Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 829735, 21 pages
doi:10.1155/2009/829735

Research Article

Positive Solutions for Boundary Value Problems of Second-Order Functional Dynamic Equations on Time Scales

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Received 3 February 2009; Accepted 26 February 2009

Recommended by Alberto Cabada

Criteria are established for existence of least one or three positive solutions for boundary value problems of second-order functional dynamic equations on time scales. By this purpose, we use a fixed-point index theorem in cones and Leggett-Williams fixed-point theorem.

1. Introduction

In a recent paper [1], by applying a fixed-point index theorem in cones, Jiang and Weng studied the existence of positive solutions for the boundary value problems described by second-order functional differential equations of the form

$$\begin{gathered} y''(x) + r(x) f(y(\omega(x))) = 0, \quad 0 < x < 1, \\ \alpha y(x) - \beta y'(x) = \xi(x), \quad a \le x \le 0, \\ \gamma y(x) + \delta y'(x) = \eta(x), \quad 1 \le x \le b. \end{gathered} \tag{1.1}$$

Aykut [2] applied a cone fixed-point index theorem in cones and obtained sufficient conditions for the existence of positive solutions of the boundary value problems of functional difference equations of the form

$$\begin{gathered} -\Delta^2 y(n-1) = f(n, y(\omega(n))), \quad n \in [a,b], \\ \alpha y(n-1) - \beta \Delta y(n-1) = \xi(n), \quad n \in [\tau_1, a], \\ \gamma y(n) + \delta \Delta y(n) = \eta(n), \quad n \in [b, \tau_2]. \end{gathered} \tag{1.2}$$

In this article, we are interested in proving the existence and multiplicity of positive solutions for the boundary value problems of a second-order functional dynamic equation of the form

$$\begin{gathered} -y^{\Delta\nabla}(t) + q(t)y(t) = f\left(t, y(\omega(t))\right), \quad t \in [a,b], \\ \alpha y(\rho(t)) - \beta y^{\Delta}(\rho(t)) = \xi(t), \quad t \in [\tau_1, a], \\ \gamma y(t) + \delta y^{\Delta}(t) = \eta(t), \quad t \in [b, \tau_2]. \end{gathered} \tag{1.3}$$

Throughout this paper we let $\mathbb{T}$ be any time scale (nonempty closed subset of $\mathbb{R}$) and $[a,b]$ be a subset of $\mathbb{T}$ such that $[a,b] = \{t \in \mathbb{T},\ a \le t \le b\}$, and for $t \in [\tau_1, a], t$ is not right scattered and left dense at the same time.

Some preliminary definitions and theorems on time scales can be found in books [3, 4] which are excellent references for calculus of time scales.

We will assume that the following conditions are satisfied.

(H1) $q(t) \in C[a,b]$, $q(t) \ge 0$.

(H2) $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to y and $f(t,y) \ge 0$ for $y \in \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

(H3) $\omega(t)$ defined on $[a,b]$ satisfies

$$\begin{gathered} c = \inf\left\{\omega(t) : a \le t \le b\right\} < b, \\ d = \sup\left\{\omega(t) : a \le t \le b\right\} > a. \end{gathered} \tag{1.4}$$

Let $E_1 := \{t \in E : a \le \omega(t) \le \rho(b)\}$ be nonempty subset of

$$E := \left\{t \in [a,b] : a \le \omega(t) \le b\right\}. \tag{1.5}$$

(H4) $\alpha, \beta, \gamma, \delta \ge 0$, $\alpha + \beta > 0$, $\gamma + \delta > 0$;

if $q(t) \equiv 0$ $(a \le t \le b)$, then $\alpha + \gamma > 0$; $-\gamma/\delta \in \mathcal{R}^+$ for $\delta > 0$, where $\mathcal{R}^+$ denotes the set of all positively regressive and rd-continuous functions.

(H5) $\xi(t)$ and $\eta(t)$ are defined on $[\tau_1, \sigma(a)]$ and $[b, \sigma(\tau_2)]$, respectively, where $\tau_1 := \min\{a, c\}, \tau_2 := \max\{b, d\}$; furthermore, $\xi(a) = \eta(b) = 0$;

$$\begin{gathered} \xi(\sigma(t)) \ge 0, \quad \text{for } t \in [\tau_1, a] \text{ as } \beta = 0; \\ \int_t^a \xi(s) e_{\ominus(\alpha/\beta)}(s,0) \nabla s \ge 0, \quad \text{for } t \in [\tau_1, a] \text{ as } \beta > 0; \\ \eta(t) \ge 0, \quad \text{for } t \in [b, \tau_2] \text{ as } \delta = 0; \\ e_{-\gamma/\delta}(t,0) \int_b^t \eta(s) e_{\ominus(-\gamma/\delta)}(\sigma(s),0) \Delta s \ge 0, \quad \text{for } t \in [b, \tau_2] \text{ as } \delta > 0. \end{gathered} \tag{1.6}$$

There have been a number of works concerning of at least one and multiple positive solutions for boundary value problems recent years. Some authors have studied the problem for ordinary differential equations, while others have studied the problem for difference equations, while still others have considered the problem for dynamic equations on a time scale [5–10]. However there are fewer research for the existence of positive solutions of the boundary value problems of functional differential, difference, and dynamic equations [1, 2, 11–13].

Our problem is a dynamic analog of the BVPs (1.1) and (1.2). But it is more general than them. Moreover, conditions for the existence of at least one positive solution were studied for the BVPs (1.1) and (1.2). In this paper, we investigate the conditions for the existence of at least one or three positive solutions for the BVP (1.3). The key tools in our approach are the following fixed-point index theorem [14], and Leggett-Williams fixed-point theorem [15].

Theorem 1.1 (see [14]). *Let E be Banach space and $K \subset E$ be a cone in E. Let $r > 0$, and define $\Omega_r = \{x \in K : \|x\| < r\}$. Assume $A : \overline{\Omega}_r \to K$ is a completely continuous operator such that $Ax \neq x$ for $x \in \partial\Omega_r$.*

(i) *If $\|Au\| \leq \|u\|$ for $u \in \partial\Omega_r$, then $i(A, \Omega_r, K) = 1$.*

(ii) *If $\|Au\| \geq \|u\|$ for $u \in \partial\Omega_r$, then $i(A, \Omega_r, K) = 0$.*

Theorem 1.2 (see [15]). *Let $\mathcal{P}$ be a cone in the real Banach space E. Set*

$$\begin{aligned} \mathcal{P}_r &:= \{x \in \mathcal{P} : \|x\| < r\}, \\ \mathcal{P}(\psi, p, q) &:= \{x \in \mathcal{P} : p \leq \psi(x),\ \|x\| \leq q\}. \end{aligned} \tag{1.7}$$

Suppose that $A : \overline{\mathcal{P}}_r \to \overline{\mathcal{P}}_r$ is a completely continuous operator and ψ is a nonnegative continuous concave functional on $\mathcal{P}$ with $\psi(x) \leq \|x\|$ for all $x \in \overline{\mathcal{P}}_r$. If there exists $0 < p < q < s \leq r$ such that the following conditions hold:

(i) *$\{x \in \mathcal{P}(\psi, q, s) : \psi(x) > q\} \neq \{\}$ and $\psi(Ax) > q$ for all $x \in \mathcal{P}(\psi, q, s)$;*

(ii) *$\|Ax\| < p$ for $\|x\| \leq p$;*

(iii) *$\psi(Ax) > q$ for $x \in \mathcal{P}(\psi, q, r)$ with $\|Ax\| > s$.*

Then A has at least three fixed points x_1, x_2, and x_3 in $\overline{\mathcal{P}}_r$ satisfying

$$\|x_1\| < p, \quad \psi(x_2) > q, \quad p < \|x_3\| \quad \text{with } \psi(x_3) < q. \tag{1.8}$$

2. Preliminaries

First, we give the following definitions of solution and positive solution of BVP (1.3).

