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Research Article

Existence of Positive Solutions in Generalized Boundary Value Problem for p-Laplacian Dynamic Equations on Time Scales

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We analytically establish the conditions for the existence of at least two or three positive solutions in the generalized m-point boundary value problem for the p-Laplacian dynamic equations on time scales by means of the Avery-Henderson fixed point theorem and the five functionals fixed point theorem. Furthermore, we illustrate the possible application of our analytical results with a concrete and nontrivial dynamic equation on specific time scales.

1. Introduction

Since the seminal work by Stefan Hilger in 1988, there has been a rapid development in the research of dynamic equations on time scales. The gradually maturing theory of dynamic equations not only includes the majority of the existing analytical results on both differential equations and difference equations with uniform time-steps but also establishes a solid foundation for the research of hybrid equations on different kinds of time scales. More importantly, with this foundation and those ongoing investigations, concrete applications of dynamic equations on time scales in mathematical modeling of real processes and phenomena, such as population dynamics, economic evolutions, chemical kinetics, and neural signal processing, have been becoming fruitful [1–8].

Recently, among the topics in the research of dynamic equations on time scales, the investigation of the boundary value problems for some specific dynamic equations on time scales has become a focal one that attained a great deal of attention from many mathematicians. In fact, systematic framework has been established for the study of the positive solutions in the boundary value problems for the second-order equations on time scales [9–15]. In particular, some results have been analytically obtained on the existence of positive solutions in some specific boundary value problems for the p-Laplacian dynamic equations on time scales [16–19].

More specifically, He and Li [19], investigated the existence of at least triple positive solutions to the following p-Laplacian boundary value problem:

$$\begin{gathered}\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla}+h(t)f(u)=0, \quad t\in[0,T]_{\mathbb{T}}, \\ u(0)-B_0\left(u^{\Delta}(0)\right)=0, \qquad u^{\Delta}(T)=0.\end{gathered} \tag{1.1}$$

Here and throughout, $\mathbb{T}$ is supposed to be a time scale, that is, $\mathbb{T}$ is any nonempty closed subset of real numbers in $\mathbb{R}$ with order and topological structure defined in a canonical way. The closed interval in $\mathbb{T}$ is defined as $[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$. Accordingly, the open interval and the half-open interval could be defined, respectively. In addition, it is assumed that $0,\,T\in\mathbb{T}$, $\eta\in(0,\rho(T))_{\mathbb{T}}$, $f\in C_{\mathrm{ld}}([0,\infty),[0,\infty))$, $h\in C_{\mathrm{ld}}((0,T)_{\mathbb{T}},(0,\infty))$, and $\underline{b}x\leqslant B_0(x)\leqslant\overline{b}x$ for some positive constants $\underline{b}$ and $\overline{b}$. Moreover, $\phi_p(u)$ is supposed to be the p-Laplacian operator, that is, $\phi_p(u)=|u|^{p-2}u$ and $(\phi_p)^{-1}=\phi_q$, in which $p>1$ and $1/p+1/q=1$. With these configurations and with the aid of the five functionals fixed point theorem [20], they established the criteria for the existence of at least triple positive solutions of the above boundary value problem.

Later on, Yaslan [21], investigated the following boundary value problem:

$$\begin{gathered}u^{\Delta\nabla}(t)+h(t)f(t,u(t))=0, \quad t\in[t_1,t_3]_{\mathbb{T}}\subset\mathbb{T}, \\ \alpha u(t_1)-\beta u^{\Delta}(t_1)=u^{\Delta}(t_2), \qquad u^{\Delta}(t_3)=0,\end{gathered} \tag{1.2}$$

in which $0\leqslant t_1<t_2<t_3$, $\alpha>0$, and $\beta>1$. Indeed, Yaslan analytically established the conditions for the existence of at least two or three positive solutions in the above boundary value problem by means of the Avery-Henderson fixed point theorem and the Leggett-Williams fixed point theorem [22]. As a matter of fact, these analytical results are even new for those special equations on time scales, such as the difference equations with uniform time-step and the ordinary differential equations. Following the work in [21], Sun and Wang [23], further extended the results to the following boundary value problem:

$$\begin{gathered}\left(\phi\left(u^{\Delta}(t)\right)\right)^{\nabla}+h(t)f(t,u(t))=0, \quad t\in(0,T)_{\mathbb{T}}, \\ u(0)-\beta u^{\Delta}(0)=\gamma u^{\Delta}(\eta), \qquad u^{\Delta}(T)=0.\end{gathered} \tag{1.3}$$

In this paper, inspired by the aforementioned results and methods in dealing with those boundary value problems on time scales, we intend to analytically discuss the possible existence of multiple positive solutions for the following one-dimensional p-Laplacian dynamic equation:

$$\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla}+h(t)f(t,u(t))=0, \quad t\in(0,T]_{\mathbb{T}}, \tag{1.4}$$

with m-point boundary value conditions:

$$
\begin{gathered}
u(0) - \beta B_0\left(u^{\Delta}(0)\right) = \sum_{i=1}^{m-2} B\left(u^{\Delta}(\xi_i)\right), \\
\phi_p\left(u^{\Delta}(T)\right) = \sum_{i=1}^{m-2} a_i \phi_p\left(u^{\Delta}(\xi_i)\right).
\end{gathered}
\tag{1.5}
$$

In the following discussion, we impose the following three hypotheses.

(H1) $0 \leqslant \beta$, $0 \leqslant a_i$ for $i = 1, \dots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < T$, and $\rho = 1 - d > 0$, where $d = \sum_{i=1}^{m-2} a_i$.

(H2) $h : [0,T]_{\mathbb{T}} \mapsto [0,\infty)$ is left dense continuous (ld-continuous), and there exists a $t_0 \in [0,T]_{\mathbb{T}}$ such that $h(t_0) \neq 0$. $f : [0,T]_{\mathbb{T}} \times [0,\infty) \mapsto [0,\infty)$ is continuous.

(H3) Both B_0 and B are continuously odd functions defined on $\mathbb{R}$. There exist two positive numbers $\underline{b}$ and $\overline{b}$ such that, for any $v > 0$,

$$
\underline{b} v \leqslant B_0(v), \qquad B(v) \leqslant \overline{b} v. \tag{1.6}
$$

Note that the definition of the ld-continuous function will be described in Definition 2.3 of Section 2. Also note that, together with conditions (1.5) and the above hypotheses (H1)–(H3), the dynamic equation (1.4) with conditions (1.5) not only includes the above-mentioned specific boundary value problems in literature but also nontrivially extends the situation to a much wider class of boundary value problems on time scales. A question naturally appears: "can we still establish some criteria for the existence of at least double or triple positive solutions in the generalized boundary value problems (1.4) and (1.5)?" In this paper, we will give a positive answer to this question by virtue of the Avery-Henderson fixed point theorem and the five functionals fixed point theorem. Particularly, those obtained criteria will significantly extend the results in literature [19, 21, 23].

The rest of paper is organized as follows. In Section 2, we preliminarily import some definitions and properties of time scales and introduce some useful lemmas which will be utilized in the following discussion. In Section 3, we analytically present a criteria for the existence of at least two positive solutions in the boundary value problems (1.4) and (1.5) by virtue of the Avery-Henderson fixed point theorem. In Section 4, we provide some sufficient conditions for the existence of at least three positive solutions in light of the five functionals fixed point theorem. Finally, we further provides concrete and nontrivial example to illustrate the possible application of the obtained analytical results on dynamic equations on time scales in Section 5.

