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Research Article

A Global Description of the Positive Solutions of Sublinear Second-Order Discrete Boundary Value Problems

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Let $T \in \mathbb{N}$ be an integer with $T > 1$, $\mathbb{T} := \{1, \ldots, T\}$, $\widehat{\mathbb{T}} := \{0, 1, \ldots, T+1\}$. We consider boundary value problems of nonlinear second-order difference equations of the form $\Delta^2 u(t-1) + \lambda a(t) f(u(t)) = 0$, $t \in \mathbb{T}$, $u(0) = u(T+1) = 0$, where $a : \mathbb{T} \to \mathbb{R}^+$, $f \in C([0,\infty), [0,\infty))$ and, $f(s) > 0$ for $s > 0$, and $f_0 = f_\infty = 0$, $f_0 = \lim_{s \to 0^+} f(s)/s$, $f_\infty = \lim_{s \to +\infty} f(s)/s$. We investigate the global structure of positive solutions by using the Rabinowitz's global bifurcation theorem.

1. Introduction

Let $T \in \mathbb{N}$ be an integer with $T > 1$, $\mathbb{T} := \{1, \ldots, T\}$, $\widehat{\mathbb{T}} := \{0, 1, \ldots, T+1\}$. We study the global structure of positive solutions of the problem

$$\begin{gathered} \Delta^2 u(t-1) + \lambda a(t) f(u(t)) = 0, \quad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0. \end{gathered} \tag{1.1}$$

Here λ is a positive parameter, $a : \mathbb{T} \to \mathbb{R}^+$ and $f : [0,\infty) \to [0,\infty)$ are continuous. Denote $f_0 = \lim_{s \to 0^+} f(s)/s$ and $f_\infty = \lim_{s \to +\infty} f(s)/s$.

There are many literature dealing with similar difference equations subject to various boundary value conditions. We refer to Agarwal and Henderson [1], Agarwal and O'Regan [2], Agarwal and Wong [3], Rachunkova and Tisdell [4], Rodriguez [5], Cheng and Yen [6], Zhang and Feng [7], R. Ma and H. Ma [8], Ma [9], and the references therein. These results were usually obtained by analytic techniques, various fixed point theorems, and global bifurcation techniques. For example, in [8], the authors investigated the global structure of sign-changing solutions of some discrete boundary value problems in the case that $f_0 \in (0,\infty)$. However, relatively little result is known about the global structure of solutions in the case that $f_0 = 0$, and no global results were found in the available literature when $f_0 = 0 = f_\infty$. The likely reason is that the Rabinowitz's global bifurcation theorem [10] cannot be used directly in this case.

In the present work, we obtain a direct and complete description of the global structure of positive solutions of (1.1) under the assumptions:

(A1) $a : \mathbb{T} \to (0,\infty)$;

(A2) $f : [0,\infty) \to [0,\infty)$ is continuous and $f(s) > 0$ for $s > 0$;

(A3) $f_0 = 0$, where $f_0 = \lim_{s\to 0^+} f(s)/s$;

(A4) $f_\infty = 0$, where $f_\infty = \lim_{s\to +\infty} f(s)/s$.

Let Y denote the Banach space defined by

$$Y = \{y \mid y : \mathbb{T} \longrightarrow \mathbb{R}\} \tag{1.2}$$

equipped with the norm

$$\|y\|_Y = \max_{t\in\mathbb{T}} |y(t)|. \tag{1.3}$$

Let E denote the Banach space defined by

$$E = \left\{u : \widehat{\mathbb{T}} \longrightarrow \mathbb{R} \mid u(0) = u(T+1) = 0\right\} \tag{1.4}$$

equipped with the norm

$$\|u\|_0 = \max_{t\in\mathbb{T}} |u(t)|. \tag{1.5}$$

Define an operator $L : E \to Y$ by

$$(Lu)(t) = -\Delta^2 u(t-1), \quad t \in \mathbb{T}. \tag{1.6}$$

To state our main results, we need the spectrum theory of the linear eigenvalue problem

$$\begin{gathered} \Delta^2 u(t-1) + \lambda a(t) u(t) = 0, \quad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0. \end{gathered} \tag{1.7}$$

Lemma 1.1 ([5, 11]). *Let (A1) hold. Then there exists a sequence* $\{\lambda_n\}_{n=1}^T \in (0,\mathbb{R})$ *satisfying that*

(i) $\{\lambda_n \mid n \in \{1,2,\ldots,T\}\}$ *is the set of eigenvalues of* (1.7);

(ii) $\lambda_{n+1} > \lambda_n$ *for* $n \in \{1,2,\ldots,T-1\}$;

(iii) *for* $k \in \{1, 2, \ldots, T\}$, $\ker(L - \lambda_k I)$ *is one-dimensional subspace of* E;

(iv) *for each* $k \in \{1, 2, \ldots, T\}$, *if* $v \in \ker(L - \lambda_k I) \setminus \{0\}$, *then* v *has exactly* $k - 1$ *simple generalized zeros in* $[0, T]$.