Definition 2.1. We say a function $y(t)$ is a solution of BVP (1.3) if it satisfies the following.

(1) $y(t)$ is nonnegative on $[\rho(\tau_1), \sigma(\tau_2)]$.

(2) $y(t) = y(\tau_1; t)$ as $t \in [\rho(\tau_1), a]$, where $y(\tau_1; t) : [\rho(\tau_1), a] \to [0, \infty)$ is defined as

$$y(\tau_1; t) = \begin{cases} e_{\alpha/\beta}(t,0)\left[\dfrac{1}{\beta}\displaystyle\int_t^a \xi(s) e_{\ominus(\alpha/\beta)}(s,0)\nabla s + e_{\ominus(\alpha/\beta)}(a,0) y(a)\right], & \text{if } \beta > 0, \\ \dfrac{1}{\alpha}\xi(\sigma(t)), & \text{if } \beta = 0. \end{cases} \tag{2.1}$$

(3) $y(t) = y(\tau_2; t)$ as $t \in [b, \sigma(\tau_2)]$, where $y(\tau_2; t) : [b, \sigma(\tau_2)] \to [0, \infty)$ is defined as

$$y(\tau_2; t) = \begin{cases} e_{-\gamma/\delta}(t,0)\left[\dfrac{1}{\delta}\displaystyle\int_b^t \eta(s) e_{\ominus(-\gamma/\delta)}(\sigma(s),0)\Delta s + e_{\ominus(-\gamma/\delta)}(b,0) y(b)\right], & \text{if } \delta > 0, \\ \dfrac{1}{\gamma}\eta(t), & \text{if } \delta = 0. \end{cases} \tag{2.2}$$

(4) y is Δ-differentiable, $y^{\Delta} : [\rho(a), b] \to \mathbb{R}$ is ∇-differentiable on $[a, b]$ and $(y^{\Delta})^{\nabla}$: $[a, b] \to \mathbb{R}$ is continuous.

(5) $-y^{\Delta\nabla}(t) + q(t)y(t) = f(t, y(\omega(t)))$, for $t \in [a, b]$.

Furthermore, a solution $y(t)$ of (1.3) is called a positive solution if $y(t) > 0$ for $t \in [a, b]$. Denote by $\varphi(t)$ and $\psi(t)$ the solutions of the corresponding homogeneous equation

$$-y^{\Delta\nabla}(t) + q(t)y(t) = 0, \quad t \in [a, b], \tag{2.3}$$

under the initial conditions

$$\begin{aligned} \varphi(\rho(a)) &= \beta, & \varphi^{\Delta}(\rho(a)) &= \alpha, \\ \psi(b) &= \delta, & \psi^{\Delta}(b) &= -\gamma. \end{aligned} \tag{2.4}$$

Set

$$D = W_t(\psi, \varphi) = \psi(t)\varphi^{\Delta}(t) - \psi^{\Delta}(t)\varphi(t). \tag{2.5}$$

Since the Wronskian of any two solutions of (2.3) is independent of t, evaluating at $t = \rho(a)$, $t = b$ and using the initial conditions (2.4) yield

$$D = \alpha\psi(\rho(a)) - \beta\psi^{\Delta}(\rho(a)) = \gamma\varphi(b) + \delta\varphi^{\Delta}(b). \tag{2.6}$$

Using the initial conditions (2.4), we can deduce from (2.3) for $\varphi(t)$ and $\psi(t)$, the following equations:

$$\varphi(t) = \beta + \alpha\left(t - \rho(a)\right) + \int_{\rho(a)}^{t}\int_{\rho(a)}^{\tau} q(s)\varphi(s)\nabla s\,\Delta\tau, \tag{2.7}$$

$$\psi(t) = \delta + \gamma(b - t) + \int_{t}^{b}\int_{\tau}^{b} q(s)\psi(s)\nabla s\,\Delta\tau. \tag{2.8}$$

(See [8].)

Lemma 2.2 (see [8]). *Under the conditions (H1) and the first part of (H4) the solutions $\varphi(t)$ and $\psi(t)$ possess the following properties:*

$$\begin{aligned}
&\varphi(t) \geq 0, \quad t \in \left[\rho(a), \sigma(b)\right], \qquad &&\psi(t) \geq 0, \quad t \in \left[\rho(a), b\right],\\
&\varphi(t) > 0, \quad t \in \left(\rho(a), \sigma(b)\right], \qquad &&\psi(t) > 0, \quad t \in \left[\rho(a), b\right),\\
&\varphi^{\Delta}(t) \geq 0, \quad t \in \left[\rho(a), b\right], \qquad &&\psi^{\Delta}(t) \leq 0, \quad t \in \left[\rho(a), b\right].
\end{aligned} \tag{2.9}$$

Let $G(t, s)$ be the Green function for the boundary value problem:

$$\begin{gathered}
-y^{\Delta\nabla}(t) + q(t)y(t) = 0, \quad t \in [a, b],\\
\alpha y\left(\rho(a)\right) - \beta y^{\Delta}\left(\rho(a)\right) = 0,\\
\gamma y(b) + \delta y^{\Delta}(b) = 0,
\end{gathered} \tag{2.10}$$

given by

$$G(t, s) = \frac{1}{D}\begin{cases} \psi(t)\varphi(s), & \text{if } \rho(a) \leq s \leq t \leq \sigma(b),\\ \psi(s)\varphi(t), & \text{if } \rho(a) \leq t \leq s \leq \sigma(b), \end{cases} \tag{2.11}$$

where $\varphi(t)$ and $\psi(t)$ are given in (2.7) and (2.8), respectively. It is obvious from (2.6), (H1) and (H4), that $D > 0$ holds.

Lemma 2.3. *Assume the conditions (H1) and (H4) are satisfied. Then*

(i) $0 \leq G(t, s) \leq G(s, s)$ *for* $t, s \in [\rho(a), b]$,

(ii) $G(t, s) \geq \Gamma G(s, s)$ *for* $t \in [a, \rho(b)]$ *and* $s \in [\rho(a), b]$,

where

$$\Gamma = \min\left\{I_1, I_2\right\}, \tag{2.12}$$

in which

$$I_1 = \{\delta + (b-\rho(b))[\gamma + \delta q(b)(b-\rho(b))]\} \cdot \left\{\delta + \gamma(b-\rho(a)) + \int_{\rho(a)}^{b}\int_{\tau}^{b} q(s)\psi(s)\nabla s\,\Delta\tau\right\}^{-1},$$
$$I_2 = \{\beta + \alpha(a-\rho(a))\} \cdot \left\{\beta + \alpha(b-\rho(a)) + \int_{\rho(a)}^{b}\int_{\rho(a)}^{\tau} q(s)\varphi(s)\nabla s\,\Delta\tau\right\}^{-1}. \tag{2.13}$$