2. Preliminaries

2.1. Time Scales

For the sake of self-consistency, we import some necessary definitions and lemmas on time scales. More details can be found in [4] and reference therein. First of all, a time scale $\mathbb{T}$ is any nonempty closed subset of real numbers $\mathbb{R}$ with order and topological structure defined in a canonical way, as mentioned above. Then, we have the following definition of the categories of points on time scales.

Definition 2.1. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator $\sigma : \mathbb{T} \mapsto \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \mapsto \mathbb{T}$, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(r) = \sup\{s \in \mathbb{T} : s < r\}, \quad \forall t,\ r \in \mathbb{T}. \tag{2.1}$$

Then, the graininess operator $\mu : \mathbb{T} \mapsto [0, \infty)$ is defined as $\mu(t) = \sigma(t) - t$. In addition, if $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbb{T}$ has a right scattered minimum m, denote by $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, denote by $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$.

The following definitions describe the categories of functions on time scales and the basic computations of integral and derivative.

Definition 2.2. Assume that $f : \mathbb{T} \mapsto \mathbb{R}$ is a function and that $t \in \mathbb{T}^k$. $f^{\Delta}(t)$ is supposed to be the number (provided it exists) with the property that given any $\epsilon > 0$; there is a neighborhood $U \subset \mathbb{T}$ of t satisfying

$$\Big| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \Big| \leqslant \epsilon |\sigma(t) - s|, \tag{2.2}$$

for all $s \in U$. Then $f^{\Delta}(t)$ is said to be the delta derivative of f at t. Similarly, assume that $f : \mathbb{T} \mapsto \mathbb{R}$ is a function and that $t \in \mathbb{T}_k$. Denote by $f^{\nabla}(t)$ the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $V \subset \mathbb{T}$ of t such that

$$\Big| [f(\rho(t)) - f(s)] - f^{\nabla}(t)[\rho(t) - s] \Big| \leqslant \epsilon \big|\rho(t) - s\big|, \tag{2.3}$$

for all $s \in V$. Then $f^{\nabla}(t)$ is said to be the nabla derivative of f at t.

Definition 2.3. A function $f : \mathbb{T} \mapsto \mathbb{R}$ is left dense continuous (ld-continuous) provided that it is continuous at all left dense points of $\mathbb{T}$, and its right side limits exists (being finite) at right dense points of $\mathbb{T}$. Denote by $C_{\mathrm{ld}} = C_{\mathrm{ld}}(\mathbb{T})$ the set of all left dense continuous functions on $\mathbb{T}$.

Definition 2.4. Let $f : \mathbb{T} \mapsto \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \mapsto \mathbb{R}$ such that $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, then F is a delta antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a), \quad \forall a, b \in \mathbb{T}. \tag{2.4}$$

Analogously, let $f : \mathbb{T} \mapsto \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $\Phi : \mathbb{T} \mapsto \mathbb{R}$ such that $\Phi^{\nabla}(t) = f(t)$ for all $t \in \mathbb{T}_k$, then F is a nabla antiderivative of f. In this case, the integral is given by the formula

$$\int_a^b f(\tau)\nabla\tau = \Phi(b) - \Phi(a), \quad \forall a, b \in \mathbb{T}. \tag{2.5}$$

2.2. Main Lemmas

This subsection aims to establish several lemmas which are useful in the proof of the main results in this paper. In particular, these lemmas focus on the following linear boundary value problems:

$$\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + g(t) = 0, \quad t \in (0, T]_{\mathbb{T}}, \tag{2.6}$$

$$u(0) - \beta B_0\left(u^{\Delta}(0)\right) = \sum_{i=1}^{m-2} B\left(u^{\Delta}(\xi_i)\right), \qquad \phi_p\left(u^{\Delta}(T)\right) = \sum_{i=1}^{m-2} a_i \phi_p\left(u^{\Delta}(\xi_i)\right). \tag{2.7}$$

Lemma 2.5. *If $\rho = 1 - \sum_{i=1}^{m-2} a_i \neq 0$, then, for $g \in C_{\mathrm{ld}}([0, \sigma(T)]_{\mathbb{T}}, \mathbb{R})$, the linear boundary value problems* (2.6) *and* (2.7) *have a unique solution satisfying*

$$\begin{aligned} u(t) = {} & \int_0^t \left[\phi_q \circ (\mathcal{F} - \mathcal{A})g\right](s)\Delta s + \beta B_0\left(\left[\phi_q \circ \mathcal{F}g\right](t)\right) \\ & + \sum_{i=1}^{m-2} B\left(\left[\phi_q \circ (\mathcal{F} - \mathcal{A})g\right](\xi_i)\right). \end{aligned} \tag{2.8}$$

$t \in [0, \sigma(T)]_{\mathbb{T}}$. *Here,*

$$[\mathcal{A}g](t) = \int_0^t g(\tau)\nabla\tau, \qquad [\mathcal{F}g](t) = \frac{1}{\rho}\left([\mathcal{A}g](T) - \sum_{i=1}^{m-2} a_i [\mathcal{A}g](\xi_i)\right). \tag{2.9}$$

Proof. It follows from (2.8) that

$$u^{\Delta}(t) = \left[\phi_q \circ (\mathcal{F} - \mathcal{A})g\right](t). \tag{2.10}$$

Thus, we obtain that

$$\phi_p\left(u^{\Delta}(t)\right) = \left[(\mathcal{F} - \mathcal{A})g\right](t), \tag{2.11}$$

and that

$$\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} = -g(t). \tag{2.12}$$

Then, $u(t)$ satisfies (2.6), which verifies that $u(t)$ is a solution of the problems (2.6) and (2.7). Furthermore, in order to show the uniqueness, we suppose that both $u_1(t)$ and $u_2(t)$ are the solutions of the problems (2.6) and (2.7). Then, we have

$$\left(\phi_p\left(u_1^{\Delta}(t)\right)\right)^{\nabla} - \left(\phi_p\left(u_2^{\Delta}(t)\right)\right)^{\nabla} = 0, \quad t \in (0,T]_{\mathbb{T}}, \tag{2.13}$$

$$\phi_p\left(u_1^{\Delta}(T)\right) - \phi_p\left(u_2^{\Delta}(T)\right) = \sum_{i=1}^{m-2} a_i\left[\phi_p\left(u_1^{\Delta}(\xi_i)\right) - \phi_p\left(u_2^{\Delta}(\xi_i)\right)\right], \tag{2.14}$$

$$u_1(0) - u_2(0) - \beta B_0\left(u_1^{\Delta}(0)\right) + \beta B_0\left(u_2^{\Delta}(0)\right) = \sum_{i=1}^{m-2}\left[B\left(u_1^{\Delta}(\xi_i)\right) - B\left(u_2^{\Delta}(\xi_i)\right)\right]. \tag{2.15}$$

In fact, (2.13) further yields

$$\phi_p\left(u_1^{\Delta}(t)\right) - \phi_p\left(u_2^{\Delta}(t)\right) = \overline{c}, \quad t \in [0,T]_{\mathbb{T}}. \tag{2.16}$$

Hence, from (2.14) and (2.16), the assumption $\rho = 1 - d \neq 0$, and the definition of the p-Laplacian operator, it follows that

$$u_1^{\Delta}(t) - u_2^{\Delta}(t) \equiv 0, \quad t \in [0,T]_{\mathbb{T}}. \tag{2.17}$$

This equation, with (2.15), further implies

$$u_1(t) \equiv u_2(t), \quad t \in [0,\sigma(T)]_{\mathbb{T}}, \tag{2.18}$$

which consequently leads to the completion of the proof, that is, $u(t)$ specified in (2.8) is the unique solution of the problems (2.6) and (2.7). □