Let Σ *denote the closure of set of positive solutions of* (1.1) *in* $[0, \infty) \times E$.

Let M *be a subset of* E. *A component of* M *is meant a maximal connected subset of* M, *that is, a connected subset of* M *which is not contained in any other connected subset of* M.

The main results of this paper are the following theorem.

Theorem 1.2. *Let (A1)–(A4) hold. Then there exists a component* ζ *in* Σ *which joins* (∞, θ) *with* (∞, ∞), *and*

$$\operatorname{Proj}_{\mathbb{R}} \zeta = [\rho^*, \infty) \tag{1.8}$$

for some $\rho^* > 0$. *Moreover, there exists* $\lambda^* \geq \rho^* > 0$ *such that* (1.1) *has at least two positive solutions for* $\lambda \in (\lambda^*, \infty)$.

We will develop a bifurcation approach to treat the case $f_0 = 0$ directly. Crucial to this approach is to construct a sequence of functions $\{f^{[n]}\}$ which is asymptotic linear at 0 and satisfies

$$f^{[n]} \longrightarrow f, \qquad \left(f^{[n]}\right)_0 > 0, \qquad \left(f^{[n]}\right)_0 \longrightarrow 0. \tag{1.9}$$

By means of the corresponding auxiliary equations, we obtain a sequence of unbounded components $\{C_+^{[n]}\}$ via Rabinnowitz's global bifurcation theorem [10], and this enables us to find an unbounded component $\mathcal{C}$ satisfying

$$\mathcal{C} \subset \limsup_{n \to \infty} C_+^{[n]}. \tag{1.10}$$

2. Some Preliminaries

In this section, we give some notations and preliminary results which will be used in the proof of our main results.

Definition 2.1 (see [12]). Let X be a Banach space, and let $\{C_n \mid n = 1, 2, \ldots\}$ be a family of subsets of X. Then the *superior limit* $\mathfrak{D}$ of $\{C_n\}$ is defined by

$$\mathfrak{D} := \limsup_{n \to \infty} C_n = \{x \in X \mid \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \longrightarrow x\}. \tag{2.1}$$

Definition 2.2 (see [12]). A *component* of a set M is meant a maximal connected subset of M.

Lemma 2.3 ([12, Whyburn]). *Suppose that* Y *is a compact metric space,* A *and* B *are nonintersecting closed subsets of* Y, *and no component of* Y *interests both* A *and* B. *Then there exist two disjoint compact subsets* Y_A *and* Y_B, *such that* $Y = Y_A \cup Y_B$, $A \subset Y_A$, $B \subset Y_B$.

Using the above Whyburn's lemma, Ma and An [13] proved the following lemma.

Lemma 2.4 ([13, Lemma 2.2]). *Let X be a Banach space, and let $\{C_n\}$ be a family of connected subsets of X. Assume that*

(i) *there exist $z_n \in C_n$, $n = 1, 2, \ldots$, and $z^* \in X$, such that $z_n \to z^*$;*

(ii) $\lim_{n\to\infty} r_n = \infty$, *where $r_n = \sup\{\|x\| \mid x \in C_n\}$;*

(iii) *for every $R > 0$, $(\cup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of X, where*

$$B_R = \{x \in X \mid \|x\| \le R\}. \tag{2.2}$$

Then there exists an unbounded component $\mathcal{C}$ in $\mathfrak{D}$ and $z^* \in \mathcal{C}$.

