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Research Article

Existence of Solutions for m-point Boundary Value Problems on a Half-Line

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By using the Leray-Schauder continuation theorem, we establish the existence of solutions for m-point boundary value problems on a half-line $x''(t) + f(t, x(t), x'(t)) = 0, 0 < t < +\infty, x(0) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \lim_{t\to+\infty} x'(t) = 0$, where $\alpha_i \in R, \sum_{i=1}^{m-2} \alpha_i \neq 1$ and $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < +\infty$ are given.

1. Introduction

Multipoint boundary value problems (BVPs) for second-order differential equations in a finite interval have been studied extensively and many results for the existence of solutions, positive solutions, multiple solutions are obtained by use of the Leray-Schauder continuation theorem, Guo-Krasnosel'skii fixed point theorem, and so on; for details see [1–4] and the references therein.

In the last several years, boundary value problems in an infinite interval have been arisen in many applications and received much attention; see [5, 6]. Due to the fact that an infinite interval is noncompact, the discussion about BVPs on the half-line is more complicated, see [5–14] and the references therein. Recently, in [15], Lian and Ge studied the following three-point boundary value problem:

$$\begin{gathered} x''(t) + f\big(t, x(t), x'(t)\big) = 0, \quad 0 < t < +\infty, \\ x(0) = \alpha x(\eta), \qquad \lim_{t\to+\infty} x'(t) = 0, \end{gathered} \tag{1.1}$$

where $\alpha \in R, \alpha \neq 1$, and $\eta \in (0,+\infty)$ are given. In this paper, we will study the following m-point boundary value problems:

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty,$$
$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \qquad \lim_{t \to +\infty} x'(t) = 0, \tag{1.2}$$

where $\alpha_i \in R$, $\sum_{i=1}^{m-2} \alpha_i \neq 1, \alpha_i$ have the same signal, and $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < +\infty$ are given. We first present the Green function for second-order multipoint BVPs on the half-line and then give the existence results for (1.2) using the properties of this Green function and the Leray-Schauder continuation theorem.

We use the space $C^1_\infty[0,+\infty) = \{x \in C^1[0,+\infty), \lim_{t\to+\infty} x(t) \text{ exists}, \lim_{t\to+\infty} x'(t) \text{ exists}\}$ with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|\cdot\|_\infty$ is supremum norm on the half-line, and $L^1[0,+\infty) = \{x : [0,+\infty) \to R \text{ is absolutely integrable on } [0,+\infty)\}$ with the norm $\|x\|_{L^1} = \int_0^\infty |x(t)|dt$.

We set

$$P = \int_0^{+\infty} p(s)ds, \qquad P_1 = \int_0^{+\infty} sp(s)ds, \qquad Q = \int_0^{+\infty} q(s)dt, \tag{1.3}$$

and we suppose $\alpha_i, i = 1, 2, \ldots, m-2$ are the same signal in this paper and we always assume $\alpha = \sum_{i=1}^{m-2} \alpha_i$.

2. Preliminary Results

In this section, we present some definitions and lemmas, which will be needed in the proof of the main results.

Definition 2.1 (see [15]). It holds that $f : [0,+\infty) \times R^2 \longmapsto R$ is called an S-Carathéodory function if and only if

(i) for each $(u, v) \in R^2, t \mapsto f(t, u, v)$ is measurable on $[0,+\infty)$,

(ii) for almost every $t \in [0,+\infty), (u, v) \mapsto f(t, u, v)$ is continuous on R^2,

(iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1[0,+\infty)$ with $t\varphi_r(t) \in L^1[0,+\infty), \varphi_r(t) > 0$ on $(0,+\infty)$ such that $\max\{|u|, |v|\} \le r$ implies $|f(t, u, v)| \le \varphi_r(t)$, for a.e. $t \in [0,+\infty)$.