Lemma 2.6. *Suppose that $\rho = 1 - d > 0$. If $g \in C_{\mathrm{ld}}([0,\sigma(T)]_{\mathbb{T}}, [0,\infty))$, then the unique solution of the problems* (2.6) *and* (2.7) *satisfies*

$$u(t) \geqslant 0, \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \tag{2.19}$$

Proof. Observe that, for any $t \in [0,T]_{\mathbb{T}}$,

$$\begin{aligned}
\left[(\mathcal{F} - \mathcal{H})g\right](t) &\geqslant \left[(\mathcal{F} - \mathcal{H})g\right](T) \\
&= \frac{1}{\rho}\left(d\left[\mathcal{H}g\right](T) - \sum_{i=1}^{m-2} a_i\left[\mathcal{H}g\right](\xi_i)\right) \\
&\geqslant \frac{1}{\rho}\left(d\left[\mathcal{H}g\right](T) - \sum_{i=1}^{m-2} a_i\left[\mathcal{H}g\right](T)\right) \\
&= 0.
\end{aligned} \tag{2.20}$$

Thus, by (2.8) specified in Lemma 2.5, we get

$$u^{\Delta}(t) = [\phi_q \circ (\mathcal{F} - \mathcal{A})g](t) \geqslant 0, \quad t \in [0, T]_{\mathbb{T}}. \tag{2.21}$$

Thus, $u(t)$ is nondecreasing in the interval $[0, \sigma(T)]_{\mathbb{T}}$. In addition, notice that

$$u(0) = \beta B_0([\phi_q \circ \mathcal{F}g](t)) + \sum_{i=1}^{m-2} B([\phi_q \circ (\mathcal{F} - \mathcal{A})g](\xi_i)) \geqslant 0. \tag{2.22}$$

The last term in the above estimation is no less than zero owing to those assumptions. Therefore, from the monotonicity of $u(t)$, we get

$$u(t) \geqslant u(0) \geqslant 0, \quad t \in [0, \sigma(T)]_{\mathbb{T}}, \tag{2.23}$$

which consequently completes the proof. □

Lemma 2.7. *Suppose that $\rho = 1 - \sum_{i=1}^{m-2} a_i > 0$. If $g \in C_{\mathrm{ld}}([0, \sigma(T)]_{\mathbb{T}}, [0, \infty))$, then the unique positive solution of the problems* (2.6) *and* (2.7) *satisfies*

$$\beta \underline{b}[\phi \circ \mathcal{A}g](T) \leqslant u(t) \leqslant \left(\tau + \overline{b}\beta + (m-2)\overline{b}\right)\left[\phi_q \circ \left(\frac{1}{\rho}\mathcal{A}\right)g\right](T) \tag{2.24}$$

for $t, \tau \in [0, \sigma(T)]_{\mathbb{T}}$ with $t \leqslant \tau$.

Proof. Since $u(t)$ is nondecreasing in the interval $[0, \sigma(T)]_{\mathbb{T}}$,

$$\begin{aligned}
u(t) &\leqslant u(\tau) \\
&= \int_0^{\tau} [\phi_q \circ (\mathcal{F} - \mathcal{A})g](s)\Delta s + \beta B_0([\phi_q \circ \mathcal{F}g](t)) + \sum_{i=1}^{m-2} B([\phi_q \circ (\mathcal{F} - \mathcal{A})g](\xi_i)) \\
&\leqslant \int_0^{\tau} [\phi_q \circ \mathcal{F}g](s)\Delta s + \beta B_0([\phi_q \circ \mathcal{F}g](t)) + \sum_{i=1}^{m-2} B([\phi_q \circ \mathcal{F}g](\xi_i)) \\
&\leqslant \left(\tau + \overline{b}\beta + (m-2)\overline{b}\right)\left[\phi_q \circ \left(\frac{1}{\rho}\mathcal{A}\right)g\right](T).
\end{aligned} \tag{2.25}$$

On the other hand,

$$\begin{aligned}
[\mathcal{F}g](t) &= \frac{1}{\rho}\left([\mathcal{A}g](T) - \sum_{i=1}^{m-2} a_i [\mathcal{A}g](\xi_i)\right) \\
&\geqslant \frac{1}{\rho}\left([\mathcal{A}g](T) - \sum_{i=1}^{m-2} a_i [\mathcal{A}g](T)\right) \\
&= [\mathcal{A}g](T).
\end{aligned} \tag{2.26}$$

Hence,

$$u(t) \geqslant \beta B_0\left(\left[\phi_q \circ \mathcal{F} g\right](t)\right) \geqslant \beta \underline{b}\left[\phi \circ \mathcal{H} g\right](T). \tag{2.27}$$

This completes the proof. □

Now, denote by $\mathcal{E} = C_{\mathrm{ld}}[0, \sigma(T)]_{\mathbb{T}}$ and by $\|u\| = \sup_{t\in[0,\sigma(T)]_{\mathbb{T}}}|u(t)|$, where $u \in \mathcal{E}$. Then, it is easy to verify that $\mathcal{E}$ endowed with $\|\cdot\|$ becomes a Banach space. Furthermore, define a cone, denoted by $\mathcal{P}$, through

$$\begin{aligned} \mathcal{P} = \Big\{ u \in \mathcal{E} \mid u(t) \geqslant 0 \text{ for } t \in [0, \sigma(T)]_{\mathbb{T}},\ u^{\Delta}(t) \geqslant 0 \text{ for } t \in [0, T]_{\mathbb{T}}, \\ u^{\Delta\nabla}(t) \leqslant 0 \text{ for } t \in (0, \sigma(T))_{\mathbb{T}} \Big\}. \end{aligned} \tag{2.28}$$

Also, for a given positive real number r, define a function set $\mathcal{P}_r$ by

$$\mathcal{P}_r = \{u \in \mathcal{P} \mid \|u\| < r\}. \tag{2.29}$$

Naturally, we denote by $\overline{\mathcal{P}}_r = \{u \in \mathcal{P} \mid \|u\| \leqslant r\}$ and by $\partial\mathcal{P}_r = \{u \in \mathcal{P} \mid \|u\| = r\}$. With these settings and notations, we are in a position to have the following properties.

Lemma 2.8. *If $u \in \mathcal{P}$, then (i) $u(t) \geqslant (t/T)\|u\|$ for any $t \in [0, T]_{\mathbb{T}}$; (ii) $su(t) \geqslant tu(s)$ for any pair of $s, t \in [0, T]_{\mathbb{T}}$ with $t \geqslant s$.*

The proof of this lemma, which could be found in [19, 21], is directly from the specific construction of the set $\mathcal{P}$. Next, let us construct a map $\mathfrak{A} : \mathcal{P} \to \mathcal{E}$ through

$$\begin{aligned} [\mathfrak{A}u](t) = \int_0^t \left[\phi_q \circ (\mathcal{F} - \mathcal{H})\overline{f}\right](s)\Delta s + \beta B_0\left(\left[\phi_q \circ \mathcal{F}\overline{f}\right](t)\right) \\ + \sum_{i=1}^{m-2} B\left(\left[\phi_q \circ (\mathcal{F} - \mathcal{H})\overline{f}\right](\xi_i)\right), \end{aligned} \tag{2.30}$$

for any $u \in \mathcal{P}$. Here, $\overline{f}(t) = h(t)f(t, u(t))$. Thus, we obtain the following properties on this map.

