Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 569803, 22 pages
doi:10.1155/2009/569803

Research Article

Some Basic Difference Equations of Schrödinger Boundary Value Problems

Andreas Ruffing,[1,2] Maria Meiler,[2] and Andrea Bruder[3]

[1] *Center for Applied Mathematics and Theoretical Physics (CAMTP), University of Maribor, Krekova Ulica 2, 2000 Maribor, Slovenia*
[2] *Department of Mathematics, Technische Universität München, Boltzmannstraße 3, 85747 Garching, Germany*
[3] *Department of Mathematics, Baylor University, One Bear Place 97328, Waco, TX 76798-7328, USA*

Correspondence should be addressed to Andreas Ruffing, ruffing@ma.tum.de

Received 1 April 2009; Accepted 28 August 2009

Recommended by Alberto Cabada

We consider special basic difference equations which are related to discretizations of Schrödinger equations on time scales with special symmetry properties, namely, so-called basic discrete grids. These grids are of an adaptive grid type. Solving the boundary value problem of suitable Schrödinger equations on these grids leads to completely new and unexpected analytic properties of the underlying function spaces. Some of them are presented in this work.

1. Introduction

It is well known that solving Schrödinger's equation is a prominent $\mathcal{L}^2$-boundary value problem. In this article, we want to become familiar with some of the dynamic equations that arise in context of solving the Schrödinger equation on a suitable time scale where the expression time scale is in the context of this article related to the *spatial variables.*

The Schrödinger equation is the partial differential equation ($t \in \mathbb{R}$)

$$\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+V(x,y,z)\right)\psi(x,y,z,t)=i\frac{\partial}{\partial t}\psi(x,y,z,t), \tag{1.1}$$

where the function $V:\Omega\subseteq\mathbb{R}^3\to\mathbb{R}$ yields information on the corresponding physically relevant *potential*. The solutions of the Schrödinger equation play a probabilistic role, being modeled by suitable $\mathcal{L}^2$-functions. For the convenience of the reader, let us first cite some of the fundamental facts on Schrödinger's equation. To do so, let us denote by $C^2(\Omega)$ all complex-valued functions which are defined on Ω and which are twice differentiable in each of their variables.

Definition 1.1. Let $\psi : \Omega \to \mathbb{C}$ be twice partially differentiable in its three variables. Let moreover $V : \Omega \to \mathbb{R}$ be a piecewise continuous function, $P(\Omega)$ denoting the space of piecewise continuous functions with values in $\mathbb{C}$. The linear map $H : C^2(\Omega) \to P(\Omega)$, given by

$$(H\psi)(x,y,z) := \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + V(x,y,z)\right)\psi(x,y,z) \tag{1.2}$$

is called *Schrödinger Operator* in $C^2(\Omega)$.

The following lemma makes a statement on the separation ansatz of the conventional Schrödinger partial differential equation where we throughout the sequel assume $\Omega = \mathbb{R}^3$.

Lemma 1.2 (Separation Ansatz). *Let the Schrödinger equation* (1.1) *be given, fulfilling the assertions of Definition 1.1 where $\Omega = \mathbb{R}^3$. In addition, the function V will have the property*

$$V : \mathbb{R}^3 \longrightarrow \mathbb{R}, \qquad (x,y,z) \longmapsto V(x,y,z) = f(x) + g(y) + h(z), \tag{1.3}$$

where f, g, h are continuous. Let now $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be such that there exist eigenfunctions $\psi_1, \psi_2, \psi_3 \in \mathcal{L}^2(\mathbb{R}) \cap C^2(\mathbb{R})$ with

$$\begin{aligned} -\psi_1''(x) + f(x)\ \psi_1(x) &= \lambda_1\psi_1(x), \quad x \in \mathbb{R}, \\ -\psi_2''(y) + g(y)\ \psi_2(y) &= \lambda_2\psi_2(y), \quad y \in \mathbb{R}, \\ -\psi_3''(z) + h(z)\ \psi_3(z) &= \lambda_3\psi_3(z), \quad z \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Then the function $\varphi \in (\mathcal{L}^2(\mathbb{R}^3) \times C^2(\mathbb{R})) \cap C^2(\mathbb{R}^4)$, given by

$$(x,y,z,t) \longmapsto \varphi(x,y,z,t) := \psi_1(x)\psi_2(y)\psi_3(z)e^{-i\lambda_1 t - i\lambda_2 t - i\lambda_3 t} \tag{1.5}$$

is a solution to Schrödinger's equation (1.1), *revealing a completely separated structure of the variables.*

A fascinating topic which has led to the results to be presented in this article is discretizing the Schrödinger equation on particular suitable time scales. This might be of importance for applications and numerical investigations of the underlying eigenvalue and spectral problems. Let us therefore restrict to the *purely discrete case*, that is, we are going to focus on a so-called *basic discrete quadratic grid* resp. on its closure which is a special time scale $\mathbb{T}$ with fascinating symmetry properties.

Definition 1.3. Let $0 < q < 1$ as well as $c_1, c_2, c_3 \in \mathbb{R}$. The set

$$\mathbb{R}_q^{(c_1,c_2,c_3)} := \left\{ c_1 q^n + c_2 q^{-n} + c_3 \mid n \in \mathbb{Z} \right\} \tag{1.6}$$

denotes the *basic discrete quadratic grid* where $c_1^2 + c_2^2 \neq 0$. For $n \in \mathbb{Z}$, we abbreviate $\gamma_n := c_1 q^n + c_2 q^{-n} + c_3$ and $-\mathbb{R}_q^{(c_1,c_2,c_3)} := \{ t \in \mathbb{R} \mid -t \in \mathbb{R}_q^{(c_1,c_2,c_3)} \}$. Define the set

$$\mathbb{T} := \overline{\mathbb{R}_q^{(c_1,c_2,c_3)} \cup -\mathbb{R}_q^{(c_1,c_2,c_3)}} \tag{1.7}$$

as well as the set $\mathbb{T}^*$ by

$$\mathbb{T}^* = \mathbb{T} \setminus \{0\}. \tag{1.8}$$

The boundary conditions on the functions we need for the discretized version of Schrödinger's equation are then given by the requirement

$$\mathcal{L}^2(\mathbb{T}^*) := \left\{ f : \mathbb{T}^* \longrightarrow \mathbb{C} \mid (f, f) < \infty \right\}, \tag{1.9}$$

where the scalar product (f, g) for two suitable functions $f, g : \mathbb{T}^* \to \mathbb{R}$ will be specified by

$$(f, g) := \sum_{n=-\infty}^{\infty} |\gamma_{n+1} - \gamma_n| \Big(f(\gamma_n) \overline{g(\gamma_n)} + f(-\gamma_n) \overline{g(-\gamma_n)} \Big). \tag{1.10}$$