Lemma 2.2. *Suppose $\sum_{i=1}^{m-2} \alpha_i \neq 1$, if for any $v(t) \in L^1[0,+\infty)$ with $tv(t) \in L^1[0,+\infty)$, then the BVP,*

$$x''(t) + v(t) = 0, \quad 0 < t < +\infty,$$
$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \qquad \lim_{t \to +\infty} x'(t) = 0, \tag{2.1}$$

has a unique solution. Moreover, this unique solution can be expressed in the form

$$x(t) = \int_0^{+\infty} G(t,s)v(s)ds, \tag{2.2}$$

where $G(t,s)$ *is defined by*

$$G(t,s) = \frac{1}{\Lambda}\begin{cases} \sum_{i=1}^{m-2}\alpha_i s + \Lambda s, & s \le \eta_1,\ s \le t, \\ \sum_{i=1}^{m-2}\alpha_i s + \Lambda t, & s \le \eta_1,\ t \le s, \\ \sum_{k=1}^{i}\alpha_k\eta_k + \sum_{k=i+1}^{m-2}\alpha_k s + \Lambda s, & 0 < \eta_i \le s \le \eta_{i+1},\ s \le t,\ i = 1,2,\ldots,m-3, \\ \sum_{k=1}^{i}\alpha_k\eta_k + \sum_{k=i+1}^{m-2}\alpha_k s + \Lambda t, & 0 < \eta_i \le s \le \eta_{i+1},\ t \le s,\ i = 1,2,\ldots,m-3, \\ \sum_{i=1}^{m-2}\alpha_i\eta_i + \Lambda s, & s \ge \eta_{m-2},\ s \le t, \\ \sum_{i=1}^{m-2}\alpha_i\eta_i + \Lambda t, & s \ge \eta_{m-2},\ t \le s, \end{cases} \tag{2.3}$$

here note $\Lambda = 1 - \sum_{i=1}^{m-2}\alpha_i$.

Proof. Integrate the differential equation from t to $+\infty$, noticing that $v(t), tv(t) \in L^1[0,+\infty)$, then from 0 to t and one has

$$x(t) = x(0) + \int_0^t\int_s^{+\infty} v(\tau)d\tau\, ds. \tag{2.4}$$

Since $x(0) = \sum_{i=1}^{m-2}\alpha_i x(\eta_i)$, from (2.4), it holds that

$$x(t) = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\sum_{i=1}^{m-2}\alpha_i\eta_i\int_{\eta_i}^{+\infty} v(s)ds + \sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i} sv(s)ds\right] + t\int_t^{+\infty} v(s)ds + \int_0^t sv(s)ds. \tag{2.5}$$

For $0 \le t \le \eta_1$, the unique solution of (2.1) can be stated by

$$\begin{aligned} x(t) = {}& \int_0^t\left(\frac{\sum_{i=1}^{m-2}\alpha_i s}{1-\sum_{i=1}^{m-2}\alpha_i} + s\right)v(s)ds + \int_t^{\eta_1}\left(\frac{\sum_{i=1}^{m-2}\alpha_i s}{1-\sum_{i=1}^{m-2}\alpha_i} + t\right)v(s)ds \\ & + \sum_{i=1}^{m-3}\int_{\eta_i}^{\eta_{i+1}}\left(\frac{\sum_{k=1}^{i}\alpha_k\eta_k + \sum_{k=i+1}^{m-2}\alpha_k s + \Lambda t}{1-\sum_{i=1}^{m-2}\alpha_i}\right)v(s)ds + \int_{\eta_{m-2}}^{+\infty}\left(\frac{\sum_{i=1}^{m-2}\alpha_i\eta_i}{1-\sum_{i=1}^{m-2}\alpha_i} + t\right)v(s)ds. \end{aligned} \tag{2.6}$$