Lemma 2.9. *Assume that the hypotheses* (H1)–(H3) *are all fulfilled. Then, $\mathfrak{A}(\mathcal{P}) \subset \mathcal{P}$ and $\mathfrak{A} : \overline{\mathcal{P}}_r \to \mathcal{P}$ is completely continuous.*

Proof. At first, arbitrarily pick up $u \in \mathcal{P}$. Then it directly follows from Lemma 2.6 that $[\mathfrak{A}u](t) \geqslant 0$ for all $t \in [0, \sigma(T)]_{\mathbb{T}}$. Moreover, direct computation yields

$$[\mathfrak{A}u]^{\Delta}(t) = \left[\phi_q \circ (\mathcal{F} - \mathcal{H})\overline{f}\right](t) \geqslant 0, \tag{2.31}$$

for all $t \in (0, \sigma(T))_{\mathbb{T}}$, and

$$\left[\phi_p\left([\mathfrak{A}u]^{\Delta}(t)\right)\right]^{\nabla} = -\overline{f}(t) \leqslant 0 \tag{2.32}$$

for all $t \in (0, \sigma(T))_{\mathbb{T}}$. Thus, the latter inequality implies that $[\mathfrak{A}u]^{\Delta}(t)$ is decreasing on $[0, \sigma(T))_{\mathbb{T}}$. This implies that $[[\mathfrak{A}u]^{\Delta}(t)]^{\nabla} \leqslant 0$ for $t \in (0, \sigma(T))_{\mathbb{T}}$. Consequently, we complete the proof of the first part of the conclusion that $\mathfrak{A}u \in \mathcal{P}$ for any $u \in \mathcal{P}$.

Secondly, we are to validate the complete continuity of the map $\mathfrak{A}$. To approach this aim, we have to verify that $\mathfrak{A}(\overline{\mathcal{P}}_r)$ is bounded, where $\overline{\mathcal{P}}_r$ is obviously bounded. It follows from the proof of Lemma 2.7 that

$$\begin{aligned} \|\mathfrak{A}u\| &= [\mathfrak{A}u](\sigma(T)) \\ &\leqslant \left(\sigma(T) + \overline{b}\beta + (m-2)\overline{b}\right)\left[\phi_q \circ \left(\frac{1}{\rho}\mathcal{H}\right)\overline{f}\right](T) \\ &\leqslant \left(\sigma(T) + \overline{b}\beta + (m-2)\overline{b}\right)\phi_q\left(\frac{1}{\rho}\int_0^T M_0 h(\tau)\nabla\tau\right), \end{aligned} \tag{2.33}$$

where $M_0 = \max\{f(t,u) \mid t \in [0,T]_{\mathbb{T}},\ 0 \leqslant u \leqslant r\}$. This manifests the uniform boundedness of the set $\mathfrak{A}(\overline{\mathcal{P}}_r)$. In addition, for any given $t_1, t_2 \in [0, \sigma(T)]_{\mathbb{T}}$ with $t_1 < t_2$, we have the following estimation:

$$\begin{aligned} |[\mathfrak{A}u](t_1) - [\mathfrak{A}u](t_2)| &= \left|\int_{t_1}^{t_2}\left[\phi_q \circ (\mathcal{F} - \mathcal{H})\overline{f}\right](s)\Delta s\right| \\ &\leqslant \left|\int_{t_1}^{t_2}\left[\phi_q \circ \mathcal{F}\overline{f}\right](s)\Delta s\right| \\ &\leqslant \left|\int_{t_1}^{t_2}\left[\phi_q \circ \left(\frac{1}{\rho}\mathcal{H}\right)\overline{f}\right](T)\Delta s\right| \\ &\leqslant |t_1 - t_2| \cdot \phi_q\left(\frac{1}{\rho}\int_0^T M_0 h(\tau)\nabla\tau\right). \end{aligned} \tag{2.34}$$

This validates the equicontinuity of the elements in the set $\mathfrak{A}(\overline{\mathcal{P}}_r)$. Therefore, according to the Arzelà-Ascoli theorem on time scales [2], we conclude that $\mathfrak{A}(\overline{\mathcal{P}}_r)$ is relatively compact. Now, let $\{u_n\}_{n=1}^{\infty} \subset \overline{\mathcal{P}}_r$ with $\|u_n - u\| \to 0$. Then $|u_n(t)| \leqslant r$ for all $t \in [0, \sigma(T)]_{\mathbb{T}}$ and $n = 1, 2, \ldots$. Also, $|u_n(t) - u(t)| \to 0$ is uniformly valid on $[0, \sigma(T)]_{\mathbb{T}}$. These, with the uniform continuity of $f(t,u)$ on the compact set $[0,T]_{\mathbb{T}} \times [0,r]$, leads to a conclusion that $|f(t, u_n(t)) - f(t, u(t))| \to 0$ is uniformly valid on $[0,T]_{\mathbb{T}}$. Hence, it is easy to verify that $\|\mathfrak{A}u_n - \mathfrak{A}u\| \to 0$ as n tends toward positive infinity. As a consequence, we complete the whole proof. □

3. At Least Two Positive Solutions in Boundary Value Problems

This section aims to prove the existence of at least two positive solutions in the boundary value problems (1.4) and (1.5) in light of the well-known Avery-Henderson fixed point theorem. Firstly, we introduce the Avery-Henderson fixed point theorem as follows.

Theorem 3.1 ([24]). *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{E}$. For each $d > 0$, set $\mathcal{P}(\psi, d) = \{x \in \mathcal{P} \mid \psi(x) < d\}$. If α and γ are increasing nonnegative continuous functional on $\mathcal{P}$, and let θ be a nonnegative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,*

$$\gamma(x) \leqslant \theta(x) \leqslant \alpha(x), \qquad \|x\| \leqslant H\gamma(x), \tag{3.1}$$

for all $x \in \overline{\mathcal{P}(\gamma, c)}$. Suppose that there exist a completely continuous operator $\mathfrak{A} : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$ and three positive numbers $0 < a < b < c$ such that

$$\theta(\lambda x) \leqslant \lambda\theta(x), \quad 0 \leqslant \lambda \leqslant 1,\ x \in \partial\mathcal{P}(\theta, b), \tag{3.2}$$

and (i) $\gamma(\mathfrak{A}x) > c$ for all $x \in \partial\mathcal{P}(\gamma, c)$; (ii) $\theta(\mathfrak{A}x) < b$ for all $x \in \partial\mathcal{P}(\theta, b)$; (iii) $\mathcal{P}(\alpha, a) \neq \emptyset$ and $\alpha(\mathfrak{A}x) > a$ for all $x \in \partial\mathcal{P}(\alpha, a)$. Then, the operator $\mathfrak{A}$ has at least two fixed points, denoted by x_1 and x_2, belonging to $\overline{\mathcal{P}(\gamma, c)}$ and satisfying $a < \alpha(x_1)$ with $\theta(x_1) < b$ and $b < \theta(x_2)$ with $\gamma(x_2) < c$.

Secondly, let $t^\star = \min\{t \in \mathbb{T} \mid T/2 \leqslant t \leqslant T\}$ and select $t_\star \in \mathbb{T}$ satisfying $0 < t_\star < t^\star$. Furthermore, set, respectively,

$$\begin{aligned} L &= \frac{t^\star}{T}\beta\underline{b}\phi_q\left(\int_{t^\star}^{T} h(\tau)\nabla\tau\right), \\ M &= \frac{\underline{b}\beta t_\star}{T} \cdot \phi_q\left(\int_{t_\star}^{T} h(\tau)\nabla\tau\right), \\ N &= \left(T + \beta\overline{b} + (m-2)\overline{b}\right)\phi_q\left(\frac{1}{\rho}\int_{0}^{T} h(\tau)\nabla\tau\right)\Delta s. \end{aligned} \tag{3.3}$$

Then, we arrive at the following results.