Proof. $\varphi(t) \geq 0$, for $t \in [\rho(a), \sigma(b)]$, and $\psi(t) \geq 0$, for $t \in [\rho(a), b]$. Besides, $\varphi(t)$ is nondecreasing and $\psi(t)$ is nonincreasing, for $t \in [\rho(a), b]$. Therefore, we have

$$0 \leq \frac{1}{D}\begin{cases} \psi(t)\varphi(s), & \text{if } \rho(a) \leq s \leq t \leq \sigma(b) \\ \psi(s)\varphi(t), & \text{if } \rho(a) \leq t \leq s \leq \sigma(b) \end{cases} \leq G(s,s). \tag{2.14}$$

So statement (i) of the lemma is proved. If $G(s,s) = 0$, for a given $s \in [\rho(a), b]$, then statement (ii) of the lemma is obvious for such values. Now, $s \in [\rho(a), b]$ and $G(s,s) \neq 0$. Consequently, $G(s,s) > 0$, for all such s. Let us take any $t \in [a, \rho(b)]$. Then we have for $s \in [\rho(a), t]$,

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{\psi(t)}{\psi(s)} \geq \frac{\psi(\rho(b))}{\psi(\rho(a))} \\ &= \{\delta + (b-\rho(b))[\gamma + \delta q(b)(b-\rho(b))]\} \\ &\quad \cdot \left\{\delta + \gamma(b-\rho(a)) + \int_{\rho(a)}^{b}\int_{\tau}^{b} q(s)\psi(s)\nabla s\,\Delta\tau\right\}^{-1}, \end{aligned} \tag{2.15}$$

and we have for $s \in [t, b]$,

$$\frac{G(t,s)}{G(s,s)} = \frac{\varphi(t)}{\varphi(s)} \geq \frac{\varphi(a)}{\varphi(b)} = \{\beta + \alpha(a-\rho(a))\} \cdot \left\{\beta + \alpha(b-\rho(a)) + \int_{\rho(a)}^{b}\int_{\rho(a)}^{\tau} q(s)\varphi(s)\nabla s\,\Delta\tau\right\}^{-1}. \tag{2.16}$$

Let $\mathcal{B} = C[\rho(\tau_1), \sigma(\tau_2)]$ be endowed with maximum norm $\|y\| := \max_{\rho(\tau_1)\leq t\leq\sigma(\tau_2)}|y(t)|$ for $y \in \mathcal{B}$, and let $K \subset \mathcal{B}$ be a cone defined by

$$K = \left\{y \in \mathcal{B} : \min_{a\leq t\leq\rho(b)} y(t) \geq \Gamma\|y\|\right\}, \tag{2.17}$$

where Γ is as in (2.12).

Suppose that $y(t)$ is a solution of (1.3), then it can be written as

$$y(t) = \begin{cases} y(\tau_1; t), & \text{if } \rho(\tau_1) \leq t \leq a, \\ \displaystyle\int_{\rho(a)}^{b} G(t,s) f(s, y(\omega(s)))\nabla s, & \text{if } a \leq t \leq b, \\ y(\tau_2; t), & \text{if } b \leq t \leq \sigma(\tau_2), \end{cases} \tag{2.18}$$

where

$$y(\tau_1;t)=\begin{cases}e_{\alpha/\beta}(t,0)\left[\dfrac{1}{\beta}\displaystyle\int_t^a \xi(s)e_{\ominus(\alpha/\beta)}(s,0)\nabla s+e_{\ominus(\alpha/\beta)}(a,0)y(a)\right], & \text{if } \rho(\tau_1)\le t\le a,\ \beta>0,\\ \dfrac{1}{\alpha}\xi(\sigma(t)), & \text{if } \rho(\tau_1)\le t\le a,\ \beta=0,\end{cases}$$

$$y(\tau_2;t)=\begin{cases}e_{-\gamma/\delta}(t,0)\left[\dfrac{1}{\delta}\displaystyle\int_b^t \eta(s)e_{\ominus(-\gamma/\delta)}(\sigma(s),0)\Delta s\right.\\ \qquad\qquad\left.+e_{\ominus(-\gamma/\delta)}(b,0)y(b)\right], & \text{if } b\le t\le\sigma(\tau_2),\ \delta>0,\\ \dfrac{1}{\gamma}\eta(t), & \text{if } b\le t\le\sigma(\tau_2),\ \delta=0.\end{cases}\tag{2.19}$$

Throughout this paper we assume that $u_0(t)$ is the solution of (1.3) with $f\equiv 0$. Clearly, $u_0(t)$ can be expressed as follows:

$$u_0(t)=\begin{cases}u_0(\tau_1;t), & \text{if } \rho(\tau_1)\le t\le a,\\ 0, & \text{if } a\le t\le b,\\ u_0(\tau_2;t), & \text{if } b\le t\le\sigma(\tau_2),\end{cases}\tag{2.20}$$

where

$$u_0(\tau_1;t)=\begin{cases}\dfrac{1}{\beta}e_{\alpha/\beta}(t,0)\displaystyle\int_t^a \xi(s)e_{\ominus(\alpha/\beta)}(s,0)\nabla s, & \text{if } \rho(\tau_1)\le t\le a,\ \beta>0,\\ \dfrac{1}{\alpha}\xi(\sigma(t)), & \text{if } \rho(\tau_1)\le t\le a,\ \beta=0,\end{cases}$$
$$u_0(\tau_2;t)=\begin{cases}\dfrac{1}{\delta}e_{-\gamma/\delta}(t,0)\displaystyle\int_b^t \eta(s)e_{\ominus(-\gamma/\delta)}(\sigma(s),0)\Delta s, & \text{if } b\le t\le\sigma(\tau_2),\ \delta>0,\\ \dfrac{1}{\gamma}\eta(t), & \text{if } b\le t\le\sigma(\tau_2),\ \delta=0.\end{cases}\tag{2.21}$$

Let $y(t)$ be a solution of (1.3) and $u(t)=y(t)-u_0(t)$. Noting that $u(t)\equiv y(t)$ for $t\in[a,b]$, we have

$$u(t)=\begin{cases}u(\tau_1;t), & \text{if } \rho(\tau_1)\le t\le a,\\ \displaystyle\int_{\rho(a)}^b G(t,s)f(s,u(\omega(s))+u_0(\omega(s)))\nabla s, & \text{if } a\le t\le b,\\ u(\tau_2;t), & \text{if } b\le t\le\sigma(\tau_2),\end{cases}\tag{2.22}$$

where

$$u(\tau_1;t) = \begin{cases} e_{\alpha/\beta}(t,a)y(a), & \text{if } \rho(\tau_1) \le t \le a,\ \beta > 0, \\ 0, & \text{if } \rho(\tau_1) \le t \le a,\ \beta = 0, \end{cases}$$
$$u(\tau_2;t) = \begin{cases} e_{-\gamma/\delta}(t,b)y(b), & \text{if } b \le t \le \sigma(\tau_2),\ \delta > 0, \\ 0, & \text{if } b \le t \le \sigma(\tau_2),\ \delta = 0. \end{cases} \tag{2.23}$$