In this context, we assume that $|\gamma_{n+1} - \gamma_n| \neq 0$ for all $n \in \mathbb{Z}$. By construction, it is clear that $\mathcal{L}^2(\mathbb{T}^*)$ is a Hilbert space over $\mathbb{C}$ as it is a weighted sequence space, one of its orthogonal bases being given by all functions $\tilde{e}_n^{\sigma} : \mathbb{T} \setminus \{0\} \to \{0, 1\}$ which are specified by $\tilde{e}_n^{\sigma}(\tau q^m) := \delta_{mn}\, \delta_{\sigma\tau}$ with $m, n \in \mathbb{Z}$ and $\sigma, \tau \in \{1, -1\}$.

Already now, we can say that the separation ansatz for the discretized Schrödinger equation will lead us to looking for eigensolutions of a given Schrödinger operator in the threefold product space $\mathcal{L}^2(\mathbb{T}^*) \times \mathcal{L}^2(\mathbb{T}^*) \times \mathcal{L}^2(\mathbb{T}^*)$.

Hence we come to the conclusion that in case of the separation ansatz for the Schrödinger equation, the following *rationale* applies:

> *The solutions of a Schrödinger equation on a basic discrete quadratic grid are directly related to the spectral behavior of the Jacobi operators acting in the underlying weighted sequence spaces.*

Before presenting discrete versions of the Schrödinger equation on basic quadratic grids, let us first come back to the situation of Lemma 1.2 where we now assume that the potential is given by the requirement

$$f(x) = x^2, \quad g(y) = y^2, \quad h(z) = z^2, \quad (x, y, z) \in \mathbb{R}^3. \tag{1.11}$$

Hence, it is sufficient to look at the one-dimensional Schrödinger equation

$$-\psi''(x) + x^2 \psi(x) = \lambda \psi(x). \tag{1.12}$$

For the convenience of the reader, let us refer to the following fact. let the sequence of functions $(\psi_n)_{n\in\mathbb{N}_0}$ be recursively given by the requirement

$$\psi_{n+1}(x) := -\psi_n'(x) + x\psi_n(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}_0, \tag{1.13}$$

where $\psi_0 : \mathbb{R} \to \mathbb{R}, x \mapsto \psi_0(x) := e^{-(1/2)x^2}$. We then have $\psi_n \in \mathcal{L}^2(\mathbb{R}) \cap C^2(\mathbb{R})$ for $n \in \mathbb{N}_0$ and moreover

$$-\psi_n''(x) + x^2\psi_n(x) = (2n+1)\ \psi_n(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}_0. \tag{1.14}$$

This result reflects the celebrated so-called *Ladder Operator Formalism*. We first review a main result in discrete Schrödinger theory that is a basic analog of the just described continuous situation. Let us therefore state in a next step some more useful tools for the discrete description.

Definition 1.4. Let $0 < q < 1$ and let $\mathbb{T}$ be a nonempty closed set with the properties

$$\forall x \in \mathbb{T}:\ qx \in \mathbb{T}, q^{-1}x \in \mathbb{T}, -x \in \mathbb{T}. \tag{1.15}$$

Let for any $f : \mathbb{T} \to \mathbb{R}$ the right-shift resp. left-shift operation be fixed through

$$(Rf)(x) := f(qx), \quad (Lf)(x) := f\left(q^{-1}x\right), \quad x \in \mathbb{T}, \tag{1.16}$$

respectively, the right-hand resp. left-hand basic difference operation will for any function $f : \mathbb{T}^* \to \mathbb{R}$ be given by

$$(D_q f)(x) := \frac{f(qx) - f(x)}{qx - x}, \quad (D_{q^{-1}} f)(x) := \frac{f(q^{-1}x) - f(x)}{q^{-1}x - x}, \quad x \in \mathbb{T} \setminus \{0\}. \tag{1.17}$$

Let moreover $\alpha > 0$ and let

$$g : \mathbb{T} \setminus \{0\} \longrightarrow \mathbb{R}, \quad x \longmapsto g(x) := \frac{\sqrt{\varphi(qx)} - \sqrt{\varphi(x)}}{\sqrt{\varphi(x)}\ (q-1)x} = \frac{\sqrt{1 + \alpha(1-q)x^2} - 1}{qx - x}, \tag{1.18}$$

where the positive even function $\varphi : \mathbb{T} \to \mathbb{R}^+$ is chosen as a solution to the basic difference equation

$$\varphi(qx) = \left(1 + \alpha(1-q)x^2\right)\varphi(x), \quad x \in \mathbb{T}. \tag{1.19}$$

The *creation operation* A^*resp. *annihilation operation* A are then introduced by their actions on any $\psi : \mathbb{T}^* \to \mathbb{R}$ as follows:

$$A^*\psi = (-D_q + g(X)R)\psi, \quad A\psi = q^{-1}(LD_q + Lg(X))\psi. \tag{1.20}$$

We refer to the discrete Schrödinger equation with an oscillator potential on $\mathbb{T}^*$ by

$$q^{-1}\left(-D_q + g(X)R\right)\left(LD_q + Lg(X)\right)\psi = \lambda\psi. \tag{1.21}$$

The following result reveals that the discrete Schrödinger equation with an oscillator potential on $\mathbb{T}^*$ shows similar properties than its classical analog does.

