relative to $f(z)$, and $R(z, f(z))$ is a rational function in $f(z)$ with coefficients which are small functions relative to $f(z)$.

1. Introduction

We assume that the readers are familiar with the basic notations of Nevanlinna's value distribution theory; see [1–3].

Recent interest in the problem of integrability of difference equations is a consequence of the enormous activity on Painlevé differential equations and their discrete counterparts during the last decades. Many people study this topic and obtain some results; see [4–15]. In [4], Ablowitz et al. obtained a typical result as follows.

Theorem A. *If a complex difference equation*

$$f(z+1) + f(z-1) = R(z, f(z)) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_q(z)f(z)^q}, \tag{1.1}$$

with rational coefficients $a_i(z)$ $(i = 0, 1, \dots, p)$ and $b_j(z)$ $(j = 0, 1, \dots, q)$ admits a transcendental meromorphic solution of finite order, then $\deg_f R(z, f(z)) \le 2$.

In [10], Heittokangas et al. extended and improved the above result to higher-order difference equations of more general type. However, by inspecting the proofs in [4], we can find a more general class of complex difference equations by making use of a similar technique; see [10, 15].

In this paper, we mention the above details, used in [4, 10, 15], with equations of the form

$$\sum_{\{J\}}\alpha_J(z)\left(\prod_{j\in J}f(z+c_j)\right)=R(z,f(z)),\tag{1.2}$$

where $\{J\}$ is a collection of all subsets of $\{1,2,\dots,n\}$, c_j $(j\in J)$ are distinct, nonzero complex numbers, $f(z)$ is a transcendental meromorphic function, $\alpha_J(z)$'s are small functions relative to $f(z)$ and $R(z,f(z))$ is a rational function in $f(z)$ with coefficients which are small functions relative to $f(z)$.

2. Main Results

In [10], Heittokangas et al. considered the complex difference equations of the form

$$\prod_{j=1}^{n}f(z+c_j)=R(z,f(z))=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_q(z)f(z)^q},\tag{2.1}$$

with rational coefficients $a_i(z)$ $(i=0,1,\dots,p)$ and $b_j(z)$ $(j=0,1,\dots,q)$. They obtained the following theorem.

Theorem B. *Let $c_1,c_2,\dots,c_n\in\mathbb{C}\setminus\{0\}$. If the difference equation* (2.1) *with rational coefficients $a_i(z)$ $(i=0,1,\dots,p)$ and $b_j(z)$ $(j=0,1,\dots,q)$ admits a transcendental meromorphic solution of finite order $\rho(f)$, then $d\le n$, where $d=\deg_f R(z,f(z))=\max\{p,q\}$.*

It is obvious that the left-hand side of (2.1) is just a product only. If we consider the left-hand side of (2.1) is a product sum, we also have the following theorem.

Theorem 2.1. *Suppose that $c_1,c_2,\dots,c_n$ are distinct, nonzero complex numbers and that $f(z)$ is a transcendental meromorphic solution of*

$$\sum_{\{J\}}\alpha_J(z)\left(\prod_{j\in J}f(z+c_j)\right)=R(z,f(z))=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_q(z)f(z)^q},\tag{2.2}$$

with coefficients $\alpha_J(z)$'s, $a_i(z)$ $(i=0,1,\dots,p)$ and $b_j(z)$ $(j=0,1,\dots,q)$ are small functions relative to $f(z)$. If the order $\rho(f)$ is finite, then $d\le n$, where $d=\deg_f R(z,f(z))=\max\{p,q\}$.

It seems that the equivalent proposition is a known fact. In [15], Laine et al. obtain the similar result to the following Corollary 2.2. Here, for the convenience for the readers, we list it, that is, we have the following corollary.