Let

$$G(t,s) = \frac{1}{T+1}\begin{cases}(T+1-t)s, & 0 \le s \le t \le T+1,\\ t(T+1-s), & 0 \le t \le s \le T+1.\end{cases} \tag{2.3}$$

It is easy to see that

$$G(t,s) \ge \frac{1}{T+1} G(s,s), \quad (t,s) \in \mathbb{T} \times \widehat{\mathbb{T}}. \tag{2.4}$$

Denote the cone K in E by

$$K = \left\{x \in E \mid u(t) \ge 0 \text{ on } \widehat{\mathbb{T}}, \text{ and } \min_{t\in\mathbb{T}} u(t) \ge \frac{1}{T+1}\|u\|\right\}. \tag{2.5}$$

Now we define a map $A_\lambda : K \to Y$ by

$$(A_\lambda u)(t) = \lambda \sum_{s=1}^{T} G(t,s) a(s) f(u(s)), \quad t \in \mathbb{T}. \tag{2.6}$$

Define an operator $j : Y \to E$ by

$$j((y_1, \ldots, y_T)) = (0, y_1, \ldots, y_T, 0), \quad \forall (y_1, \ldots, y_T) \in Y. \tag{2.7}$$

Then the operator $T_\lambda := j \circ A_\lambda$ satisfies $T_\lambda : E \to E$.

For $r > 0$, let

$$\Omega_r = \{u \in K \mid \|u\| < r\}. \tag{2.8}$$

Using the standard arguments, we may prove the following lemma.

Lemma 2.5. *Assume that (A1)–(A2) hold. Then* $T_\lambda(K) \subseteq K$ *and* $T_\lambda : K \to K$ *is completely continuous.*

Lemma 2.6. *Assume that (A1)–(A2) hold. If* $u \in \partial\Omega_r,\ r > 0$, *then*

$$\|A_\lambda u\|_0 \geq \lambda \widehat{m}_r \sum_{s=1}^{T} G(1,s)a(s), \tag{2.9}$$

where

$$\widehat{m}_r = \min_{r/(T+1) \leq x \leq r} \{f(x)\}. \tag{2.10}$$

Proof. Since $f(u(t)) \geq \widehat{m}_r$ for $t \in \mathbb{T}$, it follows that

$$\|A_\lambda u\|_0 \geq \lambda \sum_{s=1}^{T} G(1,s)a(s)f(u(s)) \geq \lambda \widehat{m}_r \sum_{s=1}^{T} G(1,s)a(s). \tag{2.11}$$

□

3. Proof of the Main Results

Define $f^{[n]} : [0,\infty) \to [0,\infty)$ by

$$f^{[n]}(s) = \begin{cases} f(s), & s \in \left(\dfrac{1}{n}, \infty\right), \\ nf\left(\dfrac{1}{n}\right)s, & s \in \left[0, \dfrac{1}{n}\right]. \end{cases} \tag{3.1}$$

Then $f^{[n]} \in C([0,\infty),[0,\infty))$ with

$$f^{[n]}(s) > 0, \quad \forall s \in (0,\infty), \qquad \left(f^{[n]}\right)_0 = nf\left(\frac{1}{n}\right). \tag{3.2}$$

By (A3), it follows that

$$\lim_{n\to\infty} \left(f^{[n]}\right)_0 = 0. \tag{3.3}$$

To apply the global bifurcation theorem, we extend f to be an odd function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(s) = \begin{cases} f(s), & s \geq 0, \\ -f(-s), & s < 0. \end{cases} \tag{3.4}$$

Similarly we may extend $f^{[n]}$ to be an odd function $g^{[n]} : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$.

Now let us consider the auxiliary family of the equations

$$\begin{gathered}\Delta^2 u(t-1)+\lambda a(t) g^{[n]}(u)=0, \quad t \in \mathbb{T}, \\ u(0)=u(T+1)=0.\end{gathered} \tag{3.5}$$

Let $\xi^{[n]} \in C(\mathbb{R})$ be such that

$$g^{[n]}(u)=\left(g^{[n]}\right)_0 u+\xi^{[n]}(u)=n f\left(\frac{1}{n}\right) u+\xi^{[n]}(u). \tag{3.6}$$

Then

$$\lim_{|u| \rightarrow 0} \frac{\xi^{[n]}(u)}{u}=0. \tag{3.7}$$

Let us consider

$$L u-\lambda a(t)\left(g^{[n]}\right)_0 u=\lambda a(t) \xi^{[n]}(u) \tag{3.8}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Equation (3.8) can be converted to the equivalent equation

$$\begin{aligned} u(t) & =\sum_{s=1}^T G(t, s)\left[\lambda a(s)\left(g^{[n]}\right)_0 u(s)+\lambda a(s) \xi^{[n]}(u(s))\right] \\ & :=\lambda L^{-1}\left[a(\cdot)\left(g^{[n]}\right)_0 u(\cdot)\right](t)+\lambda L^{-1}\left[a(\cdot) \xi^{[n]}(u(\cdot))\right](t).\end{aligned} \tag{3.9}$$

Further we note that $\|L^{-1}[a(\cdot)\xi^{[n]}(u(\cdot))]\| = o(\|u\|)$ for u near θ in E.