Lemma 1.5. *Let the function φ be specified like in Definition 1.4, satisfying the basic difference equation* (1.19) *on $\mathbb{R}_q^{(c_1,0,0)}$ with $c_1 \neq 0$. Let moreover $\psi_0 : \mathbb{R}_q^{(c_1,0,0)} \to \mathbb{R}, x \mapsto \psi_0(x) := \sqrt{\varphi(x)}$. For $n \in \mathbb{N}_0$, the functions $\psi_n : \mathbb{R}_q^{(c_1,0,0)} \to \mathbb{R}$, given by $\psi_n(x) := ((A^*)^n\psi_0)(x)$ (while $x \in \mathbb{R}_q^{(c_1,0,0)}$) are well defined in $\mathcal{L}^2(\mathbb{R}_q^{(c_1,0,0)})$ and solve the basic Schrödinger equation* (1.21) *in the following sense:*

$$\begin{gathered}
q^{-1}\left(-D_q + g(X)R\right)\left(LD_q + Lg(X)\right)\psi_n = \frac{q^n-1}{q-1}\,\psi_n, \\
A^*\psi_n = \psi_{n+1}, \quad A\psi_n = \frac{q^n-1}{q-1}\,\psi_{n-1}, \quad \psi_n(x) = H_n^q(x)\,\psi_0(x), \\
H_{n+1}^q(x) - \alpha q^n x H_n^q(x) + \alpha\frac{q^n-1}{q-1}H_{n-1}^q(x) = 0, \quad H_0^q(x) = 1, \quad H_1^q(x) = \alpha x.
\end{gathered} \tag{1.22}$$

These relations apply for $x \in \mathbb{R}_q^{(c_1,0,0)}$ and $n \in \mathbb{N}_0$ where one set $\psi_{-1} := 0, H_{-1}^q := 0$. Moreover, the corresponding moments of the orthogonality measure for the polynomials $(H_n^q)_{n\in\mathbb{N}_0}$, arising from (1.19), *are given by*

$$\mu_{2n+2} = \frac{q^{-2n-1}-1}{\alpha(1-q)}\mu_{2n}, \quad \mu_{2n+1} = 0, n \in \mathbb{N}_0. \tag{1.23}$$

The proof for the lemma is straightforward and obeys the techniques in [1].

The following central question concerning the functions spaces behind the Schrödinger equation (1.21) *is open* and shall be partially attacked in the sequel.

1.1. Central Problem

What are the relations between the linear span of all functions $\psi_n, n \in \mathbb{N}_0$ arising from Lemma 1.5 and the function space $\mathcal{L}^2(\mathbb{R}_q^{(c_1,0,0)})$?

In contrast to the fact that the corresponding question in the Schrödinger differential equation scenario is very well understood, *the basic discrete scenario* reveals much more structure which is going to be presented throughout the sequel of this article.

All the stated questions are closely connected to solutions of the equation

$$\varphi(qx) = \left(1 + \alpha(1-q)x^2\right)\varphi(x), \quad x \in \mathbb{T} \tag{1.24}$$

which originated in context of basic discrete ladder operator formalisms. We are going to investigate the rich analytic structure of its solutions in Section 2 and are going to exploit new facts on the corresponding moment problem in Section 3 of this article.

Let us remark finally that we will—throughout the presentation of our results in this article—repeatedly make use of the suffix *basic*. The meaning of it will always be related to the basic discrete grids that we have introduced so far.

The following results will shed some new light on function spaces which are behind basic difference equations. They are not only of interest to applications in mathematical physics but their functional analytic impact will speak for itself. The results altogether show that solving the boundary value problems of Schrödinger equations on time scales (that have the structure of adaptive grids) is a wide new research area. A lot of work still has to be invested into this direction.

For more physically related references on the topic, we invite the interested reader to consider also the work in [2–5].

For the more mathematical context, see, for instance, [1, 6–12].

2. Completeness and Lack of Completeness

In the sequel, we will make use of the basic discrete grid:

$$\mathbb{R}_q := \{\pm q^n \mid n \in \mathbb{Z}\}, \tag{2.1}$$

and we will consider the Hilbert space

$$\mathcal{L}^2(\mathbb{R}_q) := \left\{ f : \mathbb{R}_q \longrightarrow \mathbb{C} \mid \sum_{n=-\infty}^{\infty} q^n \left(f(q^n)\overline{f(q^n)} + f(-q^n)\overline{f(-q^n)} \right) < \infty \right\}. \tag{2.2}$$

Theorem 2.1. *Let $0 < q < 1$ and $\alpha > 0$ as well as φ an even positive solution of*

$$\varphi(qx) = \left(1 + \alpha(1-q)x^2\right)\varphi(x) \tag{2.3}$$

on the basic discrete grid $\mathbb{R}_q$. Let the sequences of functions $(\varphi_m)_{m\in\mathbb{Z}}$ be given by shifted versions of the φ-function as follows:

$$\begin{gathered} \varphi_m \in \mathcal{L}^2(\mathbb{R}_q) \quad \varphi_m : \mathbb{R}_q \longrightarrow \mathbb{R}, \quad x \longmapsto \varphi_m(x) := \varphi(q^m x) = \varphi(-q^m x), \quad m \in \mathbb{Z}, \\ \psi_m \in \mathcal{L}^2(\mathbb{R}_q) \quad \psi_m : \mathbb{R}_q \longrightarrow \mathbb{R}, \quad x \longmapsto \psi_m(x) := x \ \ \varphi(q^m x), \quad m \in \mathbb{Z}. \end{gathered} \tag{2.4}$$

The finite linear complex span of precisely all the functions $(\varphi_m)_{m\in\mathbb{Z}}$ and $(\psi_m)_{m\in\mathbb{Z}}$ is then dense in $\mathcal{L}^2(\mathbb{R}_q)$.

Proof. Let $\varphi \in \mathcal{L}^2(\mathbb{R}_q)$ be a positive and even solution to

$$\varphi(qx) = \left(1 + \alpha(1-q)x^2\right)\varphi(x), \quad x \in \mathbb{R}_q. \tag{2.5}$$

One can easily show that an $\mathcal{L}^2(\mathbb{R}_q)$-solution with these properties uniquely exists, up to a positive factor, moreover all the functions defined by (2.4) are well defined in $\mathcal{L}^2(\mathbb{R}_q)$. Let us refer by the sequence $(P_n)_{n\in\mathbb{N}_0}$ to all the orthonormal polynomials $P_n : \mathbb{R}_q \to \mathbb{R}$ which arise from the Gram-Schmidt procedure with respect to the function φ^2. They satisfy a three-term recurrence relation

$$P_{n+1}(x) - \alpha_n x P_n(x) + \beta_n P_{n-1}(x) = 0, \quad P_{-1}(x) = 0, x \in \mathbb{R}_q, n \in \mathbb{N}_0, \tag{2.6}$$

where for $n \in \mathbb{N}_0$ the coefficients α_n, β_n may be determined by standard methods through the moments resulting from (2.5). From the basic difference equation (2.5) we may also conclude that the polynomials $(P_n)_{n\in\mathbb{N}_0}$ are subject to an indeterminate moment problem, we come back to this in Section 3.