If $\eta_i \le t \le \eta_{i+1}, 1 \le i \le m-3$, the unique solution of (2.1) can be stated by

$$\begin{aligned}
x(t) = {} & \int_0^{\eta_1} \left(\frac{\sum_{i=1}^{m-2} \alpha_i s}{1 - \sum_{i=1}^{m-2} \alpha_i} + s \right) v(s) ds \\
& + \sum_{j=1}^{i-1} \int_{\eta_j}^{\eta_{j+1}} \left(\frac{\sum_{k=1}^{j} \alpha_k \eta_k + \sum_{k=j+1}^{m-2} \alpha_k s + \Lambda s}{1 - \sum_{i=1}^{m-2} \alpha_i} \right) v(s) ds \\
& + \int_{\eta_i}^{t} \left(\frac{\sum_{k=1}^{i} \alpha_k \eta_k + \sum_{k=i+1}^{m-2} \alpha_k s + \Lambda s}{1 - \sum_{i=1}^{m-2} \alpha_i} \right) v(s) ds \\
& + \int_{t}^{\eta_{i+1}} \left(\frac{\sum_{k=1}^{i} \alpha_k \eta_k + \sum_{k=i+1}^{m-2} \alpha_k s + \Lambda t}{1 - \sum_{i=1}^{m-2} \alpha_i} \right) v(s) ds \\
& + \sum_{j=i+1}^{m-3} \int_{\eta_j}^{\eta_{j+1}} \left(\frac{\sum_{k=1}^{j} \alpha_k \eta_k + \sum_{k=j+1}^{m-2} \alpha_k s + \Lambda t}{1 - \sum_{i=1}^{m-2} \alpha_i} \right) v(s) ds \\
& + \int_{\eta_{m-2}}^{+\infty} \left(\frac{\sum_{i=1}^{m-2} \alpha_i \eta_i}{1 - \sum_{i=1}^{m-2} \alpha_i} + t \right) v(s) ds.
\end{aligned} \tag{2.7}$$

If $\eta_{m-2} \le t < +\infty$, the unique solution of (2.1) can be stated by

$$\begin{aligned}
x(t) = {} & \int_0^{\eta_1} \left(\frac{\sum_{i=1}^{m-2} \alpha_i s}{1 - \sum_{i=1}^{m-2} \alpha_i} + s \right) v(s) ds + \sum_{i=1}^{m-3} \int_{\eta_i}^{\eta_{i+1}} \left(\frac{\sum_{k=1}^{i} \alpha_k \eta_k + \sum_{k=i+1}^{m-2} \alpha_k s + \Lambda s}{1 - \sum_{i=1}^{m-2} \alpha_i} \right) v(s) ds \\
& + \int_{\eta_{m-2}}^{t} \left(\frac{\sum_{i=1}^{m-2} \alpha_i \eta_i}{1 - \sum_{i=1}^{m-2} \alpha_i} + s \right) v(s) ds + \int_{t}^{+\infty} \left(\frac{\sum_{i=1}^{m-2} \alpha_i \eta_i}{1 - \sum_{i=1}^{m-2} \alpha_i} + t \right) v(s) ds.
\end{aligned} \tag{2.8}$$

We note $\Lambda = 1 - \sum_{i=1}^{m-2} \alpha_i$, then

$$G(t,s) = \frac{1}{\Lambda} \begin{cases}
\displaystyle\sum_{i=1}^{m-2} \alpha_i s + \Lambda s, & s \le \eta_1,\ s \le t, \\
\displaystyle\sum_{i=1}^{m-2} \alpha_i s + \Lambda t, & s \le \eta_1,\ t \le s, \\
\displaystyle\sum_{k=1}^{i} \alpha_k \eta_k + \sum_{k=i+1}^{m-2} \alpha_k s + \Lambda s, & 0 < \eta_i \le s \le \eta_{i+1},\ s \le t,\ i = 1, 2, \ldots, m-3, \\
\displaystyle\sum_{k=1}^{i} \alpha_k \eta_k + \sum_{k=i+1}^{m-2} \alpha_k s + \Lambda t, & 0 < \eta_i \le s \le \eta_{i+1},\ t \le s,\ i = 1, 2, \ldots, m-3, \\
\displaystyle\sum_{i=1}^{m-2} \alpha_i \eta_i + \Lambda s, & s \ge \eta_{m-2},\ s \le t, \\
\displaystyle\sum_{i=1}^{m-2} \alpha_i \eta_i + \Lambda t, & s \ge \eta_{m-2},\ t \le s.
\end{cases} \tag{2.9}$$