Define an operator $A : K \to K$ as follows:

$$(Au)(t) := \begin{cases} (A_1u)(t), & \text{if } \rho(\tau_1) \le t \le a, \\ \displaystyle\int_{\rho(a)}^{b} G(t,s) f(s, u(\omega(s)) + u_0(\omega(s)))\nabla s, & \text{if } a \le t \le b, \\ (A_2u)(t), & \text{if } b \le t \le \sigma(\tau_2), \end{cases} \tag{2.24}$$

where

$$(A_1u)(t) := \begin{cases} e_{\alpha/\beta}(t,a)\displaystyle\int_{\rho(a)}^{b} G(a,s) f(s, u(\omega(s)) + u_0(\omega(s)))\nabla s, & \text{if } \rho(\tau_1) \le t \le a,\ \beta > 0, \\ 0, & \text{if } \rho(\tau_1) \le t \le a,\ \beta = 0, \end{cases}$$
$$(A_2u)(t) := \begin{cases} e_{-\gamma/\delta}(t,b)\displaystyle\int_{\rho(a)}^{b} G(b,s) f(s, u(\omega(s)) + u_0(\omega(s)))\nabla s, & \text{if } b \le t \le \sigma(\tau_2),\ \delta > 0, \\ 0, & \text{if } b \le t \le \sigma(\tau_2),\ \delta = 0. \end{cases} \tag{2.25}$$

It is easy to derive that y is a positive solution of BVP (1.3) if and only if $u = y - u_0$ is a nontrivial fixed point u of $A : K \to K$, where u_0 be defined as before. □

Lemma 2.4. *$A(K) \subset K$.*

Proof. For $u \in K$, we have $Au(t) \ge 0$, $t \in [\rho(\tau_1), \sigma(\tau_2)]$. Moreover, we have from definition of A that $Au(t) \le Au(a)$ and $Au(t) \le Au(b)$, for $t \in [\rho(\tau_1), a]$ and $t \in [b, \sigma(\tau_2)]$, respectively. Thus, $\|Au\| = \|Au\|_{[a,b]}$, where $\|Au\|_{[a,b]} = \max\{|Au(t)| : a \le t \le b\}$. It follows from the definition K and Lemma 2.3 that

$$\begin{aligned} \min_{a \le t \le \rho(b)} (Au)(t) &= \min_{a \le t \le \rho(b)} \int_{\rho(a)}^{b} G(t,s) f(s, u(\omega(s)) + u_0(\omega(s)))\nabla s \\ &\ge \Gamma \int_{\rho(a)}^{b} G(s,s) f(s, u(\omega(s)) + u_0(\omega(s)))\nabla s \\ &\ge \Gamma \max_{a \le t \le b} \int_{\rho(a)}^{b} G(t,s) f(s, u(\omega(s)) + u_0(\omega(s)))\nabla s \\ &\ge \Gamma \|Au\|_{[a,b]} = \Gamma \|Au\|, \quad t \in [a,b]. \end{aligned} \tag{2.26}$$

Thus, $A(K) \subset K$. □

Lemma 2.5. *$A : K \to K$ is completely continuous.*

Lemma 2.6. *If*

$$\lim_{v\to 0^+}\frac{f(t,v)}{v}=\infty,\qquad \lim_{v\to +\infty}\frac{f(t,v)}{v}=\infty, \tag{2.27}$$

for all $t\in[a,b]$, then there exist $0<r_0<R_0<\infty$ such that $i(A,K_r,K)=0$, for $0<r\le r_0$ and $i(A,K_R,K)=0$, for $R\ge R_0$.

Proof. Choose $M>0$ such that

$$\Gamma^2 M\int_{E_1} G(s,s)\nabla s>1. \tag{2.28}$$

By using the first equality of (2.27), we can choose $r_0>0$ such that

$$f(t,v)\ge Mv,\quad 0\le v\le r_0. \tag{2.29}$$

If $u\in\partial K_r$ $(0<r\le r_0)$, then for $t_0\in[a,\rho(b)]$, we have

$$\begin{aligned}
(Au)(t_0)&=\int_{\rho(a)}^{b} G(t_0,s)f\big(s,u(\omega(s))+u_0(\omega(s))\big)\nabla s\\
&\ge \Gamma\int_{\rho(a)}^{b} G(s,s)f\big(s,u(\omega(s))+u_0(\omega(s))\big)\nabla s\\
&\ge \Gamma\int_{E_1} G(s,s)f\big(s,u(\omega(s))\big)\nabla s\\
&\ge \Gamma M\int_{E_1} G(s,s)u(\omega(s))\nabla s\\
&\ge \Gamma^2\|u\|M\int_{E_1} G(s,s)\nabla s\\
&>\|u\|.
\end{aligned} \tag{2.30}$$

Therefore we get

$$\|Au\|>\|u\|,\quad \forall u\in\partial K_r. \tag{2.31}$$

Thus, we have from Theorem 1.1, $i(A,K_r,K)=0$, for $0<r\le r_0$. On the other hand, the second equality of (2.27) implies for every $M>0$, there is an $R_0>r_0$, such that

$$f(t,v)\ge Mv,\quad v\ge \Gamma R_0. \tag{2.32}$$

Here we choose $M > 0$ satisfying (2.28). For $u \in \partial K_R$, $R \geq R_0$, we have definition of K_R that

$$u(t) \geq \Gamma\|u\| = \Gamma R, \quad t \in [a, \rho(b)]. \tag{2.33}$$

It follows from (2.32) that

$$\begin{aligned}
(Au)(t_0) &= \int_{\rho(a)}^{b} G(t_0, s) f(s, u(\omega(s)) + u_0(\omega(s)))\nabla s \\
&\geq \Gamma \int_{\rho(a)}^{b} G(s, s) f(s, u(\omega(s)) + u_0(\omega(s)))\nabla s \\
&\geq \Gamma \int_{E_1} G(s, s) f(s, u(\omega(s)))\nabla s \\
&\geq \Gamma M \int_{E_1} G(s, s) u(\omega(s)))\nabla s \\
&\geq \Gamma^2 M R \int_{E_1} G(s, s)\nabla s \\
&> R = \|u\|.
\end{aligned} \tag{2.34}$$

This shows that

$$\|Au\| > \|u\|, \quad \forall u \in K_R. \tag{2.35}$$

Thus, by Theorem 1.1, we conclude that $i(A, K_R, K) = 0$ for $R \geq R_0$. The proof is therefore complete. □

3. Existence of One Positive Solution

In this section, we investigate the conditions for the existence of at least one positive solution of the BVP (1.3).

In the next theorem, we will also assume that the following condition on $f(t, v)$.

(H6):

$$\liminf_{v \to 0^+} \min_{t \in [a,b]} \frac{f(t, v)}{v} > k\lambda_1, \qquad \limsup_{v \to +\infty} \max_{t \in [a,b]} \frac{f(t, v)}{v} < q\lambda_1, \tag{3.1}$$

where $k > 0$ is large enough such that

$$k\Gamma \int_{E_1} \varphi_1(s)\nabla s \geq \int_{\rho(a)}^{b} \varphi_1(s)\nabla s, \tag{3.2}$$

and $q > 0$ is small enough such that

$$\Gamma\int_{\rho(a)}^{\rho(b)}\varphi_1(s)\nabla s \geq q\int_{\rho(a)}^{b}\varphi_1(s)\nabla s, \tag{3.3}$$

where $\varphi_1(t)$ ($\varphi_1(t) > 0$, $t \in [a,b]$) is the eigenfunction related to the smallest eigenvalue λ_1 ($\lambda_1 > 0$) of the eigenvalue problem:

$$\begin{gathered} -\varphi_1^{\Delta\nabla}(t) + q(t)\varphi_1(t) = \lambda\varphi_1(t), \\ \alpha\varphi_1(\rho(a)) - \beta\varphi_1^{\Delta}(\rho(a)) = 0, \qquad \gamma\varphi_1(b) + \delta\varphi_1^{\Delta}(b) = 0. \end{gathered} \tag{3.4}$$