For $n \in \mathbb{N}_0$ and $x \in \mathbb{R}_q$, the functions given by $P_n(x)\varphi(x)$ may now be normalized, let us denote their norms by ρ_n where n is running in $\mathbb{N}_0$. Let us for $n \in \mathbb{N}_0$ moreover denote the normalized versions of the functions $P_n\varphi$ by u_n.

The following observation is essential. the $\mathbb{C}$-linear finite span of all functions given by

$$(R^m\varphi)(x), \quad x(R^m\varphi)(x), \quad x \in \mathbb{R}_q, \ m \in \mathbb{Z} \tag{2.7}$$

is the same than the $\mathbb{C}$-linear finite span of all functions specified by

$$(R^m u_n)(x), \quad x(R^m u_n)(x), \quad x \in \mathbb{R}_q, \ m \in \mathbb{Z}, \ n \in \mathbb{N}_0. \tag{2.8}$$

As a consequence of (2.6), we conclude that the functions $(u_n)_{n\in\mathbb{N}_0}$ fulfill the recurrence relation:

$$\rho_{n+1}u_{n+1}(x) - \alpha_n\rho_n x u_n(x) + \beta_n\rho_{n-1}u_{n-1}(x) = 0, \quad u_{-1}(x) = 0, x \in \mathbb{R}_q, \ n \in \mathbb{N}_0. \tag{2.9}$$

How ever (2.9) can be brought into the standard form which is of relevance for considering the corresponding Jacobi operator,

$$Xu_n = a_{n+1}u_{n+1} + a_n u_{n-1}, \quad n \in \mathbb{N}_0, \tag{2.10}$$

where the coefficients are given by $a_0 = 0, a_{n+1} = \sqrt{\beta_{n+1}/\alpha_n\alpha_{n+1}}, n \in \mathbb{N}_0$.

The representation (2.10) results from the fact that the functions $(u_n)_{n\in\mathbb{N}_0}$ constitute a system of orthonormal functions and due to the fact that X, acting as a multiplication operator, requires to be a formally symmetric linear operator on the finite linear span of the orthonormal system $(u_n)_{n\in\mathbb{N}_0}$. Let us now consider the Hilbert space:

$$\mathcal{H} := \left\{ \sum_{n=0}^{\infty} c_n u_n \mid \sum_{n=0}^{\infty} |c_n|^2 < \infty, c_n \in \mathbb{C},\ n \in \mathbb{N}_0 \right\}. \tag{2.11}$$

As for the definition range of X in $\mathcal{H}$, let us choose X as a densely defined linear operator in $\mathcal{H}$ where we assume that

$$D(X) := \left\{ \sum_{n=0}^{k} c_n u_n \mid c_n \in \mathbb{C}, k \in \mathbb{N}_0 \right\}. \tag{2.12}$$

Let the expansion for a possible eigenvector of the adjoint X^* be written as $\sum_{j=0}^{\infty} c_j \psi_j$, the eigenvalue equation being $X^* \sum_{j=0}^{\infty} c_j \psi_j = \lambda \sum_{j=0}^{\infty} c_j \psi_j$. Note that the type of moment problem behind is related to the situation that $X : D(X) \subseteq \mathcal{H} \to \mathcal{H}$ has deficiency indices $(1,1)$. This also implies that any $\lambda \in \mathbb{C}$ constitutes an eigenvalue of X^*, hence the point spectrum of X^* is $\mathbb{C}$. According to the deficiency index structure $(1,1)$ of the operator X, let us now choose the particular self-adjoint extension Y of X which allows a prescribed real-eigenvalue $\lambda = 1 \neq 0$. The corresponding situation for the eigensolution may be written as

$$Y \sum_{j=0}^{n} c_j u_j = \sum_{j=0}^{n} c_j u_j + w_n, \quad n \in \mathbb{N}_0, \tag{2.13}$$

where the sequence $(w_n)_{n\in\mathbb{N}_0}$ converges to 0 in the sense of the canonical $\mathcal{L}^2(\mathbb{R}_q)$-norm.

The element $\sum_{j=0}^{n} c_j u_j$ is in the finite linear space of all functions $u_j, j \in \mathbb{N}_0$. Applying the powers $R^k, k \in \mathbb{Z}$ of the shift operator R (being given by $(Rv)(x) := v(qx)$ for any function $v \in \mathcal{L}^2(\mathbb{R}_q), x \in \mathbb{R}_q$) to (2.13) now leads to the fact that we can construct all eigenfunctions of the operator Y belonging to $q^k, k \in \mathbb{Z}$, as a consequence of

$$R^k Y \sum_{j=1}^{n} c_j u_j = \lambda q^k R^k \sum_{j=1}^{n} c_j u_j + R^k w_n, \quad n \in \mathbb{N}_0. \tag{2.14}$$

Note that we have used in (2.14) the commutation behavior $R^k X = q^k X R^k$ which is satisfied for any fixed $k \in \mathbb{Z}$ and in addition the fact that the sequence $(R^k w_n)_{n\in\mathbb{N}_0}$ again converges to 0 in the sense of the canonical $\mathcal{L}^2(\mathbb{R}_q)$-norm for any $k \in \mathbb{Z}$. An analogous result is obtained in the case when we start with the eigenvalue $\lambda = -1 \neq 0$.