Corollary 2.2. *Suppose that $c_1, c_2, \ldots, c_n$ are distinct, nonzero complex numbers and that $f(z)$ is a transcendental meromorphic solution of* (2.2) *with rational coefficients $\alpha_J(z)$'s, $a_i(z)$ $(i = 0, 1, \ldots, p)$ and $b_j(z)$ $(j = 0, 1, \ldots, q)$. If $d = \max\{p, q\} > n$, then the order $\rho(f)$ is infinite.*

In [15], when the left-hand side of (2.1) is just a sum, Laine et al. obtained the following theorem.

Theorem C. *Suppose that $c_1, c_2, \ldots, c_n$ are distinct, nonzero complex numbers and that $f(z)$ is a transcendental meromorphic solution of*

$$\sum_{j=1}^{n} \alpha_j(z) f(z + c_j) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{2.3}$$

where the coefficients $\alpha_j(z)$'s are nonvanishing small functions relative to $f(z)$ and where $P(z, f(z))$ and $Q(z, f(z))$ are relatively prime polynomials in $f(z)$ over the field of small functions relative to $f(z)$. Moreover, one assumes that $q = \deg_f Q(z, f(z)) > 0$,

$$n = \max\{p, q\} = \max\left\{\deg_f P(z, f(z)), \deg_f Q(z, f(z))\right\}, \tag{2.4}$$

and that, without restricting generality, $Q(z, f(z))$ is a monic polynomial. If there exists $\alpha \in [0, n)$ such that for all r sufficiently large,

$$\overline{N}\left(r, \sum_{j=1}^{n} \alpha_j(z) f(z + c_j)\right) \leq \alpha \overline{N}(r + C, f(z)) + S(r, f), \tag{2.5}$$

where $C = \max_{1 \leq j \leq n}\{|c_j|\}$, then either the order $\rho(f) = +\infty$, or

$$Q(z, f(z)) \equiv (f(z) + h(z))^q, \tag{2.6}$$

where $h(z)$ is a small meromorphic function relatively to $f(z)$.

They obtained Theorem C and presented a problem that whether the result will be correct if we replace the left-hand side of (2.3) by a product sum as in Theorem 2.1. Here, under the new hypothesis, we consider the left-hand side of (2.3) is a product sum and obtain what follows.

Theorem 2.3. *Suppose that $c_1, c_2, \ldots, c_n$ are distinct, nonzero complex numbers and that $f(z)$ is a transcendent meromorphic solution of*

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j \in J} f(z + c_j)\right) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{2.7}$$

where the coefficients $\alpha_J(z)$*'s are nonvanishing small functions relative to* $f(z)$ *and where* $P(z, f(z))$, $Q(z, f(z))$ *are relatively prime polynomials in* $f(z)$ *over the field of small functions relative to* $f(z)$. *Moreover, one assumes that* $q = \deg_f Q(z, f(z)) > 0$,

$$n = \max\{p, q\} = \max\left\{\deg_f P(z, f(z)), \deg_f Q(z, f(z))\right\}, \tag{2.8}$$

and that, without restricting generality, $Q(z, f(z))$ *is a monic polynomial. If there exists* $\alpha \in [0, n)$ *such that for all* r *sufficiently large,*

$$\sum_{j=1}^{n} \overline{N}(r, f(z + c_j)) \leq \alpha \overline{N}(r + C, f(z)) + S(r, f), \tag{2.9}$$

where $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. *Then either the order* $\rho(f) = +\infty$, *or*

$$Q(z, f(z)) \equiv (f(z) + h(z))^q, \tag{2.10}$$

where $h(z)$ *is a small meromorphic function relative to* $f(z)$.