Therefore, the unique solution of (2.1) is $x(t) = \int_0^{+\infty} G(t,s)v(s)ds$, which completes the proof. □

*Remark of Lemma 2.2.*Obviously $G(t,s)$ satisfies the properties of a Green function, so we call $G(t,s)$ the Green function of the corresponding homogeneous multipoint BVP of (2.1) on the half-line.

Lemma 2.3. *For all $t, s \in [0, +\infty)$, it holds that*

$$|G(t,s)| \leq \begin{cases} s, & \sum_{i=1}^{m-2} \alpha_i < 0, \\ \dfrac{s}{\Lambda}, & 0 \leq \sum_{i=1}^{m-2} \alpha_i < 1, \\ \max\left\{ \dfrac{\sum_{i=1}^{m-2} \alpha_i s}{-\Lambda}, \dfrac{\sum_{i=1}^{m-2} \alpha_i \eta_{m-2}}{-\Lambda} \right\}, & \sum_{i=1}^{m-2} \alpha_i > 1. \end{cases} \tag{2.10}$$

Proof. For each $s \in [0, +\infty)$, $G(t,s)$ is nondecreasing in t. Immediately, we have

$$\begin{aligned} &\min\left\{ \frac{\sum_{i=1}^{m-2} \alpha_i s}{\Lambda}, \frac{\sum_{k=1}^{i} \alpha_k \eta_k + \sum_{k=i+1}^{m-2} \alpha_k s}{\Lambda}, \frac{\sum_{i=1}^{m-2} \alpha_i \eta_i}{\Lambda} \right\} \leq G(t,s) \leq G(s,s) \\ &= \frac{1}{\Lambda} \begin{cases} s, & s \leq \eta_1, \\ \displaystyle\sum_{k=1}^{i} \alpha_k \eta_k + \left(\sum_{k=i+1}^{m-2} \alpha_k + \Lambda \right) s, & \eta_i \leq s \leq \eta_{i+1} < +\infty,\ i = 1, 2, \ldots, m-3, \\ \displaystyle\sum_{i=1}^{m-2} \alpha_i \eta_i + \Lambda s, & s \geq \eta_{m-2}. \end{cases} \end{aligned} \tag{2.11}$$

Further, we have

$$\begin{gathered} \frac{\sum_{i=1}^{m-2} \alpha_i s}{\Lambda} \leq G(t,s) \leq s, \quad \sum_{i=1}^{m-2} \alpha_i < 0, \\ 0 < \min\left\{ \frac{\sum_{i=1}^{m-2} \alpha_i s}{\Lambda}, \frac{\sum_{i=1}^{m-2} \alpha_i \eta_1}{\Lambda} \right\} \leq G(t,s) \leq \frac{s}{\Lambda}, \quad 0 \leq \sum_{i=1}^{m-2} \alpha_i < 1, \\ \min\left\{ \frac{\sum_{i=1}^{m-2} \alpha_i s}{\Lambda}, \frac{\sum_{i=1}^{m-2} \alpha_i \eta_{m-2}}{\Lambda} \right\} \leq G(t,s) \leq s, \quad \sum_{i=1}^{m-2} \alpha_i > 1. \end{gathered} \tag{2.12}$$