Summing up the stated facts, we see that the self-adjoint operator Y, interpreted now as the multiplication operator, acting on a dense domain in $\mathcal{L}^2(\mathbb{R}_q)$, has precisely the point spectrum $\{q^n, -q^n \mid n \in \mathbb{Z}\}$ in the sense of

$$Y e_n^{\sigma} = \sigma q^n e_n^{\sigma}, \quad n \in \mathbb{Z},\ \sigma \in \{+1, -1\}, \tag{2.15}$$

the functions $e_n^\sigma,\ n \in \mathbb{Z}$ with norm 1 being fixed by

$$e_n^\sigma(\tau q^m) := \frac{1}{q^{n/2}\sqrt{1-q}}\,\delta_{\sigma\tau}\delta_{mn}, \quad m,n \in \mathbb{Z},\ \sigma,\tau \in \{+1,-1\}. \tag{2.16}$$

Let us recall what we had stated at the beginning: the $\mathbb{C}$-linear finite span of all functions

$$(R^m\varphi)(x), \quad x(R^m\varphi)(x), \quad x \in \mathbb{R}_q,\ m \in \mathbb{Z} \tag{2.17}$$

is the same than the $\mathbb{C}$-linear finite span of all functions:

$$(R^m u_n)(x), \quad x(R^m u_n)(x), \quad x \in \mathbb{R},\ m \in \mathbb{Z},\ n \in \mathbb{N}_0. \tag{2.18}$$

Taking the observations together, we conclude that the $\mathbb{C}$-linear span of all functions

$$\varphi_m(x) = (R^m\varphi)(x), \quad \psi_m(x) = x(R^m\varphi)(x), \quad x \in \mathbb{R}_q,\ m \in \mathbb{Z} \tag{2.19}$$

is dense in the original Hilbert space $\mathcal{L}^2(\mathbb{R}_q)$.

We finally want to show that the $\mathbb{C}$-linear span of *precisely all* the functions in (2.19) is dense in $\mathcal{L}^2(\mathbb{R}_q)$. This can be seen as follows. taking away one of the functions $R^n\varphi$ or $XR^n\varphi$ $(n \in \mathbb{N}_0)$ would already remove the completeness of the smaller Hilbert space $\mathcal{H} \subseteq \mathcal{L}^2(\mathbb{R}_q)$. According to the property that the functions from (2.19) are dense in $\mathcal{L}^2(\mathbb{R}_q)$, it follows, for instance, that for any $k \in \mathbb{Z}$, there exists a double sequence $(c_j^k)_{k,j\in\mathbb{Z}}$ such that in the sense of the canonically induced $\mathcal{L}^2(\mathbb{R}_q)$-norm:

$$\lim_{l\longrightarrow\infty}\left\| e_{k+1}^{+1} + e_{k+1}^{-1} - \sum_{j=-l}^{l} c_j^k\varphi_j \right\|_{\mathcal{L}^2(\mathbb{R}_q)=0}, \quad k \in \mathbb{Z}. \tag{2.20}$$

Suppose that there exists a specific $i \in \mathbb{Z}$ such that $c_i^k = 0$ for all $k \in \mathbb{Z}$.

Let us consider the following expression where $l \in \mathbb{N}$ and $-l < i < l$:

$$Re_{k+1}^{+1} + Re_{k+1}^{-1} - \sum_{j=-l}^{l} c_j^k R\varphi_j = e_k^{+1} + e_k^{-1} - \sum_{j=-l}^{l} c_j^k\varphi_{j+1}, \quad k \in \mathbb{Z}. \tag{2.21}$$

The last expression now may be rewritten as

$$e_k^{+1} + e_k^{-1} - \sum_{j=-l+2}^{l+1} c_{j-1}^k\varphi_j, \quad k \in \mathbb{Z}. \tag{2.22}$$

Successive application of R^m to (2.21) resp. (2.22) with $m \in \mathbb{N}_0$ resp. $-m \in \mathbb{N}_0$ shows that the existence of such a specific $c_i^k = 0$ for all $k \in \mathbb{Z}$ would finally imply that For all $j,k \in \mathbb{Z} : c_j^k = 0$.

This however would lead to a contradiction. Therefore, it becomes apparent that the complex finite linear span of precisely all the functions $R^m\varphi$ resp. $XR^m\varphi$ (where m is running in $\mathbb{Z}$) is dense in the Hilbert space $\mathcal{L}^2(\mathbb{R}_q)$. Summing up all facts, the basis property stated in the theorem finally follows according to (2.15).

Let us now focus on the following situation to move on towards the second main result of this article.

let P be a positive symmetric polynomial, that is,

$$P : \mathbb{R} \longrightarrow \mathbb{R}^+, \quad x \longmapsto P(x) = P(-x) > 0, \quad P(x) = \sum_{j=0}^{N} \gamma_j x^{2j}, \quad \gamma_j > 0, \tag{2.23}$$

where $N \in \mathbb{N}_0$. □

Definition 2.2. Let $f \in \mathcal{L}^1(\mathbb{R}) \cap C^1(\mathbb{R})$ with finite moments. Then

$$P_f := \left\{ \sum_{j=0}^{N} c_j X^j f \mid c_j \in \mathbb{R}, N \in \mathbb{N}_0 \right\} \tag{2.24}$$

is called the *real polynomial hull* of f.

Theorem 2.3. *Let $0 < q < 1$ and moreover $f(qx) = P(x)f(x)$, $x \in \mathbb{R}$, $f \in C^1(\mathbb{R})$. Then $P_{\sqrt{f}}$ is not dense in $\mathcal{L}^2(\mathbb{R})$.*

Proof. For $n \in \mathbb{N}_0$, the nth moment μ_n of f can be calculated from the prerequisites of Theorem 2.3, namely;

$$\int_{-\infty}^{\infty} x^n f(qx)dx = \int_{-\infty}^{\infty} x^n \sum_{j=0}^{N} \gamma_j x^{2j} f(x)dx, \tag{2.25}$$

where $N \in \mathbb{N}_0$—written in short:

$$q^{-n-1}\mu_n = \sum_{j=0}^{N} \gamma_j \mu_{n+2j}. \tag{2.26}$$

Discretization and integration on the basic grid $\mathbb{R}_q := \{\pm q^k \mid k \in \mathbb{Z}\}$ gives

$$\sum_{k=-\infty}^{\infty} q^k \left(q^k\right)^n f\left(q^{k+1}\right) \sum_{k=-\infty}^{\infty} q^k \left(q^k\right)^n \sum_{j=0}^{N} \gamma_j \left(q^k\right)^{2j} f\left(q^k\right). \tag{2.27}$$

and, if k is changed into $k+1$ on the left-hand side,

$$\sum_{k=-\infty}^{\infty} q^{k-1} \left(q^{k-1}\right)^n f\left(q^k\right) \sum_{j=0}^{N} \gamma_j \sum_{k=-\infty}^{\infty} q^k \left(q^k\right)^{n+2j} f\left(q^k\right). \tag{2.28}$$

Define for $n \in \mathbb{N}_0$:

$$\widehat{\mu}_n := \sum_{k=-\infty}^{\infty} q^k \left(q^k\right)^n f\left(q^k\right). \tag{2.29}$$

Then, we have

$$q^{-n-1}\widehat{\mu}_n = \sum_{j=0}^{N} \gamma_j \widehat{\mu}_{n+2j}. \tag{2.30}$$

If two densities generate the same moments then the induced orthogonal polynomials are the same. this is an isometry situation. According to the constructions of the two different types of moments, namely, on the one, hand side, the moments of type μ_n and on the other hand the moments of type $\widehat{\mu}_n$ and by comparing (2.26) and (2.30), we see that here, the mentioned isometry situation is matched—provided the initial conditions for the respective moments are chosen in the same way.