The results of Rabinowitz [10] for (3.8) can be stated as follows. For each integer $n \geq 1$, $\nu \in \{+,-\}$, there exists a continuum $C_\nu^{[n]}$ of solutions of (3.8) joining $(\lambda_1/(g^{[n]})_0, \theta)$ to infinity in $([0,\infty) \times \nu K)$. Moreover, $C_\nu^{[n]} \setminus \{(\lambda_1/(g^{[n]})_0, \theta)\} \subset ([0,\infty) \times \nu(\operatorname{int} K))$.

Lemma 3.1. *Let (A1)–(A4) hold. Then, for each fixed n, $C_+^{[n]}$ joins $(\lambda_1/(g^{[n]})_0, \theta)$ to (∞, ∞) in $[0,\infty) \times K$.*

Proof. We divide the proof into two steps.

Step 1. We show that $\sup\{\lambda \mid (\lambda, u) \in C_+^{[n]}\} = \infty$.

Assume on the contrary that $\sup\{\lambda \mid (\lambda, u) \in C_+^{[n]}\} =: c_0 < \infty$. Let $\{(\eta_k, y_k)\} \subset C_+^{[n]}$ be such that

$$|\eta_k| + \|y_k\|_0 \longrightarrow \infty. \tag{3.10}$$

Then $\|y_k\|_0 \to \infty$. This together with the fact

$$\min_{t\in\mathbb{T}} y_k(t) \geq \frac{1}{T+1}\|y_k\|_0 \tag{3.11}$$

implies that

$$\lim_{k\to\infty} y_k(t) = \infty, \quad \text{uniformly for } t \in \mathbb{T}. \tag{3.12}$$

Since $(\eta_k, y_k) \in C_+^{[n]}$, we have that

$$\begin{gathered} \Delta^2 y_k(t-1) + \eta_k a(t) g^{[n]}(y_k(t)) = 0, \quad t \in \mathbb{T}, \\ y_k(0) = y_k(T+1) = 0. \end{gathered} \tag{3.13}$$

Set $v_k(t) = y_k(t)/\|y_k\|_0$. Then

$$\|v_k\|_0 = 1. \tag{3.14}$$

Now, choosing a subsequence and relabelling if necessary, it follows that there exists $(\eta_*, v_*) \in [0, c_0] \times E$ with

$$\|v_*\|_0 = 1, \tag{3.15}$$

such that

$$\lim_{k\to\infty} (\eta_k, v_k) = (\eta_*, v_*), \quad \text{in } \mathbb{R} \times E. \tag{3.16}$$

Moreover, using (3.13), (3.12), and the assumption $f_\infty = 0$, it follows that

$$\begin{gathered} \Delta^2 v_*(t-1) + \eta_* a(t) \cdot 0 = 0, \quad t \in \mathbb{T}, \\ v_*(0) = v_*(T+1) = 0, \end{gathered} \tag{3.17}$$

and consequently, $v_*(t) \equiv 0$ for $t \in \widehat{\mathbb{T}}$. This contradicts (3.15). Therefore

$$\sup\{\lambda \mid (\lambda, y) \in \mathcal{C}\} = \infty. \tag{3.18}$$

Step 2. We show that $\sup\{\|u\|_0 \mid (\lambda, u) \in C_+^{[n]}\} = \infty$.

Assume on the contrary that $\sup\{\|u\|_0 \mid (\lambda, u) \in C_+^{[n]}\} =: M_0 < \infty$. Let $\{(\eta_k, y_k)\} \subset C_+^{[n]}$ be such that

$$\eta_k \longrightarrow \infty, \quad \|y_k\|_0 \leq M_0. \tag{3.19}$$

Since $(\eta_k, y_k) \in C_+^{[n]}$, for any $t \in \mathbb{T}$, we have from (2.6) that

$$\begin{aligned} y_k(t) &= \eta_k \sum_{s=1}^{T} G(t,s) a(s) g^{[n]}(y_k(s)) \\ &\geq \frac{\eta_k}{T+1} \sum_{s=1}^{T} G(s,s) a(s) \frac{g^{[n]}(y_k(s))}{y_k(s)} \; y_k(s) \\ &\geq \frac{\eta_k}{(T+1)^2} \sum_{s=1}^{T} G(s,s) a(s) \frac{g^{[n]}(y_k(s))}{y_k(s)} \|y_k\|_0 \\ &\geq \frac{\eta_k}{(T+1)^2} \sum_{s=1}^{T} G(s,s) a(s) b_* \|y_k\|_0, \end{aligned} \tag{3.20}$$

(where $b_* := \inf\{(g^{[n]}(x))/x \mid x \in (0, M_0]\} > 0$), which yields that $\{\eta_k\}$ is bounded. However, this contradicts (3.19).