3. The Proofs of Theorems

Lemma 3.1 (see [3, 9]). *Let* $f(z)$ *be a meromorphic function. Then for all irreducible rational functions in* $f(z)$,

$$R(z, f(z)) = \frac{a_0(z) + a_1(z) f(z) + \cdots + a_p(z) f(z)^p}{b_0(z) + b_1(z) f(z) + \cdots + b_q(z) f(z)^q}, \tag{3.1}$$

with meromorphic coefficients $a_i(z)$ $(i = 0, 1, \ldots, p)$ *and* $b_j(z)$ $(j = 0, 1, \ldots, q)$, *the characteristic function of* $R(z, f(z))$ *satisfies*

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)), \tag{3.2}$$

where $d = \max\{p, q\}$ *and*

$$\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}. \tag{3.3}$$

In the particular case when

$$\begin{aligned} T(r, a_i) &= S(r, f), \quad i = 0, 1, \ldots, p, \\ T(r, b_j) &= S(r, f), \quad j = 0, 1, \ldots, q, \end{aligned} \tag{3.4}$$

we have

$$T(R(z, f(z))) = dT(r, f(z)) + S(r, f). \tag{3.5}$$

Lemma 3.2. *Given distinct complex numbers $c_1, c_2, \ldots, c_n$, a meromorphic function $f(z)$ and meromorphic functions $\alpha_J(z)$'s, one has*

$$T\left(r, \sum_{\{J\}} \alpha_J(z) \left(\prod_{j\in J} f(z+c_j)\right)\right) \le \sum_{j=1}^{n} T(r, f(z+c_j)) + O(\Psi(r)), \tag{3.6}$$

where $\Psi(r) = T(r, \alpha_J(z))$. In the particular case when

$$T(r, \alpha_J(z)) = S(r, f), \tag{3.7}$$

one has

$$T\left(r, \sum_{\{J\}} \alpha_J(z) \left(\prod_{j\in J} f(z+c_j)\right)\right) \le \sum_{j=1}^{n} T(r, f(z+c_j)) + S(r, f). \tag{3.8}$$

Remark 3.3. Observe that the term $S(r, f)$ does not appear in (3.6). This follows by a careful inspection of the proof of [16, Proposition B.15, Theorem B.16].

Remark 3.4. Note that the inequality (3.6) remains true, if we replace the characteristic function T by the proximity function m (or by the counting function N).

Lemma 3.5 (see [12, Theorem 2.1]). *Let $f(z)$ be a nonconstant meromorphic function of finite order, $c \in \mathbb{C}$, and $0 < \delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{\delta}}\right) \tag{3.9}$$

for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E dr/r < +\infty$.

Lemma 3.6 (see [12, Lemma 2.2]). *Let $T : (0, +\infty) \to (0, +\infty)$ be a nondecreasing continuous function, $s > 0$, $0 < \alpha < 1$, and let $F \subset \mathbb{R}^+$ be the set of all r such that*

$$T(r) \le \alpha T(r+s). \tag{3.10}$$

If the logarithmic measure of F is infinite, that is, $\int_F dr/r = +\infty$, then

$$\overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = \infty. \tag{3.11}$$

Proof of Theorem 2.1. Since the coefficients $\alpha_J(z)$'s, $a_i(z)$ $(i = 0,1,\ldots,p)$ and $b_j(z)$ $(j = 0,1,\ldots,q)$ in (2.2) are small functions relative to $f(z)$, that is,

$$\begin{aligned} T(r,a_i) &= S(r,f), \quad i = 0,1,\ldots,p, \\ T(r,b_j) &= S(r,f), \quad j = 0,1,\ldots,q, \\ T(r,\alpha_J(z)) &= S(r,f), \quad J \subset \{1,2,\ldots,n\} \end{aligned} \tag{3.12}$$

hold for all r outside of a possible exceptional set E_1 with finite logarithmic measure $\int_{E_1} dr/r < +\infty$.

Let $f(z)$ be a finite order meromorphic solution of (2.2). According to Lemma 3.5, we have, for any $\epsilon > 0$,

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\epsilon}}\right) =: \widehat{S}(r,f), \tag{3.13}$$

where the exceptional set E_2 associated to $\widehat{S}(r,f)$ is of finite logarithmic measure $\int_{E_2} dr/r < +\infty$.

It follows from Lemma 3.6 that

$$N(r+s,f) = N(r,f) + \widehat{S}(r,f), \tag{3.14}$$

for any $s > 0$.