Theorem 3.2. *Suppose that the hypotheses* (H1)–(H3) *all hold, and that there exist positive real numbers a, b, c such that*

$$0 < a < b < c, \quad a < \frac{L}{N}b < \frac{Lt^\star}{TN}c. \tag{3.4}$$

In addition, suppose that f satisfies the following conditions:

(B1) *$f(t, u) > \phi_p(c/M)$ for $t \in [t_\star, T]_{\mathbb{T}}$ and $u \in [c, (T/t_\star)c]$;*

(B2) *$f(t, u) < \phi_p(b/N)$ for $t \in [0, T]_{\mathbb{T}}$ and $u \in [0, (T/t^\star)b]$;*

(B3) *$f(t, u) > \phi_p(a/L)$ for $t \in [t^\star, T]_{\mathbb{T}}$ and $u \in [0, a]$.*

Then, the boundary value problems (1.4) *and* (1.5) *have at least two positive solutions* u_1 *and* u_2 *such that*

$$\begin{gathered}\min_{t\in[t_\star,t^\star]_{\mathbb{T}}} u_i(t) \leqslant c, \quad (i=1,2),\\ a<\max_{t\in[0,t^\star]_{\mathbb{T}}} u_1(t) \quad \text{with} \max_{t\in[0,t_\star]_{\mathbb{T}}} u_1(t)<b,\\ b<\max_{t\in[0,t_\star]_{\mathbb{T}}} u_2(t) \quad \text{with} \min_{t\in[t_\star,t^\star]_{\mathbb{T}}} u_2(t)<c.\end{gathered} \tag{3.5}$$

Proof. Construct the cone $\mathcal{P}$ and the operator $\mathfrak{A}$ as specified in (2.28) and (2.30), respectively. In addition, define the increasing, nonnegative, and continuous functionals γ, θ, and α on $\mathcal{P}$, respectively, by

$$\begin{aligned}\gamma(u) &= \min_{t\in[t_\star,t^\star]_{\mathbb{T}}} u(t)=u(t_\star),\\ \theta(u) &= \max_{t\in[0,t_\star]_{\mathbb{T}}} u(t)=u(t_\star),\\ \alpha(u) &= \max_{t\in[0,t^\star]_{\mathbb{T}}} u(t)=u(t^\star).\end{aligned} \tag{3.6}$$

Obviously, $\gamma(u)=\theta(u)\leqslant\alpha(u)$ for each $u\in\mathcal{P}$.

Moreover, Lemma 2.8 manifests that $\gamma(u)=u(t_\star)\geqslant(t_\star/T)\|u\|$ for each $u\in\mathcal{P}$. Hence, we have

$$\|u\|\leqslant\frac{T}{t_\star}\gamma(u) \tag{3.7}$$

for each $u\in\mathcal{P}$. Also, notice that $\theta(\lambda u)=\lambda\theta(u)$ for $\lambda\in[0,1]$ and $u\in\partial\mathcal{P}(\theta,b)$. Furthermore, from Lemma 2.9, it follows that the operator $\mathfrak{A}:\overline{\mathcal{P}(\gamma,c)}\to\mathcal{P}$ is completely continuous.

Next, we are to verify the validity of all the conditions in Theorem 3.1 with respect to the operator $\mathfrak{A}$.

Let $u\in\partial\mathcal{P}(\gamma,c)$. Then, $\gamma(u)=\min_{t\in[t_\star,t^\star]_{\mathbb{T}}}u(t)=u(t_\star)=c$. This implies $u(t)\geqslant c$ for $t\in[t_\star,t^\star]_{\mathbb{T}}$, which, combined with (3.7), yields

$$c\leqslant u(t)\leqslant\frac{T}{t_\star}c \tag{3.8}$$

for $t\in[t_\star,T]_{\mathbb{T}}$. Noticing the assumption (B1), we have $f(t,u(t))>\phi_p(c/M)$ for $t\in[t_\star,\mathrm{T}]_{\mathbb{T}}$. Also noticing the particular form in (2.30), Lemma 2.8, the property $\mathfrak{A}u\in\mathcal{P}$, and the proof of

Lemma 2.7, we get

$$\begin{aligned}\gamma(\mathfrak{A}u) &= [\mathfrak{A}u](t_\star)\\ &\geqslant \frac{t_\star}{T}\|\mathfrak{A}u\|\\ &= \frac{t_\star}{T}[\mathfrak{A}u](\sigma(T))\\ &\geqslant \frac{b\beta t_\star}{T}\phi_q\left(\left[\mathscr{L}\overline{f}\right](T)\right)\\ &> \frac{b\beta t_\star}{T}\cdot\frac{c}{M}\cdot\phi_q\left(\int_{t_\star}^{T}h(\tau)\nabla\tau\right)\\ &= c.\end{aligned}\tag{3.9}$$

Therefore, condition (i) in Theorem 3.1 is satisfied.

In what follows, let us consider $u \in \partial\mathcal{P}(\theta, b)$. In such a case, we obtain $\gamma(u) = \theta(u) = \max_{t\in[0,t_\star]_{\mathbb{T}}} u(t) = u(t_\star) = b$, which means that $0 \leqslant u(t) \leqslant b$ for $t \in [0, t_\star]_{\mathbb{T}}$. Similarly, it follows from (3.7) that, for all $u \in \mathcal{P}$,

$$\|u\| \leqslant \frac{T}{t_\star}\gamma(u) = \frac{T}{t_\star}b.\tag{3.10}$$

Hence, we have $0 \leqslant u(t) \leqslant (T/t_\star)b$ for $t \in [0, T]_{\mathbb{T}}$. This, combined with the assumption (B2), yields $f(t, u(t)) < \phi_p(b/N)$ for all $t \in [0, T]_{\mathbb{T}}$. Therefore, from the proof of Lemma 2.7, we have

$$\begin{aligned}\theta(\mathfrak{A}u) &= \max_{t\in[0,t_\star]_{\mathbb{T}}}[\mathfrak{A}u](t)\\ &= [\mathfrak{A}u](t_\star)\\ &\leqslant [\mathfrak{A}u](T)\\ &\leqslant \left(T+\overline{b}\beta+(m-2)\overline{b}\right)\left[\phi_q\circ\left(\frac{1}{\rho}\right)\mathscr{L}\overline{f}\right](T)\\ &< \frac{b}{N}\cdot\left(T+\beta\overline{b}+(m-2)\overline{b}\right)\phi_q\left(\frac{1}{\rho}\int_0^T h(\tau)\nabla\tau\right)\Delta s\\ &= b,\end{aligned}\tag{3.11}$$

which consequently leads to the validity of condition (ii) in Theorem 3.1.

Last, let us notice that the constant functions $(1/2)a \in \mathcal{P}(\alpha, a)$. Then, $\mathcal{P}(\alpha, a) \neq \emptyset$. Take $u \in \partial\mathcal{P}(\alpha, a)$. We thus obtain $\alpha(u) = \max_{t\in[0,t^\star]_{\mathbb{T}}} u(t) = u(t^\star) = a$. This, with the assumption

(B3), manifests that $0 \leqslant u(t) \leqslant a$ and $f(t, u(t)) > \phi_p(a/L)$ for all $t \in [t^\star, T]_{\mathbb{T}}$. Analogously, we can get

$$\begin{aligned}
\alpha(\mathfrak{A}u) &= [\mathfrak{A}u](t^\star) \\
&\geqslant \frac{t^\star}{T}[\mathfrak{A}u](\sigma(T)) \\
&\geqslant \frac{b\beta t^\star}{T}\phi_q\left(\left[\mathscr{L}\overline{f}\right](T)\right) \\
&> \frac{a}{L}\cdot\frac{t^\star}{T}\beta\underline{b}\phi_q\left(\int_{t^\star}^{T} h(\tau)\nabla\tau\right) \\
&= a,
\end{aligned} \tag{3.12}$$

which shows the validity of condition (iii) in Theorem 3.1.