Theorem 3.1. *If (H1)–(H6) are satisfied, then the BVP* (1.3) *has at least one positive solution.*

Proof. Fix $0 < m < 1 < m_1$ and let $f_1(u) = u^m + u^{m_1}$ for $u \geq 0$. Then, $f_1(u)$ satisfies (2.27). Define $\widetilde{A} : K \to K$ by

$$(\widetilde{A}u)(t) := \begin{cases} (\widetilde{A}_1u)(t), & \text{if } \rho(\tau_1) \leq t \leq a, \\ \displaystyle\int_{\rho(a)}^{b} G(t,s)f_1(u(\omega(s)) + u_0(\omega(s)))\nabla s, & \text{if } a \leq t \leq b, \\ (\widetilde{A}_2x)(t), & \text{if } b \leq t \leq \sigma(\tau_2), \end{cases} \tag{3.5}$$

where

$$\begin{gathered} (\widetilde{A}_1u)(t) = \begin{cases} e_{\alpha/\beta}(t,a)\displaystyle\int_{\rho(a)}^{b} G(a,s)f_1(u(\omega(s)) + u_0(\omega(s)))\nabla s, & \text{if } \rho(\tau_1) \leq t \leq a,\ \beta > 0, \\ 0, & \text{if } \rho(\tau_1) \leq t \leq a,\ \beta = 0, \end{cases} \\ (\widetilde{A}_2u)(t) = \begin{cases} e_{-\gamma/\delta}(t,b)\displaystyle\int_{\rho(a)}^{b} G(b,s)f_1(u(\omega(s)) + u_0(\omega(s)))\nabla s, & \text{if } b \leq t \leq \sigma(\tau_2),\ \delta > 0, \\ 0, & \text{if } b \leq t \leq \sigma(\tau_2),\ \delta = 0. \end{cases} \end{gathered} \tag{3.6}$$

Then $\widetilde{A}$ is a completely continuous operator. One has from Lemma 2.6 that there exist $0 < r_0 < R_0 < \infty$ such that

$$0 < r \leq r_0 \text{ implies } i(\widetilde{A}, K_r, K) = 0, \tag{3.7}$$

$$R \geq R_0 \text{ implies } i(\widetilde{A}, K_R, K) = 0. \tag{3.8}$$

Define $H : [0,1] \times K \to K$ by $H(t,u) = (1-t)Au + t\widetilde{A}u$ then H is a completely continuous operator. By the first equality in (H6) and the definition of f_1, there are $\epsilon > 0$ and $0 < r_1 \leq r_0$ such that

$$\begin{gathered} f(t,u) \geq (k\lambda_1 + \epsilon)u, \quad \forall t \in [a,b],\ 0 \leq u \leq r_1, \\ f_1(u) \geq (k\lambda_1 + \epsilon)u, \quad 0 \leq u \leq r_1. \end{gathered} \tag{3.9}$$

We now prove that $H(t,u) \neq u$ for all $t \in [0,1]$ and $u \in \partial K_{r_1}$. In fact, if there exists $t_0 \in [0,1]$ and $u_1 \in \partial K_{r_1}$ such that $H(t_0,u_1) = u_1$, then $u_1(t)$ satisfies the equation

$$\begin{aligned} &-u_1^{\Delta\nabla}(t) + q(t)u_1(t) \\ &\quad = (1-t_0)f(t,u_1(\omega(t)) + u_0(\omega(t))) + t_0 f_1(u_1(\omega(t)) + u_0(\omega(t))), \quad a \le t \le b \end{aligned} \tag{3.10}$$

and the boundary conditions

$$\begin{gathered} \alpha u_1(\rho(t)) - \beta u_1^{\Delta}(\rho(t)) = 0, \quad t \in [\tau_1, a], \\ \gamma u_1(t) + \delta u_1^{\Delta}(t) = 0, \quad t \in [b, \tau_2]. \end{gathered} \tag{3.11}$$

Multiplying both sides of (3.10) by $\varphi_1(t)$, then integrating from a to b, and using integration by parts in the left-hand side two times, we obtain

$$\begin{aligned} &\lambda_1 \int_{\rho(a)}^{b} \varphi_1(s)u_1(s)\nabla s \\ &\quad = \int_{\rho(a)}^{b} \left[(1-t_0)f(s,u_1(\omega(s)) + u_0(\omega(s))) + t_0 f_1(u_1(\omega(s)) + u_0(\omega(s)))\right]\varphi_1(s)\nabla s \\ &\quad = (1-t_0)\int_{\rho(a)}^{b} f(s,u_1(\omega(s)) + u_0(\omega(s)))\varphi_1(s)\nabla s \\ &\qquad + t_0 \int_{\rho(a)}^{b} f_1(u_1(\omega(s))) + u_0(\omega(s)))\varphi_1(s)\nabla s. \end{aligned} \tag{3.12}$$

Combining (3.9) and (3.12), we get

$$\begin{aligned} \lambda_1 \int_{\rho(a)}^{b} \varphi_1(s)u_1(s)\nabla s &\ge (1-t_0)\int_{E_1} f(s,u_1(\omega(s)))\varphi_1(s)\nabla s + t_0\int_{E_1} f_1(u_1(\omega(s)))\varphi_1(s)\nabla s \\ &\ge (1-t_0)(k\lambda_1 + \epsilon)\int_{E_1} u_1(\omega(s))\varphi_1(s)\nabla s \\ &\quad + t_0(k\lambda_1 + \epsilon)\int_{E_1} u_1(\omega(s))\varphi_1(s)\nabla s \\ &= \left(\lambda_1 + \frac{\epsilon}{k}\right)k\int_{E_1} \varphi_1(s)u_1(\omega(s))\nabla s \\ &\ge \left(\lambda_1 + \frac{\epsilon}{k}\right)k\Gamma\|u_1\|\int_{E_1} \varphi_1(s)\nabla s \\ &\ge \left(\lambda_1 + \frac{\epsilon}{k}\right)\|u_1\|\int_{\rho(a)}^{b} \varphi_1(s)\nabla s. \end{aligned} \tag{3.13}$$

We also have

$$\lambda_1 \int_{\rho(a)}^{b} \varphi_1(s) u_1(s) \nabla s \leq \lambda_1 \|u_1\| \int_{\rho(a)}^{b} \varphi_1(s) \nabla s. \tag{3.14}$$

Equations (3.13) and (3.14) lead to

$$\lambda_1 \geq \lambda_1 + \frac{\epsilon}{k}. \tag{3.15}$$

This is impossible. Thus $H(t,u) \neq u$ for $u \in \partial K_{r_1}$ and $t \in [0,1]$. By (3.7) and the homotopy invariance of the fixed-point index (see [11]), we get that

$$i(A, K_{r_1}, K) = i(H(0,\cdot), K_{r_1}, K) = i(H(1,\cdot), K_{r_1}, K) = i(\widetilde{A}, K_{r_1}, K) = 0. \tag{3.16}$$

On the other hand, according to the second inequality of (H6), there exist $\epsilon > 0$ and $R' > R_0$ such that

$$f(t,u) \leq (q\lambda_1 - \epsilon)u, \quad u > R', \ t \in [a,b]. \tag{3.17}$$