Now, equating the Nevanlinna characteristic function on both sides of (2.2), and applying Lemmas 3.1 and 3.2, we have

$$\begin{aligned} dT(r,f) &= T\left(\sum_{\{J\}} \alpha_J(z) \left(\prod_{j\in J} f(z+c_j)\right)\right) + S(r,f) \\ &\leq \sum_{j=1}^{n} T(r, f(z+c_j)) + S(r,f) \\ &= \sum_{j=1}^{n} N(r, f(z+c_j)) + \sum_{j=1}^{n} m(r, f(z+c_j)) + S(r,f) \\ &\leq nN(r+C,f) + \sum_{j=1}^{n} m(r, f(z+c_j)) + S(r,f) \\ &\leq nN(r+C,f) + nm(r,f) + \sum_{j=1}^{n} m\left(r, \frac{f(z+c_j)}{f(z)}\right) + S(r,f), \end{aligned} \tag{3.15}$$

where $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$.

Therefore, by (3.13) and (3.14), it follows that

$$dT(r,f) \le nN(r,f) + nm(r,f) + \widehat{S}(r,f) + S(r,f) \\ = nT(r,f) + \widehat{S}(r,f) + S(r,f), \tag{3.16}$$

for all r outside of a possible exceptional set $E_1 \cup E_2$ with finite logarithmic measure. Dividing this by $T(r,f)$ and letting $r \to +\infty$ outside of the exceptional set E_1 and E_2 of $S(r,f)$ and $\widehat{S}(r,f)$, respectively, we have $d \le n$. The proof of Theorem 2.1 is completed. □

Example 3.7. Let $c \in \mathbb{C}$ be a constant such that $c \ne (\pi/2)m$, where $m \in \mathbb{Z}$, and let $A = \tan c, B = \tan(c/2)$. We see that $f(z) = \tan z$ solves

$$f\left(z+\frac{c}{2}\right)f(z+c) + f\left(z-\frac{c}{2}\right)f(z-c) \\ = \frac{2ABf(z)^4 + 2\left[1+(A+B)^2+A^2B^2\right]f(z)^2 + 2AB}{A^2B^2f(z)^4 - (A^2+B^2)f(z)^2 + AB}. \tag{3.17}$$

This shows that the equality $d = n = 4$ is arrived in Theorem 2.1 if $\rho(f) = 1 < +\infty$.

Example 3.8. Let $\mu = e - 1/e, \nu = e + 1/e$. We see that $f(z) = z + e^z$ solves

$$f(z-1)f(z+2) - f(z+1)f(z-2) \\ = \mu f(z)^2 + \left[\mu(\nu-3)z - \nu^2 + 2\nu + 2\right]f(z) - \mu(\nu-2)z + \nu^2 - 2\nu. \tag{3.18}$$

This shows that the case $d = 2 < n = 4$ may occur in Theorem 2.1 if $\rho(f) = 1 < +\infty$.

Lemma 3.9 (see [17]). *Let $f(z)$ be a meromorphic function and let ϕ be given by*

$$\phi = f^n + a_{n-1}f^{n-1} + \cdots + a_0, \\ T(r,a_j) = S(r,f), \quad j = 0,1,\ldots,n-1. \tag{3.19}$$

Then either

$$\phi \equiv \left(f + \frac{a_{n-1}}{n}\right)^n, \tag{3.20}$$

or

$$T(r,f) \le \overline{N}\left(r,\frac{1}{\phi}\right) + \overline{N}(r,f) + S(r,f). \tag{3.21}$$

Lemma 3.10 (see [15]). *Let $f(z)$ be a nonconstant meromorphic function and let $P(z, f(z))$, $Q(z, f(z))$ be two polynomials in $f(z)$ with meromorphic coefficients small functions relative to $f(z)$. If $P(z, f(z))$ and $Q(z, f(z))$ have no common factors of positive degree in $f(z)$ over the field of small functions relative to $f(z)$, then*

$$\overline{N}\left(r, \frac{1}{Q(z, f(z))}\right) \leq \overline{N}\left(r, \frac{P(z, f(z))}{Q(z, f(z))}\right) + S(r, f). \tag{3.22}$$