Now, in the light of Theorem 3.1, we consequently arrive to the conclusion that the boundary value problems (1.4) and (1.5) admit at least two positive solutions, denoted by u_1 and u_2, satisfying $\min_{t\in[t_\star, t^\star]_{\mathbb{T}}} u_i(t) \leqslant c$, $(i = 1, 2)$, $a < \alpha(u_1)$ with $\theta(u_1) < b$, and $b < \theta(u_2)$ with $\gamma(u_2) < c$, respectively. □

4. At Least Three Positive Solutions in Boundary Value Problems

By means of the five functionals fixed point theorem which is attributed to Avery [20], this section is to analytically prove the existence of at least three positive solutions in the boundary value problems (1.4) and (1.5).

Take γ, β, θ as nonnegative continuous convex functionals on $\mathcal{P}$. Both α and ψ are supposed to be nonnegative continuous concave functionals on $\mathcal{P}$. Then, for nonnegative real numbers h, a, b, c, and d, construct five convex sets, respectively, through

$$\begin{aligned}
\mathcal{P}(\gamma, c) &= \{x \in \mathcal{P} \mid \gamma(x) < c\}, \\
\mathcal{P}(\gamma, \alpha, a, c) &= \{x \in \mathcal{P} \mid a \leqslant \alpha(x),\ \gamma(x) \leqslant c\}, \\
\mathcal{Q}(\gamma, \beta, d, c) &= \{x \in \mathcal{P} \mid \beta(x) \leqslant d,\ \gamma(x) \leqslant c\}, \\
\mathcal{P}(\gamma, \theta, \alpha, a, b, c) &= \{x \in \mathcal{P} \mid a \leqslant \alpha(x),\ \theta(x) \leqslant b,\ \gamma(x) \leqslant c\}, \\
\mathcal{Q}(\gamma, \beta, \psi, h, d, c) &= \{x \in \mathcal{P} \mid h \leqslant \psi(x),\ \beta(x) \leqslant d,\ \gamma(x) \leqslant c\}.
\end{aligned} \tag{4.1}$$

Theorem 4.1 ([20]). *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{E}$. Suppose that α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$, and that γ, β, and θ are nonnegative continuous convex functionals on $\mathcal{P}$ such that, for some positive numbers c and M,*

$$\alpha(x) \leqslant \beta(x), \qquad \|x\| \leqslant M\gamma(x) \tag{4.2}$$

for all $x \in \overline{\mathcal{P}(\gamma,c)}$. *In addition, suppose that* $\mathfrak{A} : \overline{\mathcal{P}(\gamma,c)} \mapsto \overline{\mathcal{P}(\gamma,c)}$ *is a completely continuous operator and that there exist nonnegative real numbers* $h,\ d,\ a,\ b$ *with* $0 < d < a$ *such that*

(i) $\{x \in \mathcal{P}(\gamma,\theta,\alpha,a,b,c) \mid \alpha(x) > a\} \neq \emptyset$ and $\alpha(\mathfrak{A}x) > a$ for $x \in \mathcal{P}(\gamma,\theta,\alpha,a,b,c)$;

(ii) $\{x \in \mathcal{Q}(\gamma,\beta,\psi,h,d,c) \mid \beta(x) < d\} \neq \emptyset$ and $\beta(\mathfrak{A}x) < d$ for $x \in \mathcal{Q}(\gamma,\beta,\psi,h,d,c)$;

(iii) $\alpha(\mathfrak{A}x) > a$ for $x \in \mathcal{P}(\gamma,\alpha,a,c)$ with $\theta(\mathfrak{A}x) > b$;

(iv) $\beta(\mathfrak{A}x) < d$ for $x \in \mathcal{Q}(\gamma,\beta,d,c)$ with $\psi(\mathfrak{A}x) < h$.

Then the operator $\mathfrak{A}$ *admits at least three fixed points* $x_1,x_2,x_3 \in \overline{\mathcal{P}(\gamma,c)}$ *satisfying* $\beta(x_1) < d$, $a < \alpha(x_2)$, *and* $d < \beta(x_3)$ *with* $\alpha(x_3) < a$, *respectively.*

In the light of this theorem, we can have the following result on the existence of at least three solutions in the boundary value problems (1.4) and (1.5).

Theorem 4.2. *Suppose that the hypotheses* (H1)–(H3) *are all fulfilled. Also suppose that there exist positive real numbers* a, b, *and* c *such that*

$$0 < a < b < c, \quad a < \frac{t_\star}{T} b < \frac{t_\star t^\star}{T^2} c, \qquad Nb < Mc. \tag{4.3}$$

Furthermore, let f *satisfies the following conditions:*

(C1) $f(t,u) < \phi_p(c/N)$ *for* $t \in [0,T]_{\mathbb{T}}$ *and* $u \in [0,(T/t_\star)c]$;

(C2) $f(t,u) > \phi_p(b/M)$ *for* $t \in [t^\star,T]_{\mathbb{T}}$ *and* $u \in [b,(T^2/t_\star^2)b]$;

(C3) $f(t,u) < \phi_p(a/N)$ *for* $t \in [0,T]_{\mathbb{T}}$ *and* $u \in [0,(T/t^\star)a]$.

Then, the boundary value problems (1.4) *and* (1.5) *possess at least three solutions* $u_1(t)$, $u_2(t)$, *and* $u_3(t)$, *defined on* $[0,\sigma(T)]_{\mathbb{T}}$, *satisfying, respectively,*

$$\begin{gathered}\max_{t\in[0,t_\star]_{\mathbb{T}}} u_i(t) \leqslant c, \qquad (i = 1,2,3), \\ \max_{t\in[0,t^\star]_{\mathbb{T}}} u_1(t) < a, \qquad b < \min_{t\in[t^\star,\sigma(T)]_{\mathbb{T}}} u_2(t), \\ a < \max_{t\in[0,t^\star]_{\mathbb{T}}} u_3(t) \quad \text{with} \quad \min_{t\in[t^\star,\sigma(T)]_{\mathbb{T}}} u_3(t) < b.\end{gathered} \tag{4.4}$$

Proof. Set the cone $\mathcal{P}$ as constructed in (2.28) and the operator $\mathfrak{A}$ as defined in (2.30). Take, respectively, the nonnegative continuous concave functionals on the $\mathcal{P}$ as follows:

$$\begin{aligned}\gamma(u) = \theta(u) &= \max_{t\in[0,t_\star]_{\mathbb{T}}} u(t) = u(t_\star), \\ \alpha(u) &= \min_{t\in[t^\star,\sigma(T)]_{\mathbb{T}}} u(t) = u(t^\star), \\ \beta(u) &= \max_{t\in[0,t^\star]_{\mathbb{T}}} u(t) = u(t^\star), \\ \psi(u) &= \min_{t\in[t_\star,\sigma(T)]_{\mathbb{T}}} u(t) = u(t_\star).\end{aligned} \tag{4.5}$$

Then, we get $\alpha(u) = \beta(u)$ for $u \in \mathcal{P}$. Besides, from Lemma 2.8, it follows that

$$\|u\| \leqslant \frac{T}{t_\star}\gamma(u) \tag{4.6}$$

for u $\in \mathcal{P}$. In what follows, we aim to show the validity of all the conditions in Theorem 4.1 with respect to the operator $\mathfrak{A}$.