Therefore, $C_+^{[n]}$ joins $(\lambda_1/(g^{[n]})_0, 0)$ to (∞, ∞) in $[0,\infty) \times K$. □

Lemma 3.2. *Let (A1)–(A4) hold and let $I \subset (0,\infty)$ be a closed and bounded interval. Then there exists a positive constant M, such that*

$$\sup\left\{ \|y\|_0 \mid (\eta, y) \in C_+^{[n]},\ \eta \in I \right\} \leq M. \tag{3.21}$$

Proof. Assume on the contrary that there exists a sequence $\{(\eta_k, y_k)\} \subset C_+^{[k]} \cap (I \times K)$ such that

$$\|y_k\|_0 \longrightarrow \infty. \tag{3.22}$$

Then, (3.11), (3.12), and (3.13) hold. Set $v_k(t) = y_k(t)/\|y_k\|_0$, then

$$\|v_k\|_0 = 1. \tag{3.23}$$

Now, choosing a subsequence and relabeling if necessary, it follows that there exists $(\eta_*, v_*) \in I \times E$ with

$$\|v_*\|_0 = 1, \tag{3.24}$$

such that

$$\lim_{k\to\infty} (\eta_k, v_k) = (\eta_*, v_*), \quad \text{in } \mathbb{R} \times E. \tag{3.25}$$

Moreover, from (3.13), (3.12), and the assumption $f_\infty = 0$, it follows that

$$\begin{gathered} \Delta^2 v_*(t-1) + \eta_* a(t) \cdot 0 = 0, \quad t \in \mathbb{T}, \\ v_*(0) = v_*(T+1) = 0, \end{gathered} \tag{3.26}$$

and consequently, $v_*(t) \equiv 0$ for $t \in \widehat{\mathbb{T}}$. This contradicts (3.24). Therefore

$$\sup\left\{ \|y\|_0 \mid (\eta, y) \in C_+^{[n]},\ \eta \in I \right\} \le M. \tag{3.27}$$

□

Lemma 3.3. *Let (A1)–(A4) hold. Then there exits $\rho_* > 0$ such that*

$$\left(\cup_{n=1}^{\infty} C_+^{[n]} \right) \cap \left((0, \rho_*) \ \times K \right) = \emptyset. \tag{3.28}$$

Proof. Assume on the contrary that there exists $\{(\eta_k, y_k)\} \subset (\cup_{n=1}^{\infty} C_+^{[n]}) \cap ((0, +\infty) \times K)$ such that $\eta_k \to 0$. Then

$$y_k(t) = \eta_k \sum_{s=1}^{T} G(t,s) a(s) g^{[n]}(y_k(s)), \quad t \in \mathbb{T}. \tag{3.29}$$

Set $v_k(t) = (y_k(t))/\|y_k\|_0$, then

$$\|v_k\|_0 = 1, \tag{3.30}$$

and for all $t \in \mathbb{T}$,

$$v_k(t) = \eta_k \sum_{s=1}^{T} G(t,s) a(s) \frac{g^{[n]}(y_k(s))}{y_k(s)} \frac{y_k(s)}{\|y_k\|_0} \le \eta_k \sum_{s=1}^{T} G(s,s) a(s) B_n^* \|v_k\|_0, \tag{3.31}$$

where $B_n^* = \sup\{(g^{[n]}(x))/x \mid x \in (0,\infty), n \in \mathbb{N}\})$. Let

$$B^* = \sup\{B_n^* \mid n \in \mathbb{N}\}. \tag{3.32}$$

Then $B^* < \infty$, and

$$v_k(t) \le \eta_k \sum_{s=1}^{T} G(s,s) a(s) B^* \|v_k\|_0 \longrightarrow 0, \tag{3.33}$$

which contradicts (3.30). Therefore, there exists $\rho^* > 0$, such that

$$\left(\cup_{n=1}^{\infty} C_+^{[n]} \right) \cap \left((0, \rho^*) \times K \right) = \emptyset. \tag{3.34}$$