We use this observation now to proceed with the conclusions.

Let us make use again of the lattice

$$\mathbb{R}_q := \{\pm q^n \mid n \in \mathbb{Z}\}. \tag{2.31}$$

We define the restriction $\widehat{f}$ of f on $\mathbb{R}_q$ by

$$\widehat{f} : \mathbb{R}_q \longrightarrow \mathbb{R}, \qquad \widehat{f}(\pm q^n) := f(\pm q^n), \quad n \in \mathbb{Z}, \tag{2.32}$$

and we will use again the Hilbert space

$$\mathcal{L}^2(\mathbb{R}_q) := \left\{ f : \mathbb{R}_q \longrightarrow \mathbb{C} \mid \sum_{n=-\infty}^{\infty} q^n \left(f(q^n)\overline{f(q^n)} + f(-q^n)\overline{f(-q^n)} \right) < \infty \right\}. \tag{2.33}$$

Then the discrete analog of P_f is

$$P_{\widehat{f}} := \left\{ \sum_{j=0}^{N} c_j X^j \widehat{f} \mid c_j \in \mathbb{R}, N \in \mathbb{N}_0 \right\}. \tag{2.34}$$

In order to show that $P_{\sqrt{\widehat{f}}}$ is not dense in $\mathcal{L}^2(\mathbb{R}_q)$, we will construct a linear operator being bounded in the space $P_{\sqrt{f}}$ but unbounded when restricted to $P_{\sqrt{\widehat{f}}}$.

Let us start from a function $\varphi \in P_{\sqrt{f}}$, given by

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto \varphi(x) := \sum_{j=0}^{N} c_j x^j \sqrt{f(x)}, \tag{2.35}$$

and define generally for $\psi \in \mathcal{L}^2(\mathbb{R})$:

$$(L\psi)(x) := \psi\left(q^{-1}x\right). \tag{2.36}$$

We denote from now on the respective multiplication operator again by X and use $LX = q^{-1}XL$ to see that for a suitable operator-valued function f, the following holds:

$$(LQ(X))\sum_{j=0}^{N} c_j X^j \sqrt{f(X)} = \sum_{j=0}^{N} c_j q^{-j} X^j LQ(X)\sqrt{f(X)}. \tag{2.37}$$

Using $f(qx) = P(x)f(x)$, choose Q such that $LQ(X)\sqrt{f(X)}$ obeys:

$$\begin{aligned} LQ(X)\sqrt{f(X)} &= Q\left(q^{-1}X\right)\sqrt{f(q^{-1}X)} \\ &= Q\left(q^{-1}X\right)\sqrt{\frac{f(X)}{P(q^{-1}X)}} \\ &= Q\left(q^{-1}X\right)\sqrt{\frac{1}{P(q^{-1}X)}}\sqrt{f(X)}, \end{aligned} \tag{2.38}$$

that is, we choose $Q(X) := \sqrt{P(X)}$.

Note that for the definition of Q, we need to consider the characterization of the corresponding measure.

Now, rewrite this as

$$L\sqrt{P(X)}\sum_{j=0}^{N} c_j X^j \sqrt{f(X)} = \sum_{j=0}^{N} c_j q^{-j} x^j \sqrt{f(X)}. \tag{2.39}$$

Therefore, there exist $a_j^{(N)} \in \mathbb{R}$ $(j = 0 \cdots N)$ such that

$$L\sqrt{P(X)}\sum_{j=0}^{N} a_j^{(N)} X^j \sqrt{f(X)} = q^{-N}\sum_{j=0}^{N} a_j^{(N)} X^j \sqrt{f(X)}, \tag{2.40}$$

where

$$\varphi_N(X) := \sum_{j=0}^{N} a_j^{(N)} X^j \sqrt{f(X)}. \tag{2.41}$$

represent the eigenfunctions of $L\sqrt{P(X)}$ (not necessarily orthogonal since $L\sqrt{P(X)}$ was not required to be symmetric) and q^{-N} are the eigenvalues. However since the eigenvalues are unbounded, this implies that $L\sqrt{P(X)}$ is an unbounded operator. Let us choose its domain as the algebraic span U of the occurring eigenfunctions.

Now consider

$$L\sqrt{P(X)}\varphi_N = q^{-N}\varphi_N. \tag{2.42}$$

Define the operators T_m by

$$T_m := \sum_{j=0}^{m} \frac{\left(-L\sqrt{P(X)}\right)^j}{j!}, \qquad m \in \mathbb{N}_0. \tag{2.43}$$

For $m \in \mathbb{N}_0$, the operator T_m has the same eigenvectors as $L\sqrt{P(x)}$ and we receive

$$T_m\varphi_N = \sum_{j=0}^{m} \frac{\left(-q^{-N}\right)^j}{j!}\varphi_N. \tag{2.44}$$

Therefore, it follows with $T\varphi_N := \lim_{n\to\infty} T_m\varphi_N$:

$$T\varphi_N = e^{-q^{-N}}\varphi_N \tag{2.45}$$

T is a bounded operator on U since $e^{-q^{-N}} \to 0$ as $N \to \infty$. Let us state that

$$T : P_{\sqrt{f}} \longrightarrow P_{\sqrt{f}}, \quad T = \sum_{j=0}^{\infty} \frac{\left(-L\sqrt{P(X)}\right)^j}{j!}, \tag{2.46}$$

that is, the domain of T may be chosen as the entire span of $P_{\sqrt{f}}$. Therefore, T is a bounded operator in the space of $P_{\sqrt{f}}$.

For topological reasons, it follows that

$$\widehat{T} : P_{\sqrt{\widehat{f}}} \longrightarrow P_{\sqrt{\widehat{f}}}, \quad \widehat{T} = \sum_{j=0}^{\infty} \frac{\left(-L\sqrt{P(X)}\right)^j}{j!} \tag{2.47}$$

is also a bounded operator.