To this end, arbitrarily take a function $u \in \overline{\mathcal{P}(\gamma, c)}$. Thus, $\gamma(u) = \max_{t\in[0,t_\star]_{\mathbb{T}}} u(t) = u(t_\star) \leqslant c$, which, combined with (4.6), gives $0 \leqslant u(t) \leqslant (T/t_\star)c$ for $t \in [0,T]_{\mathbb{T}}$ and $u \in \mathcal{P}$. Hence, we have $f(t,u(t)) < \phi_p(c/N)$ for $t \in [0,T]_{\mathbb{T}}$, due to the assumption (C1). Furthermore, since $\mathfrak{A}u \in \mathcal{P}$, in the light of the proof of Lemma 2.7, we have

$$\begin{aligned}
\|\gamma(\mathfrak{A}u)\| &= [\mathfrak{A}u](t_\star) \\
&\leqslant [\mathfrak{A}u](T) \\
&\leqslant \left(T + \overline{b}\beta + (m-2)\overline{b}\right)\left[\phi_q \circ \left(\frac{1}{\rho}\mathscr{H}\right)\overline{f}\right](T) \\
&< \frac{c}{N}\cdot\left(T + \beta\overline{b} + (m-2)\overline{b}\right)\phi_q\left(\frac{1}{\rho}\int_0^T h(\tau)\nabla\tau\right)\Delta s \\
&= c.
\end{aligned} \tag{4.7}$$

So, according to Lemma 2.9, we have the complete continuity of the operator $\mathfrak{A} : \overline{\mathcal{P}(\gamma, c)} \mapsto \overline{\mathcal{P}(\gamma, c)}$.

Moreover, the set

$$\left\{u \in \mathcal{P}\left(\gamma, \theta, \alpha, b, \frac{T}{t_\star}b, c\right) \mid \alpha(u) > b\right\} \tag{4.8}$$

is not empty, since the constant function $u(t) \equiv (T/t^\star)b$ is contained in the set $\{u \in \mathcal{P}(\gamma, \theta, \alpha, b, (T/t_\star)b, c) \mid \alpha(u) > b\}$. Similarly, the set

$$\left\{u \in Q\left(\gamma, \beta, \psi, \frac{t_\star}{T}a, a, c\right) \mid \beta(u) < a\right\} \tag{4.9}$$

is nonempty because of $u(t) \equiv (t^\star/T)a \in \{u \in Q(\gamma, \beta, \psi, (t_\star/T)a, a, c) \mid \beta(u) < a\}$. For a particular $u \in \mathcal{P}(\gamma, \theta, \alpha, b, (T/t_\star)b, c)$, the implementation of (4.6) gives

$$b \leqslant \min_{t\in[t^\star,\sigma(T)]_{\mathbb{T}}} u(t) = u(t^\star) \leqslant \|u\| \leqslant \frac{T}{t_\star}\gamma(u) = \frac{T}{t_\star}\theta(u) \leqslant \frac{T^2}{t_\star^2}b \tag{4.10}$$

for $t \in [t^\star, T]_{\mathbb{T}}$. The utilization of the assumption (C2) leads us to the inequality

$$f(t,u(t)) > \phi_p\left(\frac{b}{M}\right) \quad \text{for } t \in [t^\star, T]_{\mathbb{T}}. \tag{4.11}$$

Thus, it follows from (4.11) and Lemmas 2.7 and 2.8 that

$$\begin{aligned}\alpha(\mathfrak{A}u) &= [\mathfrak{A}u](t_\star)\\ &\geqslant \frac{t_\star}{T}[\mathfrak{A}]u(\sigma(T))\\ &\geqslant \frac{t_\star}{T}\beta\underline{b}\phi_q\left(\left[\mathscr{L}\overline{f}\right](T)\right)\\ &> \frac{b}{M}\cdot\frac{t_\star}{T}\beta\underline{b}\cdot\phi_q\left(\int_{t^\star}^{T}h(\tau)\nabla\tau\right)\\ &= b.\end{aligned}\tag{4.12}$$

Clearly, we verify the validity of condition (i) in Theorem 4.1.

Next, consider $u \in Q(\gamma,\beta,\psi,(t_\star/T)a,a,c)$. In such a case, we obtain

$$0 \leqslant u(t) \leqslant \frac{T}{t^\star}a \tag{4.13}$$

for $t \in [0,T]_{\mathbb{T}}$. Imposing the assumption (C3) produces $f(t,u(t)) < \phi_p(a/N)$. Moreover, by the proof of Lemma 2.7, we obtain

$$\begin{aligned}\beta(\mathfrak{A}u) &= [\mathfrak{A}u](t^\star)\\ &\leqslant \left(T+\overline{b}\beta+(m-2)\overline{b}\right)\left[\phi_q\circ\left(\frac{1}{\rho}\right)\mathscr{L}\overline{f}\right](T)\\ &< \frac{a}{N}\cdot\left(T+\beta\overline{b}+(m-2)\overline{b}\right)\phi_q\left(\frac{1}{\rho}\int_0^T h(\tau)\nabla\tau\right)\Delta s\\ &= a.\end{aligned}\tag{4.14}$$

Therefore, we further verify the validity of condition (ii) in Theorem 4.1.

Finally, we are to validate conditions (iii) and (iv) aside from conditions (i) and (ii). For this purpose, on the one hand, let us consider $u \in \mathcal{P}(\gamma,\alpha,b,c)$ with $\theta(\mathfrak{A}u) > (T/t_\star)b$. Then, we have

$$\alpha(\mathfrak{A}u) = [\mathfrak{A}u](t^\star) \geqslant [\mathfrak{A}u](t_\star) = \theta(\mathfrak{A}u) > \frac{T}{t_\star}b > b. \tag{4.15}$$

On the other hand, consider $u \in Q(\gamma,\beta,a,c)$ with $\psi(\mathfrak{A}u) < (t_\star/T)a$. In this case, we get

$$\beta(\mathfrak{A}u) = [\mathfrak{A}u](t^\star) \leqslant \frac{t^\star}{t_\star}[\mathfrak{A}u](t_\star) = \frac{t^\star}{t_\star}\psi(\mathfrak{A}u) < \frac{t^\star}{T}a < a. \tag{4.16}$$

Accordingly, both conditions (iii) and (iv) in Theorem 4.1 are satisfied. Now, in light of Theorem 4.1, the boundary value problems (1.4) and (1.5) have at least three positive solutions circumscribed on $[0,\sigma(T)]_{\mathbb{T}}$ satisfying $\max_{t\in[t_\star,T]_{\mathbb{T}}} u_1(t) < a$, $b < \min_{t\in[0,t_\star]_{\mathbb{T}}} u_2(t)$, and $a < \max_{t\in[t_\star,T]_{\mathbb{T}}} u_3(t)$ with $\min_{t\in[t^\star,\sigma(T)]_{\mathbb{T}}} u_3(t) < b$. □

5. An Illustrative Example

This section will provide a nontrivial example to clearly illustrate the feasibility of the above-established analytical results on the dynamic equations on time scales.