□

Proof of Theorem 1.2. Take $r = 1$. Let ρ^* be as in Lemma 3.3, and let λ^* be a fixed constant satisfying $\lambda^* \geq \rho^*$ and

$$\lambda^* \widehat{m}_1^{[n]} \sum_{s=1}^{T} G(1,s)a(s) > 1, \tag{3.35}$$

where

$$\widehat{m}_1^{[n]} = \min_{1/(T+1)\leq x\leq 1} \left\{ g^{[n]}(x) \right\}. \tag{3.36}$$

It is easy to see that there exists $n_0 \in \mathbb{N}$, such that

$$\frac{1}{n_0} < \frac{1}{T+1}. \tag{3.37}$$

This implies that

$$\widehat{m}_1^{[n]} = \widehat{m}_1, \quad \forall n > n_0 \tag{3.38}$$

(see (2.10) for the definition of $\widehat{m}_1$), and accordingly, we may choose λ^* which is independent of $n > n_0$. From Lemma 2.6 and (3.35), it follows that for $\lambda > \lambda^*$,

$$\|T_\lambda u\|_0 > \|u\|_0, \quad u \in \partial\Omega_1. \tag{3.39}$$

This together with the compactness of T_λ implies that there exists $\epsilon \in (0, 1/2)$, such that

$$C_+^{[n]} \cap \left\{ (\eta, u) \mid \eta \geq \lambda^*;\ u \in K : 1 - 2\epsilon \leq \|u\|_0 \leq 1 + 2\epsilon \right\} = \emptyset, \quad \forall n > n_0. \tag{3.40}$$

Notice that $\{C_+^{[n]}\}$ satisfies all conditions in Lemma 2.4, and consequently, $\limsup_{n\to\infty} C_+^{[n]}$ contains a component $\widehat{\zeta}$ which is unbounded. However, we do not know whether $\widehat{\zeta}$ joins (∞, θ) with (∞, ∞) or not. To answer this question, we have to use the following truncation method.

Set

$$\Gamma := ([0,\infty) \times K) \setminus \left\{ (\eta, u) \mid \eta \geq \lambda^*;\ u \in K : \|u\|_0 \leq 1 + \epsilon \right\}. \tag{3.41}$$

For $n \in \mathbb{N}$ with $\lambda_1/(g^{[n]})_0 \geq \lambda^*$, we define $\zeta_0^{[n]}$ a connected subset in $C_+^{[n]}$ satisfying

(1) $\zeta_0^{[n]} \subset (C_+^{[n]} \setminus ((\lambda^*, \infty) \times \Omega_1))$;

(2) $\zeta_0^{[n]}$ joins $\{\lambda^*\} \times \Omega_1$ with infinity in Γ.

We claim that $\zeta_0^{[n]}$ satisfies all of the conditions of Lemma 2.4.

Since

$$\lim_{n\to\infty}\frac{\lambda_1}{(g^{[n]})_0}=\lim_{n\to\infty}\frac{\lambda_1}{nf(1/n)}=\infty, \tag{3.42}$$

we have from Lemmas 3.1–3.3 and (3.40) that for $n>n_0$ and $\lambda_1/(g^{[n]})_0\geq\lambda^*$,

$$\zeta_0^{[n]}\cap(\{\lambda^*\}\times\Omega_{1-\epsilon})\neq\emptyset. \tag{3.43}$$

Thus, there exists $z_{n_j}\in\zeta_0^{[n_j]}\cap(\{\lambda^*\}\times\Omega_{1-\epsilon})$, such that $z_{n_j}\to z^*$, and accordingly, condition (i) in Lemma 2.4 is satisfied. Obviously,

$$r_n=\sup\left\{|\eta|+\|y\|_0\mid(\eta,y)\in\zeta_0^{[n]}\right\}=\infty, \tag{3.44}$$

that is, condition (ii) in Lemma 2.4 holds. Condition (iii) in Lemma 2.4 can be deduced directly from the Arzelà -Ascoli theorem and the definition of $g^{[n]}$. Therefore, the superior limit of $\{\zeta_0^{[n]}\}$ contains a component ζ_0 joining $\{\lambda^*\}\times\Omega_1$ with infinity in Γ.