We define

$$C := \max_{a \leq t \leq b,\, 0 \leq u \leq R'} \left| f(t,u) - (q\lambda_1 - \epsilon)u \right| + 1, \tag{3.18}$$

then it follows that

$$f(t,u) \leq (q\lambda_1 - \epsilon)u + C, \quad u \geq 0, \ t \in [a,b]. \tag{3.19}$$

Define $H_1 : [0,1] \times K \rightarrow K$ by $H_1(t,u) = tAu$, then H_1 is a completely continuous operator. We claim that there exists $R_1 \geq R'$ such that

$$H_1(t,u) \neq u, \quad \text{for } t \in [0,1],\ u \in K,\ \|u\| \geq R_1. \tag{3.20}$$

In fact, if $H_1(t_0, u_1) = u_1$ for some $u_1 \in K$ and $0 \le t_0 \le 1$, then

$$\begin{aligned}\lambda_1 \int_{\rho(a)}^{b} u_1(s)\varphi_1(s)\nabla s &\le \int_{\rho(a)}^{b} f\big(s, u_1(\omega(s)) + u_0(\omega(s))\big)\varphi_1(s)\nabla s\\ &\le q\left(\lambda_1 - \frac{\epsilon}{q}\right)\|u_1 + u_0\| \int_{\rho(a)}^{b} \varphi_1(s)\nabla s + C\int_{\rho(a)}^{b} \varphi_1(s)\nabla s\\ &\le q\left(\lambda_1 - \frac{\epsilon}{q}\right)\|u_1\| \int_{\rho(a)}^{b} \varphi_1(s)\nabla s + C_1\int_{\rho(a)}^{b} \varphi_1(s)\nabla s,\end{aligned} \tag{3.21}$$

$$\begin{aligned}\lambda_1 \int_{\rho(a)}^{b} u_1(s)\varphi_1(s)\nabla s &\ge \lambda_1 \int_{\rho(a)}^{\rho(b)} u_1(s)\varphi_1(s)\nabla s\\ &\ge \lambda_1 \Gamma \|u_1\| \int_{\rho(a)}^{\rho(b)} \varphi_1(s)\nabla s\\ &\ge \lambda_1 q \|u_1\| \int_{\rho(a)}^{b} \varphi_1(s)\nabla s,\end{aligned} \tag{3.22}$$

where $C_1 = q(\lambda_1 - \epsilon/q)\|u_0\| + C$. Combining (3.21) with (3.22), we have

$$\|u_1\| \le \frac{C_1}{\epsilon} = \widetilde{R}_1. \tag{3.23}$$

Let $R_1 = \max\{R', \widetilde{R}_1\} + 1$. Then we get

$$H_1(t, u) \ne u, \quad \text{for } t \in [0,1],\ u \in K,\ \|u\| \ge R_1. \tag{3.24}$$

Consequently, by the homotopy invariance of the fixed-point index, we have

$$i(A, K_{R_1}, K) = i(H_1(1,\cdot), K_{R_1}, K) = i(H_1(0,\cdot), K_{R_1}, K) = i(\Theta, K_{R_1}, K) = 1, \tag{3.25}$$

where Θ is zero operator. Use (3.16) and (3.25) to conclude that

$$i(A, K_{R_1} \setminus K_{r_1}, K) = i(A, K_{R_1}, K) - i(A, K_{r_1}, K) = 1 - 0 = 1. \tag{3.26}$$

Hence, A has a fixed point in $(K_{R_1} \setminus K_{r_1})$.

Let $y(t) = u(t) + u_0(t)$. Since $y(t) = u(t)$ for $t \in [a,b]$ and $0 < r_1 \le \|u\| = \|u\|_{[a,b]} = \|y\|_{[a,b]} \le R_1$.

(H7)

$$\begin{gathered}\limsup_{v \to 0^+} \max_{t \in [a,b]} \frac{f(t,v)}{v} < q\lambda_1,\\ \liminf_{v \to +\infty} \min_{t \in [a,b]} \frac{f(t,v)}{v} > k\lambda_1, \quad \xi(t) \equiv 0,\ \eta(t) \equiv 0.\end{gathered} \tag{3.27}$$

□

Theorem 3.2. *If (H1)–(H5) and (H7) are satisfied, then the BVP* (1.3) *has at least one positive solution.*

Proof. Define $H_1 : [0,1] \times K \to K$ by $H_1(t,u) = tAu$, then H_1 is a completely continuous operator. By the first inequality in (H7), there exist $\epsilon > 0$ and $0 < r_1 \le r_0$ such that

$$f(t,v) \le (q\lambda_1 - \epsilon)v, \quad \forall t \in [a,b],\ 0 \le v \le r_1. \tag{3.28}$$

We claim that $H_1(t,u) \ne u$ for $0 \le t \le 1$ and $u \in \partial K_{r_1}$. In fact, if there exist $0 \le t_0 \le 1$ and $u_1 \in \partial K_{r_1}$ such that $H_1(t_0,u_1) = u_1$, then $u_1(t)$ satisfies the boundary condition (3.11). Since $\xi(t) \equiv 0, \eta(t) \equiv 0$, we have $u_0(t) \equiv 0$. Then we have

$$-u_1^{\Delta\nabla}(t) + q(t)u_1(t) = t_0 f(t, u_1(\omega(t))), \quad a \le t \le b. \tag{3.29}$$

Multiplying the last equation by $\varphi_1(t)$ integrating from a to b, by (3.28), we obtain

$$\begin{aligned}\lambda_1 \int_{\rho(a)}^{b} \varphi_1(s)u_1(s)\nabla s &= t_0 \int_{\rho(a)}^{b} f(s, u_1(\omega(s)))\varphi_1(s)\nabla s \\ &\le \int_{\rho(a)}^{b} f(s, u_1(\omega(s)))\varphi_1(s)\nabla s \\ &\le (q\lambda_1 - \epsilon)\|u_1\| \int_{\rho(a)}^{b} \varphi_1(s)\nabla s,\end{aligned} \tag{3.30}$$

then we have

$$\begin{aligned}\lambda_1 \int_{\rho(a)}^{b} u_1(s)\varphi_1(s)\nabla s &\ge \lambda_1 \int_{\rho(a)}^{\rho(b)} u_1(s)\varphi_1(s)\nabla s \\ &\ge \lambda_1 \Gamma \|u_1\| \int_{\rho(a)}^{\rho(b)} \varphi_1(s)\nabla s \\ &\ge \lambda_1 q \|u_1\| \int_{\rho(a)}^{b} \varphi_1(s)\nabla s.\end{aligned} \tag{3.31}$$

Equations (3.30) and (3.31) lead to

$$\lambda_1 q \le \lambda q_1 - \epsilon. \tag{3.32}$$

This is impossible. By homotopy invariance of the fixed-point index, we get that

$$i(A, K_{r_1}, K) = i(H_1(1,\cdot), K_{r_1}, K) = i(H_1(0,\cdot), K_{r_1}, K) = i(\Theta, K_{r_1}, K) = 1. \tag{3.33}$$

Define $H : [0,1] \times K \to K$ by $H(t,u) = (1-t)Au + t\widetilde{A}u$, then H is a completely continuous operator. By the second inequality in (H7), and definition of f_1, there exist $\epsilon > 0$ and $R' > R_0$ such that

$$\begin{gathered} f(t,u) \geq \left(k\lambda_1 + \epsilon\right)u, \quad u > R',\, t \in [a,b], \\ f_1(u) \geq \left(k\lambda_1 + \epsilon\right)u, \quad u > R'. \end{gathered} \tag{3.34}$$