Now consider

$$e_n(q^m) := \delta_{m,n}. \tag{2.48}$$

Then

$$(Le_n)(q^m) = e_n\left(q^{m-1}\right) = \delta_{n,m-1} = \delta_{m,n+1} = e_{n+1}(q^m). \tag{2.49}$$

In particular,

$$Le_n = e_{n+1} \tag{2.50}$$

holds. Further, we have

$$L\sqrt{P(X)}e_n = \sqrt{P(q^{n-1})}e_{n+1}. \tag{2.51}$$

Defining $\alpha_n := \sqrt{P(q^{n-1})}$ for $n \in \mathbb{Z}$ to rewrite this as

$$L\sqrt{P(X)}e_n = \alpha_n e_{n+1}, \tag{2.52}$$

and defining $u_n := L\sqrt{P(X)}e_n$ for $n \in \mathbb{Z}$, we obtain

$$\begin{aligned} -L\sqrt{P(X)}L\sqrt{P(X)}e_n &= -\alpha_n u_{n+1} \\ &= -\alpha_n L\sqrt{P(X)}e_{n+1} = \alpha_{n+1}e_{n+2} \\ &= -\alpha_n \alpha_{n+1} e_{n+2}. \end{aligned} \tag{2.53}$$

Applying $\widehat{T}$ to the e_n results in

$$\widehat{T}e_n = \sum_{m=0}^{\infty}\prod_{j=0}^{m}\frac{(-1)^m \alpha_{n+j}}{m!}e_{n+m}. \tag{2.54}$$

Therefore

$$\frac{\left\|\widehat{T}e_n\right\|^2}{\|e_n\|^2} = \frac{\left(\sum_{m=0}^{\infty}\prod_{j=0}^{m}\left((-1)^m \alpha_{n+j} q^{(m+n)/2}/m!\right)\right)^2}{\left(q^{n/2}\right)^2} \longrightarrow 0 \tag{2.55}$$

as $n \to \infty$. Thus $\widehat{T}$ is defined on any e_n and generates infinitely many "rods" on the left-hand side of e_n going toward 0. Therefore, $\widehat{T}$ is well defined on any e_n and, therefore, it is well defined on any finite linear combination of the e_n.

By hypothesis, $P_{\sqrt{\widehat{f}}}$ is dense in $\mathcal{L}^2(\mathbb{R}_q)$. Then for, $n \in \mathbb{Z}$ there exists a sequence in $P_{\sqrt{\widehat{f}}}$ which approximates e_n to any degree of accuracy in the sense of the $\mathcal{L}^2(\mathbb{R}_q)$-norm.

Now the question arises: looking at all $n \in \mathbb{Z}$, is there a lower bound for $\|\widehat{T}e_n\|^2$?

The e_{n+m} are pairwise orthogonal, that is, from

$$\widehat{T}e_n = \sum_{m=0}^{\infty}\prod_{j=0}^{m}\frac{(-1)^m \alpha_{n+j}}{m!}e_{n+m}, \tag{2.56}$$

it follows that

$$\frac{\left\|\widehat{T}e_n\right\|^2}{\|e_n\|^2} \geq \frac{\alpha_n^2\|e_{n+1}\|^2}{\|e_n\|^2}\alpha_n^2 \cdot \frac{q^{n+1}}{q^n} = q \cdot \alpha_n^2, \quad \alpha_n := \sqrt{P(q^{n-1})}, \tag{2.57}$$

and therefore we have

$$\frac{\left\|\widehat{T}e_n\right\|^2}{\|e_n\|^2} \geq q \cdot P\left(q^{n-1}\right) \longrightarrow \infty \tag{2.58}$$

as $n \to -\infty$. It follows that $\widehat{T}$ is an unbounded operator, a contradiction. Therefore, $P_{\sqrt{\hat{f}}}$ is not dense in $\mathcal{L}^2(\mathbb{R}_q)$, implying that $P_{\sqrt{f}}$ is not dense in $\mathcal{L}^2(\mathbb{R})$. □

However note that the result on the lack of completeness stated in the previous theorem should not be confusing with the fact that pointwise convergence may occur as the following theorem reveals.

Theorem 2.4. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a differentiable positive even solution to*

$$\Phi(qx) = \left(1 + \alpha(1-q)\ x^2\right)\Phi(x), \quad x \in \mathbb{R}. \tag{2.59}$$

Let moreover $H_n^q : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}_0$ be the continuous solutions to the recurrence relation

$$H_{n+1}^q(x) - \alpha q^n x H_n^q(x) + \alpha\frac{q^n-1}{q-1}H_{n-1}^q(x) = 0, \quad n \in \mathbb{N}_0, \tag{2.60}$$

with initial conditions $H_0^q(x) = 1, H_1^q(x) = \alpha x$ for all $x \in \mathbb{R}$. The closure of the finite linear span of all these continuous functions $H_n^q\Phi, n \in \mathbb{N}_0$ is a Hilbert space $\mathcal{F} \subseteq \mathcal{L}^2(\mathbb{R})$. For any element v in the finite linear span of the conventional (continuous) Hermite functions, there exists a sequence $(u_{m,k})_{k,m\in\mathbb{N}_0} \subseteq \mathcal{F}$ which converges pointwise to v.

Proof. According to the assertions of the theorem, the inverse Φ^{-1} of the function $\Phi : \mathbb{R} \to \mathbb{R}$, given by

$$\forall x \in \mathbb{R} : \ \Phi^{-1}(\Phi(x)) = x \tag{2.61}$$

is differentiable and fulfills the basic difference equation

$$\Phi^{-1}\left(q^{-1}x\right) = \left(1 + \alpha(1-q)q^{-2}x^2\right)\Phi^{-1}(x), \quad x \in \mathbb{R}. \tag{2.62}$$

The function Φ^{-1}, being extended to the whole complex plane, can be interpreted as a holomorphic function due to its growth behavior, in particular, it allows a product expansion in the whole complex plane,

$$\forall z \in \mathbb{C}: \; \Phi^{-1}(z) = a\prod_{k=0}^{\infty}(z - a_k), \tag{2.63}$$

the sequence $(a_k)_{k\in\mathbb{Z}}$ of complex numbers being uniquely fixed, a denoting a multiplicative constant. Hence, the function $h : \mathbb{C} \to \mathbb{C}$ given by

$$h(z) = e^{-(1/2)z^2}\Phi^{-1}(z), \quad z \in \mathbb{C}, \tag{2.64}$$

is also holomorphic. Inserting the corresponding power series:

$$h(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{C}. \tag{2.65}$$

we end up with the statement

$$e^{-(1/2)z^2} = h(z)\Phi(z) = \left(\sum_{n=0}^{\infty} b_n z^n\right)\Phi(z), \quad z \in \mathbb{C}. \tag{2.66}$$