Therefore, we get the result. □

Lemma 2.4. *For the Green function $G(t,s)$, it holds that*

$$\lim_{t\to+\infty} G(t,s) = \overline{G}(s) = \frac{1}{\Lambda}\begin{cases} s, & s \le \eta_1, \\ \displaystyle\sum_{k=1}^{i}\alpha_k\eta_k + \left(\sum_{k=i+1}^{m-2}\alpha_k + \Lambda\right)s, & \eta_i \le s \le \eta_{i+1} < +\infty, i = 1,2,\dots,m-3 \\ \displaystyle\sum_{i=1}^{m-2}\alpha_i\eta_i + \Lambda s, & s \ge \eta_{m-2}. \end{cases} \tag{2.13}$$

Lemma 2.5. *For the function $x \in C^1[0,+\infty)$, it is satisfied that*

$$x(0) = \sum_{i=1}^{m-2}\alpha_i x(\eta_i) \tag{2.14}$$

and α_i $(i = 1,2,\dots,m-2)$ have the same signal, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < +\infty$, then there exists $\eta \in [\eta_1, \eta_{m-2}]$ satisfying

$$x(0) = \alpha x(\eta), \tag{2.15}$$

where $\alpha = \sum_{i=1}^{m-2}\alpha_i$.

Proof. Let α_i $(i = 1,2,\dots,m-2)$ are positive, and note $M^* = \max\{x(t) \mid t \in [\eta_1,\eta_{m-2}]\}, m^* = \min\{x(t) \mid t \in [\eta_1,\eta_{m-2}]\}$, then for every i $(i = 1,2,\dots,m-2)$, we have $m^* \le x(\eta_i) \le M^*$, so $m^*\sum_{i=1}^{m-2}\alpha_i \le \sum_{i=1}^{m-2}\alpha_i x(\eta_i) \le M^*\sum_{i=1}^{m-2}\alpha_i$, that is, $m^* \le \sum_{i=1}^{m-2}\alpha_i x(\eta_i)/\sum_{i=1}^{m-2}\alpha_i x \le M^*$. Because $x(t)$ is continuous on the interval $[\eta_1,\eta_{m-2}]$, there exists $\eta \in [\eta_1,\eta_{m-2}]$ satisfying $x(0) = \alpha x(\eta)$, where $\alpha = \sum_{i=1}^{m-2}\alpha_i$. □

Theorem 2.6 (see [5]). *Let $M \subset C_\infty[0,+\infty) = \{x \in C[0,+\infty), \lim_{t\to+\infty}x(t)$ exists$\}$. Then M is relatively compact in X if the following conditions hold:*

(a) *M is uniformly bounded in $C_\infty[0,+\infty)$;*

(b) *the functions from M are equicontinuous on any compact interval of $[0,+\infty)$;*

(c) *the functions from M are equiconvergent, that is, for any given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $|f(t) - f(+\infty)| < \epsilon$, for any $t > T, f \in M$.*

3. Main Results

Consider the space $X = \{x \in C^1_\infty[0,+\infty), x(0) = \sum_{i=1}^{m-2}\alpha_i x(\eta_i), \lim_{t\to+\infty}x'(t) = 0\}$ and define the operator $T : X \times [0,1] \to X$ by

$$T(x,\lambda)(t) = \lambda\int_0^{+\infty} G(t,s)f(s,x(s),x'(s))ds, \quad 0 \le t < +\infty. \tag{3.1}$$

The main result of this paper is following.