We define

$$C := \max_{a \leq t \leq b,\, 0 \leq u \leq R'} \left| f(t,u) - \left(k\lambda_1 + \epsilon\right)u \right| + \max_{0 \leq u \leq R'} \left| f_1(u) - \left(k\lambda_1 + \epsilon\right)u \right| + 1, \tag{3.35}$$

then, it is obvious that

$$\begin{gathered} f(t,u) \geq \left(k\lambda_1 + \epsilon\right)u - C, \quad \forall t \in [a,b],\, u \geq 0, \\ f_1(u) \geq \left(k\lambda_1 + \epsilon\right)u - C, \quad u \geq 0. \end{gathered} \tag{3.36}$$

We claim that there exists $R_1 \geq R'$ such that

$$H(t,u) \neq u, \quad \text{for } t \in [0,1],\, u \in K,\, \|u\| \geq R_1. \tag{3.37}$$

In fact, if $H(t_0, u_1) = u_1$ for some $u_1 \in K$ and $0 \leq t_0 \leq 1$, then using (3.36), it is analogous to the argument of (3.13) and (3.14) that

$$\begin{aligned} \lambda_1 \int_{\rho(a)}^{b} \varphi_1(s) u_1(s) \nabla s &= \int_{\rho(a)}^{b} \left[(1-t_0) f\left(s, u_1(\omega(s)\right) + t_0 f_1\left(u_1(\omega(s))\right)\right) \right] \varphi_1(s) \nabla s \\ &\geq \int_{E_1} \left[(1-t_0) f\left(s, u_1(\omega(s)\right) + t_0 f_1\left(u_1(\omega(s))\right)\right) \right] \varphi_1(s) \nabla s \\ &\geq \int_{E_1} \left\{ (1-t_0) \left[(k\lambda_1 + \epsilon) u_1(\omega(s)) - C \right] + t_0 \left[(k\lambda_1 + \epsilon) u_1(\omega(s)) - C \right] \right\} \\ &\qquad \times \varphi_1(s) \nabla s \\ &= \int_{E_1} \left[(k\lambda_1 + \epsilon) u_1(\omega(s)) - C \right] \varphi_1(s) \nabla s \\ &\geq \left(\lambda_1 + \frac{\epsilon}{k} \right) k\Gamma \|u_1\| \int_{E_1} \varphi_1(s) \nabla s - C \int_{E_1} \varphi_1(s) \nabla s, \\ \lambda_1 \int_{\rho(a)}^{b} \varphi_1(s) u_1(s) \nabla s \leq \lambda_1 \|u_1\| \int_{\rho(a)}^{b} \varphi_1(s) \nabla s &\leq \lambda_1 \|u_1\| k\Gamma \int_{E_1} \varphi_1(s) \nabla s. \end{aligned} \tag{3.38}$$

Equation (3.38) leads to $\|u_1\| \leq C/\epsilon\Gamma = \widetilde{R}_1$. Let $R_1 = \max\{R', \widetilde{R}_1\} + 1$. Then we get

$$H_1(t,u) \neq u, \quad \text{for } t \in [0,1],\, u \in K,\, \|u\| \geq R_1. \tag{3.39}$$

Consequently, by (3.8) and the homotopy invariance of the fixed-point index, we have

$$i(A, K_{R_1}, K) = i(H(0,\cdot), K_{R_1}, K) = i(H(1,\cdot), K_{R_1}, K) = i(\widetilde{A}, K_{R_1}, K) = 0. \tag{3.40}$$

In view of (3.33) and (3.40), we obtain

$$i(A, K_{R_1} \setminus K_{r_1}, K) = i(A, K_{R_1}, K) - i(A, K_{r_1}, K) = 0 - 1 = -1. \tag{3.41}$$

Therefore, A has a fixed point in $(K_{R_1} \setminus K_{r_1})$. The proof is completed. □

Corollary 3.3. *Using the following (H8) or (H9) instead of (H6) or (H7), the conclusions of Theorems 3.1 and 3.2 are true. For* $t \in [a,b]$,

(H8)

$$\lim_{v\to 0^+}\left(\frac{f(t,v)}{v}\right) = +\infty, \quad \lim_{v\to +\infty}\left(\frac{f(t,v)}{v}\right) = 0 \quad (\textit{sublinear}); \tag{3.42}$$

(H9)

$$\lim_{v\to 0^+}\left(\frac{f(t,v)}{v}\right) = 0, \quad \lim_{v\to +\infty}\left(\frac{f(t,v)}{v}\right) = +\infty \quad (\textit{superlinear}), \quad \xi(t) \equiv 0,\ \eta(t) \equiv 0. \tag{3.43}$$

4. Existence of Three Positive Solutions

In this section, using Theorem 1.2 (the Leggett-Williams fixed-point theorem) we prove the existence of at least three positive solutions to the BVP (1.3).

Define the continuous concave functional $\psi : K \to [0,\infty)$ to be $\psi(u) := \min_{t\in[a,\rho(b)]} u(t)$, and the constants

$$M := \left[\min_{t\in[a,\rho(b)]}\int_{\rho(a)}^{\rho(b)} G(t,s)\nabla s\right]^{-1}, \tag{4.1}$$

$$N := \int_{\rho(a)}^{b} G(s,s)\nabla s. \tag{4.2}$$

Theorem 4.1. *Suppose there exists constants* $0 < p < q < q/\Gamma \le r$ *such that*

(D1) $f(t,v) < p/N$ *for* $t \in [a,b], v \in [0,p]$;

(D2) $f(t,v) \ge qM$ *for* $t \in [a,\rho(b)], v \in [q, q/\Gamma]$;

(D3) *one of the following is satisfied:*

(a) $\limsup_{v\to\infty} \max_{t\in[a,b]} (f(t,v)/v) < 1/N$,

(b) *there exists a constant* $r > q/\Gamma$ *such that* $f(t,v) \le r/N$ *for* $t \in [a,b]$ *and* $v \in [0,r]$, *where* Γ, M, *and* R, *are as defined in* (2.12), (4.1), (4.2), *respectively. Then the boundary value problem* (1.3) *has at least three positive solutions* u_1, u_2, *and* u_3 *satisfying*

$$\|u_1\| < p, \quad \min_{t\in[a,\rho(b)]}(u_2)(t) > q, \quad p < \|u_3\| \quad \textit{with} \min_{t\in[a,\rho(b)]}(u_3)(t) < q. \tag{4.3}$$

Proof. The technique here similar to that used in [5] Again the cone K, the operator A is the same as in the previous sections. For all $u \in K$ we have $\psi(u) \le \|u\|$. If $u \in \overline{K}_r$, then $\|u\| \le r$ and the condition (a) of (D3) imply that

$$\limsup_{v\to\infty} \max_{t\in[a,b]} \frac{f(t,v)}{v} < \frac{1}{N}. \tag{4.4}$$

Thus there exist a $\zeta > 0$ and $\epsilon < 1/R$ such that if $v > \zeta$, then $\max_{t\in[a,b]}(f(t,v)/v) < \epsilon$. For $\lambda := \max\{f(t,v) : v \in [0,\zeta],\, t \in [a,b]\}$, we have $f(t,v) \le \epsilon v + \lambda$ for all $v \ge 0$, for all $t \in [a,b]$. Pick any

$$r > \max\left\{\frac{q}{\Gamma}, \frac{\lambda}{1/N - \epsilon}\right\}. \tag{4.5}$$

Then $u \in \overline{K}_r$ implies that

$$\begin{aligned} \|Au\| &= \max_{t\in[a,b]} \int_{\rho(a)}^{b} G(t,s) f(s,u(\omega(s)))\nabla s \\ &\le (\epsilon\|u\| + \lambda)N < \epsilon r N + r(1-\epsilon N) = r. \end{aligned} \tag{4.6}$$

Thus $A : \overline{K}_r \to K_r$.