Similarly, for each $j\in\mathbb{N}$, we may define a connected subset, $\zeta_j^{[n]}$, in $C_+^{[n]}$ satisfying

(1) $\zeta_j^{[n]}\subset(C_+^{[n]}\setminus((\lambda^*+j,\infty)\times\Omega_1))$;

(2) $\zeta_j^{[n]}$ joins $\{\lambda^*+j\}\times\Omega_1$ with infinity in Γ,

and the superior limit of $\{\zeta_j^{[n]}\}$ contains a component ζ_j joining $\{\lambda^*+j\}\times\Omega_1$ with infinity in Γ.

It is easy to verify that

$$\zeta_k\subseteq\Sigma,\quad k=0,1,2,\ldots. \tag{3.45}$$

Now, for each $(\lambda^*,v)\in(\{\lambda^*\}\times\Omega_1)\cap\Sigma$, let $\mathcal{E}(v)$ $(\subset\Sigma)$ be a connected component containing (λ^*,v). Let

$$\mu(v):=\sup\{\lambda\mid(\lambda,u)\in\mathcal{E}(v),\ u\in\Omega_1\}. \tag{3.46}$$

Set

$$\Pi:=\{(\lambda^*,v)(\lambda^*,v)\ \in(\{\lambda^*\}\times\Omega_1)\cap\Sigma,\ \mathcal{E}(v)\ \text{ is unbounded in }\Gamma\}, \tag{3.47}$$

then $\Pi\neq\emptyset$ since

$$(\zeta_j\cap(\{\lambda^*\}\times\Omega_1))\subseteq\Pi,\quad j=0,1,2,\ldots. \tag{3.48}$$

From Lemma 2.4, it follows that Π is closed in $[0,\infty)\times E$, and furthermore, Π is compact in $[0,\infty)\times E$.

Let

$$\Sigma' := \bigcup_{(\lambda^*,\, v)\in\Pi} \mathcal{E}(v), \tag{3.49}$$

then

$$\zeta_j \subseteq \Sigma', \quad j = 0, 1, 2, \ldots. \tag{3.50}$$

If for some $(\lambda^*, v) \in \Pi$, $\mu(v) = +\infty$, then Theorem 1.2 holds.

Assume on the contrary that $\mu(v) < +\infty$ for all $(\lambda^*, v) \in \Pi$.

For every $(\lambda^*, v) \in \Pi$, let $\mathcal{E}'(v)$ be the component in $\mathcal{E}(v)\cap([\lambda^*, \infty)\times\Omega_1)$ which contains (λ^*, v). Using the standard method, we can find a bounded open set $U(v)$ in $[\lambda^*, \infty)\times\Omega_1$, such that

$$\mathcal{E}'(v) \subset U(v), \quad \partial U(v) \cap \Sigma' = \emptyset, \tag{3.51}$$

$$\sup\left\{\lambda \mid (\lambda, u) \in \overline{U}(v)\right\} < \infty, \tag{3.52}$$

where $\partial U(v)$ and $\overline{U}(v)$ are the boundary and closure of $U(v)$ in $[\lambda^*, \infty) \times \Omega_1$, respectively.

Evidently, the following family of the open sets of $\{\lambda^*\} \times \Omega_1$:

$$\{U(v) \cap (\{\lambda^*\} \times \Omega_1) \mid (\lambda^*, v) \in \Pi\} \tag{3.53}$$

is an open covering of Π. Since Π is compact set in $\{\lambda^*\} \times \Omega_1$, there exist $v_1, \ldots, v_m$ such that $(\lambda^*, v_i) \in \Pi$, $(i = 1, \ldots, m)$, and the family of open sets in $\{\lambda^*\} \times \Omega_1$:

$$\{U(v_i) \cap (\{\lambda^*\} \times \Omega_1) \mid i = 1, \ldots, m\} \tag{3.54}$$

is a finite open covering of Π. There is

$$\Pi \subseteq \{U(v_i) \cap (\{\lambda^*\} \times \Omega_1) \mid i = 1, \ldots, m\}. \tag{3.55}$$

Let

$$U_1 = \bigcup_{i=1}^{m} U(v_i). \tag{3.56}$$

Then U_1 is a bounded open set in $[\lambda^*, \infty) \times \Omega_1$,

$$\partial U_1 \cap \Sigma' = \emptyset, \tag{3.57}$$

and by (3.52), we have

$$\sup\left\{\lambda \mid (\lambda, u) \in \overline{U}_1\right\} < +\infty, \tag{3.58}$$

where ∂U_1 and $\overline{U}_1$ are the boundary and closure of U_1 in $[\lambda^*, \infty) \times \Omega_1$, respectively.