Theorem 3.1. *Let $f : [0,+\infty) \times R^2 \mapsto R$ be an S-Carathéodory function. Suppose further that there exists functions $p(t), q(t) r(t) \in L^1[0,+\infty)$ with $tp(t), tq(t) tr(t) \in L^1[0,+\infty)$ such that*

$$\left|f(t,u,v)\right| \leq p(t)|u| + q(t)|v| + r(t) \tag{3.2}$$

for almost every $t \in [0,+\infty)$ and all $(u,v) \in R^2$. Then (1.2) *has at least one solution provided:*

$$\begin{gathered}
\eta_{m-2}P + P_1 + Q < 1, \quad \alpha < 0, \\
\frac{\alpha\eta_{m-2}}{1-\alpha}P + P_1 + Q < 1, \quad 0 \leq \alpha < 1, \\
\max\left\{\frac{\alpha\eta_{m-2}}{\alpha-1}P + P_1 + Q, \frac{\alpha P_1}{\alpha-1} + \frac{\alpha\eta_{m-2}P}{\alpha-1}\right\} < 1, \quad \alpha > 1.
\end{gathered} \tag{3.3}$$

Lemma 3.2. *Let $f : [0,+\infty) \times R^2 \to R$ be an S-Carathéodory function. Then, for each $\lambda \in [0,1], T(x,\lambda)$ is completely continuous in X.*

Proof. First we show T is well defined. Let $x \in X$; then there exists $r > 0$ such that $\|x\| \leq r$. For each $\lambda \in [0,1]$, it holds that

$$\begin{aligned}
T(x,\lambda)(t) &= \lambda\int_0^{+\infty} G(t,s) f\left(s, x(s), x'(s)\right) ds \\
&\leq \int_0^{+\infty} |G(t,s)|\varphi_r(s) ds < +\infty, \quad \forall t \in [0,\infty).
\end{aligned} \tag{3.4}$$

Further, $G(t,s)$ is continuous in t so the Lebesgue dominated convergence theorem implies that

$$\begin{aligned}
|T(x,\lambda)(t_1) - T(x,\lambda)(t_2)| &\leq \lambda\int_0^{+\infty} |G(t_1,s) - G(t_2,s)|\left|f\left(s,x(s),x'(s)\right)\right| ds \\
&\leq \lambda\int_0^{+\infty} |G(t_1,s) - G(t_2,s)|\varphi_r(s) ds \\
&\longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\left|T(x,\lambda)'(t_1) - T(x,\lambda)'(t_2)\right| &\leq \lambda\int_{t_1}^{t_2} \left|f\left(s,x(s),x'(s)\right)\right| ds \\
&\leq \int_{t_1}^{t_2} \varphi_r(s) ds \longrightarrow 0 \quad \text{as } t_1 \longrightarrow t_2,
\end{aligned} \tag{3.6}$$

where $0 \leq t_1, t_2 < +\infty$. Thus, $Tx \in C^1[0,+\infty)$.

Obviously, $T(x,\lambda)(0) = \sum_{i=1}^{m-2} \alpha_i T(x,\lambda)(\eta_i)$. Notice that

$$\lim_{t\to+\infty} T(x,\lambda)'(t) = \lim_{t\to+\infty}\int_t^{+\infty} f\left(s,x(s),x'(s)\right)ds = 0, \tag{3.7}$$

so we can get $T(x,\lambda)(t) \in X$.

We claim that $T(x,\lambda)$ is completely continuous in X, that is, for each $\lambda \in [0,1]$, $T(x,\lambda)$ is continuous in X and maps a bounded subset of X into a relatively compact set.

Let $x_n \to x$ as $n \to +\infty$ in X. Next we prove that for each $\lambda \in [0,1]$, $T(x_n,\lambda) \to T(x,\lambda)$ as $n \to +\infty$ in X. Because f is a S-Carathéodory function and

$$\left|\int_0^{+\infty} \overline{G}(s)\left(f\left(s,x_n(s),x_n'(s)\right) - f\left(s,x(s),x'(s)\right)\right)ds\right| \le 2\int_0^{+\infty}\left|\overline{G}(s)\right|\varphi_{r_0}(s)ds < +\infty, \tag{3.8}$$

where $r_0 > 0$ is a real number such that $\max\{\max_{n\in N\setminus\{0\}}\|x_n\|, \|x\|\} \le r_0$, we have

$$\begin{aligned} |T(x_n,\lambda)(+\infty) - T(x,\lambda)(+\infty)| &\le \lambda\int_0^{+\infty}\left|\overline{G}(s)\right|\left|f\left(s,x_n(s),x_n'(s)\right) - f\left(s,x(s),x'(s)\right)\right|ds \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \end{aligned} \tag{3.9}$$