The condition (b) of (D3) implies that there exists a positive number r such that $f(t,v) \le r/N$ for $t \in [a,b]$ and $v \in [0,r]$. If $u \in \overline{K}_r$, then

$$\|Au\| = \max_{t\in[a,b]} \int_{\rho(a)}^{b} G(t,s) f(s,u(\omega(s)))\nabla s \le \left(\frac{r}{N}\right)N \le r. \tag{4.7}$$

Thus $A : \overline{K}_r \to \overline{K}_r$. Consequently, the assumption (D3) holds, then there exist a number r such that $r > q/\Gamma$ and $A : \overline{K}_r \to K_r$.

The remaining conditions of Theorem 1.2 will now be shown to be satisfied.

By (D1) and argument above, we can get that $A : \overline{K}_p \to K_p$. Hence, condition (ii) of Theorem 1.2 is satisfied.

We now consider condition (i) of Theorem 1.2. Choose $u_K(t) \equiv (s+q)/2$ for $t \in [a,b]$, where $s = q/\Gamma$. Then $u_K(t) \in K(\psi, q, q/\Gamma)$ and $\psi(u_K) = \psi((s+q)/2) > q$, so that $\{K(\psi, p, q/\Gamma) : \psi(u) > q\} \ne \{\}$. For $u \in K(\psi, q, q/\Gamma)$, we have $q \le u(t) \le q/\Gamma$, $t \in [a,\rho(b)]$. Combining with (D2), we get

$$f(t,u) \ge qM, \tag{4.8}$$

for $t \in [a, \rho(b)]$. Thus, we have

$$\begin{aligned}
\psi(Au) &= \min_{t\in[a,\rho(b)]} \int_{\rho(a)}^{b} G(t,s) f\big(s, u(\omega(s))\big) \nabla s \\
&> \min_{t\in[a,\rho(b)]} \int_{\rho(a)}^{\rho(b)} G(t,s) f\big(s, u(\omega(s))\big) \nabla s \geq \frac{qM}{M} = q.
\end{aligned} \tag{4.9}$$

As a result, $u \in K(\psi, q, s)$ yields $\psi(Au) > q$.

Lastly, we consider Theorem 1.2(iii). Recall that $A : K \to K$. If $u \in K(\psi, q, r)$ and $\|Au\| > q/\Gamma$, then

$$\psi(Au) = \min_{t\in[a,\rho(b)]} Au(t) \geq \Gamma \|Au\| > \Gamma \frac{q}{\Gamma} = q. \tag{4.10}$$

Thus, all conditions of Theorem 1.2 are satisfied. It implies that the TPBVP (1.3) has at least three positive solutions u_1, u_2, u_3 with

$$\|u_1\| < p, \quad \psi(u_2) > q, \quad p < \|u_3\| \quad \text{with } \psi(u_3) < q. \tag{4.11}$$

□

5. Examples

Example 5.1. Let $\mathbb{T} = \{(n/2) : n \in \mathbb{N}_0\}$. Consider the BVP:

$$\begin{gathered}
-y^{\Delta\nabla}(t) + y(t) = \sqrt{y(2t-1)}, \quad t \in [0,3] \subset \mathbb{T}, \\
y\big(\rho(t)\big) - 2y^{\Delta}\big(\rho(t)\big) = t^2, \quad t \in [-1, 0], \\
3y(t) + 4y^{\Delta}(t) = t - 3, \quad t \in [3, 5].
\end{gathered} \tag{5.1}$$

Then $a = 0$, $b = 3$, $\tau_1 = -1, \tau_2 = 5$, $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\delta = 4$, and

$$q(t) = 1, \qquad \xi(t) = t^2, \qquad \eta(t) = t - 3, \qquad f(t, v) = \sqrt{v}, \qquad \omega(t) = 2t - 1. \tag{5.2}$$

Since $\lim_{v\to 0^+}(f(t,v)/v) = +\infty$, $\lim_{v\to +\infty}(f(t,v)/v) = 0$. It is clear that (H1)–(H5) and (H8) are satisfied. Thus, by Corollary 3.3, the BVP (5.1) has at least one positive solution.

Example 5.2. Let us introduce an example to illustrate the usage of Theorem 4.1. Let

$$\mathbb{T} = \left\{ \frac{4n^2}{n^2+9} : n \in \mathbb{N}_0 \right\} \cup \{4\}. \tag{5.3}$$

Consider the TPBVP:

$$-y^{\Delta\nabla}(t) = \frac{500y^2(t^2+1)}{y^2(t^2+1)+300}, \quad t \in \left[\frac{2}{5}, 2\right],$$
$$y(0) = 0,$$
$$y^{\Delta}(t) = 0, \quad t \in [2,5]. \tag{5.4}$$

Then $a = 2/5$, $b = 2$, $\tau_1 = 2/5, \tau_2 = 5$, $\alpha = 1$, $\beta = 0$, $\delta = 1$, $\gamma = 0$, and

$$\omega(t) = t^2 + 1, \qquad \xi(t) = \left(t - \frac{2}{5}\right)^2, \qquad \eta(t) = 0, \qquad f(t,v) = \frac{500v^2}{v^2+300}. \tag{5.5}$$

The Green function of the BVP (5.4) has the form

$$G(t,s) = \begin{cases} s, & \text{if } 0 \le s \le t \le \dfrac{64}{25}, \\ t, & \text{if } 0 \le t \le s \le \dfrac{64}{25}. \end{cases} \tag{5.6}$$

Clearly, f is continuous and increasing $[0,\infty)$. We can also see that $\alpha + \gamma > 0$. By (2.12), (4.1), and (4.2), we get $\Gamma = 1/5$, $M = 65/32$, and $N = 11496/4225$.

Now we check that (D1), (D2), and (b) of (D3) are satisfied. To verify (D1), as $f(1/10) = 0.01666611113$, we take $p = 1/10$, then

$$f(y) < \frac{p}{N} = 0.03675191371, \quad y \in [0,p] \tag{5.7}$$

and (D1) holds. Note that $f(3/2) = 3.722084367$, when we set $q = 3/2$,

$$f(y) \ge qM = 3.046875000, \quad y \in [q, 5q] \tag{5.8}$$

holds. It means that (D2) is satisfied. Let $r = 1500$, we have

$$f(y) \le 500 < \frac{r}{N} = 551.2787056, \quad y \in [0,r], \tag{5.9}$$

from $\lim_{y\to\infty} f(y) = 500$, so that (b) of (D3) is met. Summing up, there exist constants $p = 1/10$, $q = 3/2$, and $r = 1500$ satisfying

$$0 < p < q < \frac{q}{\Gamma} < r. \tag{5.10}$$

Thus, by Theorem 4.1, the TPBVP (5.4) has at least three positive solutions y_1, y_2, y_3 with

$$\|y_1\| < \frac{1}{10}, \quad \varphi(y_2) > \frac{3}{2}, \quad \frac{1}{10} < \|y_3\| \quad \text{with } \varphi(y_3) < \frac{3}{2}. \tag{5.11}$$

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