Equation (3.58) together with (3.55) and (3.57) implies that

$$\sup\{\lambda \mid (\lambda, u) \in \Sigma',\ u \in \Omega_1\} < \infty. \tag{3.59}$$

However, this contradicts (3.50).

Therefore, there exists $(\lambda^*, v^*) \in \Pi$ such that $\zeta := \mathcal{E}(v^*)$ which is unbounded in both Γ and $[\lambda^*, +\infty) \times \Omega_1$.

Finally, we show that $\zeta(= \mathcal{E}(v^*))$ joins (∞, θ) with (∞, ∞). This will be done by the following three steps.

Step 1. We show that $\zeta \cap ([0, \infty) \times \{\theta\}) = \emptyset$.

Suppose on the contrary that there exists $\{(\eta_n, y_n)\} \subset \zeta$ with

$$\eta_n \longrightarrow \eta^* \geq 0, \quad \|y_n\|_0 \longrightarrow 0. \tag{3.60}$$

Then

$$\begin{aligned} y_n(t) &= \eta_n \sum_{s=1}^{T} G(t,s) a(s) f(y_n(s)) = \eta_n \sum_{s=1}^{T} G(t,s) a(s) \frac{f(y_n(s))}{y_n(s)} y_n(s) \\ &\leq \eta_n \sum_{s=1}^{T} G(s,s) a(s) \frac{f(y_n(s))}{y_n(s)} \|y_n\|_0, \end{aligned} \tag{3.61}$$

which implies

$$1 \leq \eta_n \sum_{s=1}^{T} G(s,s) a(s) \frac{f(y_n(s))}{y_n(s)}. \tag{3.62}$$

This is impossible by (A3) and the assumption $\eta_n \to \eta^*$.

Step 2. We show that $\lim_{(\lambda,u)\in\zeta,\, u\in\Omega_1,\, \lambda\to+\infty} \|u\|_0 = 0$.

Suppose on the contrary that there exists $\{(\eta_n, y_n)\} \subset \zeta$ with $y_n \in \Omega_1$ and

$$\eta_n \longrightarrow +\infty, \qquad \|y_n\|_0 \geq a \tag{3.63}$$

for some constant $a > 0$, then

$$\frac{a}{T+1} \leq y_n(s) \leq 1, \quad \forall s \in \mathbb{T}. \tag{3.64}$$

Thus

$$\begin{aligned} y_n(t) &= \eta_n \sum_{s=1}^{T} G(t,s)a(s)f\left(y_n(s)\right) \\ &\geq \frac{\eta_n}{T+1}\sum_{s=1}^{T} G(s,s)a(s)\frac{f\left(y_n(s)\right)}{y_n(s)}y_n(s) \\ &\geq \frac{\eta_n}{(T+1)^2}\sum_{s=1}^{T} G(s,s)a(s)b\left\|y_n\right\|_0, \end{aligned} \tag{3.65}$$

where $b := \inf_{a/(T+1)\leq x\leq 1}(f(x)/x)$. By (A2), it follows that $b > 0$. Obviously, (3.65) implies that $\{\eta_n\}$ is bounded. This is a contradiction.

Step 3. We show that $\lim_{(\lambda,u)\in(\zeta\cap\Gamma),\,\lambda\to+\infty}\|u\|_0 = +\infty$.

Suppose on the contrary that there exists $\{(\eta_n, y_n)\} \subset (\zeta \cap \Gamma)$ with

$$\eta_n \longrightarrow +\infty, \qquad \left\|y_n\right\|_0 \leq M \tag{3.66}$$

for some constant $M > 0$, then

$$\frac{1}{T+1} \leq y_n(s) \leq M, \quad \forall s \in \mathbb{T}. \tag{3.67}$$

Thus

$$\begin{aligned} y_n(t) &= \eta_n \sum_{s=1}^{T} G(t,s)a(s)f\left(y_n(s)\right) \\ &\geq \frac{\eta_n}{T+1}\sum_{s=1}^{T} G(s,s)a(s)\frac{f\left(y_n(s)\right)}{y_n(s)}y_n(s) \\ &\geq \frac{\eta_n}{(T+1)^2}\sum_{s=1}^{T} G(s,s)a(s)B\left\|y_n\right\|_0, \end{aligned} \tag{3.68}$$

where $B := \inf_{1/(T+1)\leq x\leq M}(f(x)/x)$. By (A2), it follows that $B > 0$. Obviously, (3.68) implies that $\{\eta_n\}$ is bounded. This is a contradiction.

To sum up, there exits a component ζ which joins (∞, θ) and (∞, ∞). □

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