Also, we can get

$$\begin{aligned} |T(x_n,\lambda)(t) - T(x_n,\lambda)(+\infty)| &\le \lambda\int_0^{+\infty}\left|G(t,s) - \overline{G}(s)\right|\left|f\left(s,x_n(s),x_n'(s)\right)\right|ds \\ &\le \int_0^{+\infty}\left|G(t,s) - \overline{G}(s)\right|\varphi_{r_0}(s)ds \\ &\longrightarrow 0, \quad \text{as } t \longrightarrow +\infty, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \left|T(x_n,\lambda)'(t) - T(x_n,\lambda)'(+\infty)\right| &\le \int_t^{+\infty}\left|f\left(s,x_n(s),x_n'(s)\right)\right|ds \\ &\le \int_t^{+\infty}\varphi_{r_0}(s)ds \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \end{aligned} \tag{3.11}$$

Similarly, we have

$$\begin{aligned} &|T(x,\lambda)(t) - T(x,\lambda)(+\infty)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty, \\ &\left|T(x,\lambda)'(t) - T(x,\lambda)'(+\infty)\right| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \end{aligned} \tag{3.12}$$

For any positive number $T_0 < +\infty$, when $t \in [0, T_0]$, we have

$$
\begin{aligned}
|T(x_n,\lambda)(t) - T(x,\lambda)(t)| &\le \int_0^{+\infty} |G(t,s)| \left| f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s)) \right| ds \\
&\longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \\
\left| T(x_n,\lambda)'(t) - T(x,\lambda)'(t) \right| &\le \int_t^{+\infty} \left| f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s)) \right| ds \\
&\longrightarrow 0, \quad \text{as } n \longrightarrow +\infty.
\end{aligned}
\tag{3.13}
$$

Combining (3.9)–(3.13), we can see that $T(\cdot,\lambda)$ is continuous. Let $B \subset X$ be a bounded subset; it is easy to prove that TB is uniformly bounded. In the same way, we can prove (3.5),(3.6), and (3.12), we can also show that TB is equicontinuous and equiconvergent. Thus, by Theorem 2.6, $T(\cdot,\lambda) : X \times [0,1] \to X$ is completely continuous. The proof is completed. □

Proof of Theorem 3.1. In view of Lemma 2.2, it is clear that $x \in X$ is a solution of the BVP (1.2) if and only if x is a fixed point of $T(\cdot, 1)$. Clearly, $T(x, 0) = 0$ for each $x \in X$. If for each $\lambda \in [0,1]$ the fixed points $T(\cdot,\lambda)$ in X belong to a closed ball of X independent of λ, then the Leray-Schauder continuation theorem completes the proof. We have known $T(\cdot,\lambda)$ is completely continuous by Lemma 3.2. Next we show that the fixed point of $T(\cdot,\lambda)$ has a priori bound M independently of λ. Assume $x = T(x,\lambda)$ and set

$$
Q_1 = \int_0^{+\infty} sq(s)ds, \qquad R = \int_0^{+\infty} r(s)ds, \qquad R_1 = \int_0^{+\infty} sr(s)dt. \tag{3.14}
$$

According to Lemma 2.5, we know that for any $x \in X$, there exists $\eta \in [\eta_1, \eta_{m-2}]$ satisfying $x(0) = \alpha x(\eta)$. Hence, there are three cases as follow.