First of all, construct a nontrivial time scale set as $\mathbb{T} = \{1-(1/2)^{\mathbb{N}^0}\} \cup [1,2] \cup \{3\}$. Set all the parameters in problems (1.4) and (1.5) as follows: $T = 2$, $0 < \sigma\ (T) = 3$, $p = 3/2$, $q = 3$, $m = 4$, $a_1 = a_2 = 1/4$, $\underline{b} = \overline{b} = 1/2$, $\beta = 2$, $\xi_1 = 1/2$, $\xi_2 = 1$, $t^\star = 1$, and $t_\star = 1/2$, so that $\rho = 1/2$. For simplicity but without loss of generality, set $h(t) \equiv 1$. we can obtain

$$M = \frac{\underline{b}\beta t_\star}{T} \cdot \phi_q\left(\int_{t_\star}^{T} h(\tau)\nabla\tau\right) = \frac{9}{16},$$
$$N = \left(T + \beta\overline{b} + (m-2)\overline{b}\right)\phi_q\left(\frac{1}{\rho}\int_{0}^{T} h(\tau)\nabla\tau\right)\Delta s = 64. \tag{5.1}$$

In particular, set the function in dynamic equation as

$$f(t,u) = \frac{2000u}{t+4000+u}, \quad t \in [0,2]_{\mathbb{T}},\ u \geqslant 0. \tag{5.2}$$

This setting allows us to properly take the other parameters as $a = 1/N$, $b = 9$, and $c = 256 \times 10^6$. It is clear that these parameters satisfy

$$0 < a < \frac{t_\star}{T} b < \frac{t_\star t^\star}{T^2} c, \qquad Nb < Mc. \tag{5.3}$$

To this end, we can verify the validity of conditions (C1)–(C3) in Theorem 4.2. As a matter of fact, direct calculations produce

$$f(t,u) \leqslant \frac{2000u}{4000+u} \leqslant \frac{4000c}{4000+2c} < 2000 = \left(\frac{c}{N}\right)^{1/2} = \phi_p\left(\frac{c}{N}\right), \tag{5.4}$$

as $t \in [0,T]_{\mathbb{T}}$ and $u \in [0,(T/t_\star)c]$,

$$f(t,u) \geqslant \frac{2000b}{T+4000+b} = \frac{18000}{4011} > 4 = \phi_p\left(\frac{b}{M}\right), \tag{5.5}$$

as $t \in [t^\star,T]_{\mathbb{T}}$ and $u \in [b,(T^2/t_\star^2)b]$, and

$$f(t,u) \leqslant \frac{2000u}{4000+u} \leqslant \frac{4000a}{4000+4a} < a = \phi_p\left(\frac{a}{N}\right), \tag{5.6}$$

$t \in [0,T]_{\mathbb{T}}$ and $u \in [0,(T/t^{\star})a]$. Accordingly, conditions (C1)–(C3) in Theorem 4.2 are satisfied for the above specified functions and parameters. Now, by virtue of Theorem 4.2, we can approach a conclusion that the dynamic equation on the specified time scales

$$\left[\left(u^{\Delta}\right)^{1/2}\right]^{\nabla} + \frac{2000u}{t+4000+u} = 0, \quad t \in (0,2]_{\mathbb{T}} \tag{5.7}$$

with the boundary conditions

$$\begin{gathered} u(0) - 2u^{\Delta}(0) = \frac{1}{2}u^{\Delta}\left(\frac{1}{2}\right) + \frac{1}{2}u^{\Delta}(1), \\ \left[u^{\Delta}(T)\right]^{1/2} = \frac{1}{4}\left[u^{\Delta}\left(\frac{1}{2}\right)\right]^{1/2} + \frac{1}{4}\left[u^{\Delta}(1)\right]^{1/2} \end{gathered} \tag{5.8}$$

possesses at least three positive solutions defined on $[0,\sigma(T)]_{\mathbb{T}}$ satisfying $\max_{t\in[0,t_{\star}]_{\mathbb{T}}} u_i(t) \leqslant c$, $(i = 1,2,3)$, $\max_{t\in[0,t^{\star}]_{\mathbb{T}}} u_1(t) < a$, $b < \min_{t\in[t^{\star},\sigma(T)]_{\mathbb{T}}} u_2(t)$, and $a < \max_{t\in[0,t^{\star}]_{\mathbb{T}}} u_3(t)$ with $\min_{t\in[0,t_{\star}]_{\mathbb{T}}} u_3(t) < b$.

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References

[1] B. Aulbach and S. Hilger, "Linear dynamic processes with inhomogeneous time scale," in *Nonlinear Dynamics and Quantum Dynamical Systems (Gaussig, 1990)*, vol. 59 of *Mathematical Research*, pp. 9–20, Akademie, Berlin, Germany, 1990.
[2] R. P. Agarwal, M. Bohner, and P. Rehák, "Half-linear dynamic equations," in *Nonlinear Analysis and Applications: To V. Lakshmikantham on His 80th Birthday. Vol. 1, 2*, pp. 1–57, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.
[3] R. P. Agarwal and D. O'Regan, "Nonlinear boundary value problems on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, no. 4, pp. 527–535, 2001.
[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
[5] M. Bohner and A. Peterosn, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.
[6] F. M. Atici, D. C. Biles, and A. Lebedinsky, "An application of time scales to economics," *Mathematical and Computer Modelling*, vol. 43, no. 7-8, pp. 718–726, 2006.
[7] J. Hoffacker and C. C. Tisdell, "Stability and instability for dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 49, no. 9-10, pp. 1327–1334, 2005.
[8] W. Zhong, W. Lin, and J. Ruan, "The generalized invariance principle for dynamic equations on time scales," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 557–565, 2007.
[9] D. R. Anderson, "Solutions to second-order three-point problems on time scales," *Journal of Difference Equations and Applications*, vol. 8, no. 8, pp. 673–688, 2002.
[10] D. Anderson, R. Avery, J. Davis, J. Henderson, and W. Yin, "Positive solutions of boundary value problems," in *Advances in Dynamic Equations on Time Scales*, pp. 189–249, Birkhäuser, Boston, Mass, USA, 2003.

[11] D. R. Anderson, "Nonlinear triple-point problems on time scales," *Electronic Journal of Differential Equations*, vol. 47, pp. 1–12, 2004.

[12] J. J. DaCunha, J. M. Davis, and P. K. Singh, "Existence results for singular three point boundary value problems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 2, pp. 378–391, 2004.

[13] E. R. Kaufmann, "Positive solutions of a three-point boundary-value problem on a time scale," *Electronic Journal of Differential Equations*, vol. 82, pp. 1–11, 2003.

[14] Z. He, "Existence of two solutions of m-point boundary value problem for second order dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 296, no. 1, pp. 97–109, 2004.

[15] İ. Yaslan, "Existence of positive solutions for nonlinear three-point problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 888–897, 2007.

[16] H.-R. Sun and W.-T. Li, "Multiple positive solutions for p-Laplacian m-point boundary value problems on time scales," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 478–491, 2006.

[17] Z. He, "Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 182, no. 2, pp. 304–315, 2005.

[18] Y.-H. Su and W.-T. Li, "Triple positive solutions of m-point BVPs for p-Laplacian dynamic equations on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 11, pp. 3811–3820, 2008.

[19] Z. He and L. Li, "Multiple positive solutions for the one-dimensional p-Laplacian dynamic equations on time scales," *Mathematical and Computer Modelling*, vol. 45, no. 1-2, pp. 68–79, 2007.

[20] R. I. Avery, "A generalization of the Leggett-Williams fixed point theorem," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 7, pp. 9–14, 1999.

[21] İ. Yaslan, "Multiple positive solutions for nonlinear three-point boundary value problems on time scales," *Computers & Mathematics with Applications*, vol. 55, no. 8, pp. 1861–1869, 2008.

[22] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.

[23] H.-R. Sun and Y.-H. Wang, "Existence of positive solutions for p-Laplacian three-point boundary-value problems on time scales," *Electronic Journal of Differential Equations*, vol. 92, pp. 1–14, 2008.

[24] R. I. Avery and J. Henderson, "Two positive fixed points of nonlinear operators on ordered Banach spaces," *Communications on Applied Nonlinear Analysis*, vol. 8, no. 1, pp. 27–36, 2001.