Proof of Theorem 2.3. Suppose that the second alternative of the conclusion is not correct. Then we have, by using Lemmas 3.9, 3.10, 3.2, (2.7), and (2.9),

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{Q(z, f(z))}\right) + \overline{N}(r, f) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{P(z, f(z))}{Q(z, f(z))}\right) + \overline{N}(r, f) + S(r, f) \\ &= \overline{N}\left(r, \sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J} f(z + c_j)\right)\right) + \overline{N}(r, f) + S(r, f) \\ &\leq \sum_{j=1}^{n} \overline{N}(r, f(z + c_j)) + \overline{N}(r, f) + S(r, f) \\ &\leq \alpha\overline{N}(r + C, f(z)) + \overline{N}(r, f) + S(r, f), \end{aligned} \tag{3.23}$$

where $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$.

Thus, we have

$$T(r, f) - \overline{N}(r, f) \leq \alpha\overline{N}(r + C, f) + S(r, f). \tag{3.24}$$

Now assuming that $\rho(f) < +\infty$, we have $S(r, f(z + c_j)) = S(r, f)$ and for all $j = 1, 2, \ldots, n$,

$$T(r, f(z + c_j)) - \overline{N}(r, f(z + c_j)) \leq \alpha\overline{N}(r + C, f(z + c_j)) + S(r, f). \tag{3.25}$$

It follows from Lemmas 3.1, 3.2, (3.23), and (2.9) we have

$$\begin{aligned} nT(r,f) &= T\left(r, \sum_{\{J\}} \alpha_J(z) \left(\prod_{j\in J} f(z+c_j)\right)\right) + S(r,f) \\ &\le \sum_{j=1}^{n} T(r, f(z+c_j)) + S(r,f) \\ &= \sum_{j=1}^{n} \left[T(r, f(z+c_j)) - \overline{N}(r, f(z+c_j))\right] + \sum_{j=1}^{n} \overline{N}(r, f(z+c_j)) + S(r,f) \\ &\le \sum_{j=1}^{n} \alpha \overline{N}(r+C, f(z+c_j)) + \alpha \overline{N}(r+C, f(z+c_j)) + S(r,f) \\ &\le (n+1)\alpha \overline{N}(r+2C, f) + S(r,f). \end{aligned} \tag{3.26}$$

From this, we have

$$T(r,f) - \overline{N}(r,f) \le \frac{n+1}{n} \alpha \overline{N}(r+2C, f) - \overline{N}(r,f) + S(r,f). \tag{3.27}$$

Together with (3.25)–(3.27), we can use method of induction and obtain, for $m \in \mathbb{N}$,

$$T(r,f) - \overline{N}(r,f) \le \frac{n+m}{n} \alpha \overline{N}(r+2mC, f) - m\overline{N}(r,f) + S(r,f). \tag{3.28}$$

Moreover, we immediately obtain from (3.28) that

$$\overline{N}(r+2mC, f) \ge \frac{nm}{(n+m)\alpha} \overline{N}(r,f) + S(r,f) \triangleq \gamma \overline{N}(r,f) + S(r,f), \tag{3.29}$$

and for sufficiently large m, we have

$$\gamma = \frac{nm}{(n+m)\alpha} > 1. \tag{3.30}$$

It also follows from Lemma 3.6 that

$$\overline{N}(r+s, f) = \overline{N}(r,f) + \widehat{S}(r,f), \tag{3.31}$$

for any $s > 0$, assuming that $f(z)$ is of finite order.

Now (3.31) combined with (3.29) and (3.30) yields an immediate contradiction if $\rho(f) < +\infty$. Therefore the only possibility is that $f(z)$ is of infinite order. The proof of Theorem 2.3 is completed. □

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