Case 1 ($\alpha < 0$). For any $x \in X$, $x(0)x(\eta) \le 0$ holds and, therefore, there exists a $t_0 \in [0, \eta]$ such that $x(t_0) = 0$. Then, we have

$$
|x(t)| = \left| \int_{t_0}^{t} x'(s)ds \right| \le (t + \eta) \|x'\|_\infty \le (t + \eta_{m-2}) \|x'\|_\infty, \quad t \in [0, \infty), \tag{3.15}
$$

and so it holds that

$$
\begin{aligned}
\|x'\|_\infty &\le \|\lambda f(t,x,x')\|_{L^1} \le \|f(t,x,x')\|_{L^1} \\
&\le \left\| p(t)|x(t)| + q(t)|x'(t)| + r(t) \right\|_{L^1} \\
&\le (\eta_{m-2} P + P_1 + Q) \|x'\|_\infty + R,
\end{aligned}
\tag{3.16}
$$

therefore,

$$
\|x'\|_\infty \le \frac{R}{1 - \eta_{m-2} P - P_1 - Q} = M_1'. \tag{3.17}
$$

At the same time, we have

$$\begin{aligned} |x(t)| &\le \lambda\left|\int_0^\infty G(t,s) f\left(s, x(s), x'(s)\right) ds\right| \\ &\le \int_0^\infty \left|s f\left(s, x(s), x'(s)\right)\right| ds \\ &\le P_1\|x\|_\infty + Q_1 M_1' + R_1, \quad t \in [0,\infty), \end{aligned} \tag{3.18}$$

and so

$$\|x\|_\infty \le \frac{Q_1 M_1' + R_1}{1 - P_1} = M_1. \tag{3.19}$$

Set $M = \max\{M_1', M_1\}$, which is independent of λ.

Case 2 ($0 \le \alpha < 1$). For any $x \in X$, we have

$$|x(t)| = \left|\alpha x(\eta) + \int_0^t x'(s) ds\right| \le \alpha\left|x(\eta)\right| + t\left\|x'\right\|_\infty, \quad t \in [0,\infty), \tag{3.20}$$

which implies that $|x(t)| \le (\alpha\eta/(1-\alpha) + t)\|x'\|_\infty \le (\alpha\eta_{m-2}/(1-\alpha) + t)\|x'\|_\infty$ for all $t \in [0,\infty)$. In the same way as for Case 1, we can get

$$\begin{aligned} \left\|x'\right\|_\infty &\le \frac{(1-\alpha)R}{(1-\alpha)(1-P_1-Q) - \alpha\eta_{m-2}P} = M_2', \\ \|x\|_\infty &\le \frac{Q_1 M_2' + R_1}{1 - \alpha - P_1} = M_2. \end{aligned} \tag{3.21}$$

Set $M = \max\{M_2', M_2\}$, which is independent of λ and is what we need.

Case 3 ($\alpha > 1$). For $x \in X$, we have

$$|x(t)| = \left|x(\eta) + \int_\eta^t x'(s) ds\right| \le \frac{1}{\alpha}|x(0)| + \left|t - \eta\right|\left\|x'\right\|_\infty, \quad t \in [0,\infty), \tag{3.22}$$

and so $|x(t)| \le (\alpha\eta/(\alpha-1) + t)\|x'\|_\infty \le (\alpha\eta_{m-2}/(\alpha-1) + t)\|x'\|_\infty$ for all $t \in [0,\infty)$.

Similarly, we obtain

$$\begin{aligned} \left\|x'\right\|_\infty &\le \frac{(\alpha-1)R}{(\alpha-1)(1-P_1-Q) - \alpha\eta_{m-2}P} = M_3', \\ \|x\|_\infty &\le \frac{\alpha\left(Q_1 M_3' + R_1\right) + \alpha\eta_{m-2}\left(Q M_3' + R\right)}{\alpha - 1 - \alpha\eta_{m-2}P} = M_3. \end{aligned} \tag{3.23}$$

Set $M = \max\{M_3', M_3\}$ and which is we need. So (1.2) has at least one solution. □

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