Any monomial $z^n, n \in \mathbb{N}_0$ may be written as

$$z^n = \left(\sum_{j=0}^{n} c_j^n H_j^{*q}(z)\right), \quad z \in \mathbb{C} \tag{2.67}$$

with a double sequence of uniquely fixed real numbers $(c_j^n)_{j,n\in\mathbb{N}_0}$. The polynomial functions $H_j^{*q}, j \in \mathbb{N}_0$ of degree j will be chosen such that

$$\int_{-\infty}^{\infty} H_i^{*q}(x)H_j^{*q}(x)\Phi(x)dx = 0 \tag{2.68}$$

for $i, j \in \mathbb{N}_0$ but $i \neq j$. Polynomials which fulfill these properties are, for instance, those fixed by (2.60), see also [1]. We may rewrite (2.66) as follows:

$$e^{-(1/2)z^2} = \left(\sum_{n=0}^{\infty}\sum_{j=0}^{n} c_j^n H_j^{*q}(z)\right)\Phi(z), \quad z \in \mathbb{C}. \tag{2.69}$$

Generalizing this result, we see that there exists a threefold sequence $(c_j^{n,m})_{j,m,n\in\mathbb{N}_0}$ of real numbers such that the classical continuous Hermite functions, given by

$$H_m(x)e^{-(1/2)x^2}, \quad x \in \mathbb{R}, m \in \mathbb{N}_0, \tag{2.70}$$

have the following representation

$$H_m(z)e^{-(1/2)z^2} = \left(\sum_{n=0}^{\infty}\sum_{j=0}^{n} c_j^{n,m} H_j^{*q}(z)\right)\Phi(z), \quad z \in \mathbb{C}. \tag{2.71}$$

We recall that the closure of the finite linear complex span of all functions $H_n^{*q}\Phi, n \in \mathbb{N}_0$ is a Hilbert space, call it $\mathcal{F}$, which is a *proper subspace* of $\mathcal{L}^2(\mathbb{R})$, hence being not dense in $\mathcal{L}^2(\mathbb{R})$—see Theorem 2.3.

For $m \in \mathbb{N}_0$, let us now consider the sequences of functions $(u_{m,k})_{k\in\mathbb{N}_0}$, given by

$$u_{m,k} : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto u_{m,k}(x) := \left(\sum_{n=0}^{k}\sum_{j=0}^{n} c_j^{n,m} H_j^{*q}(x)\right)\Phi(x), \quad k \in \mathbb{N}_0, \tag{2.72}$$

According to what we have shown it follows that each of the functions $u_{m,k} \in \mathcal{F}$ converges pointwise to the functions $h_m : \mathbb{R} \to \mathbb{R}$, given by

$$h_m : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto h_m(x) := H_m(x)e^{(-1/2)x^2}, \quad m \in \mathbb{N}_0, \tag{2.73}$$

as $k \to \infty$. □

3. Basic Difference Equations and Moment Problems

Let us make first some more general remarks on the special type of polynomials (2.60) we are considering. In literature, see, for instance, the internet reference to the Koekoek-Swarttouw online report on orthogonal polynomials http://fa.its.tudelft.nl/~koekoek/askey/ there are listed two types of deformed discrete generalizations of the classical conventional Hermite polynomials, namely, the discrete basic Hermite polynomials of type I and the discrete basic Hermite polynomials of type II. These polynomials appear in the mentioned internet report under citations 3.28 and 3.29 . Both types of polynomials, specified under the two respective citations by the symbol h_n while n is a nonnegative integer, can be succesively transformed (scaling the argument and renormalizing the coefficients) into the one and same form which is given by

$$H_{n+1}^q(x) - \alpha q^n x H_n^q(x) + \alpha\frac{q^n - 1}{q - 1} H_{n-1}^q(x) = 0, \quad n \in \mathbb{N}_0, \tag{3.1}$$

with initial conditions $H_0^q(x) = 1, H_1^q(x) = \alpha x$ for all $x \in \mathbb{R}$. Note that α is chosen as a fixed positive real number. Here, the number q may range in the set of all positive real numbers, without the number 1—the case $q = 1$ being reserved for the classical conventional Hermite polynomials. Depending on the choice of q, the two different types of discrete basic Hermite polynomials can be found. The case $0 < q < 1$ corresponds to the discrete basic Hermite polynomials of type II, the case $q > 1$ corresponds to the discrete basic Hermite polynomials of type I. Up to the late 1990 years, the perception was that both type of discrete basic Hermite polynomials have only discrete orthogonality measures. This is certainly true in the case of $q > 1$ since the existence of such an orthogonality measure was shown explicitly and since the moment problem behind the discrete basic Hermite polynomials of type I is uniquely determined.

However, it could be shown that beside the known discrete orthogonality measure, specified in the aforementioned internet report, the discrete basic Hermite polynomials of type II, hence being connected to (3.1) with $0 < q < 1$ allow also orthogonality measures with continuous support.

Let us look at this phenomenon in some more detail.

It is known as a conventional result that a symmetric orthogonality measure with discrete support for the polynomials (3.1) with $0 < q < 1$, yields moments being given by

$$\nu_{2m+2} = \frac{q^{-2m-1} - 1}{\alpha(1-q)}\nu_{2m}, \quad \nu_{2m+1} = 0, m \in \mathbb{N}_0. \tag{3.2}$$

In [1], it was shown that there exist continuous and piecewise continuous solutions to the difference equation

$$\psi(qx) = \left(1 + \alpha(1-q)x^2\right)\psi(x), \quad x \in \mathbb{R} \tag{3.3}$$

leading to the same moments (3.2). Such a behavior of the discrete basic Hermite polynomials of type II, hence being related to the scenario (3.1) with $0 < q < 1$, was quite unexpected. Vice versa, once moments ν_m with nonnegative integer m of a given weight function are given through (3.2), it can immediately be said that the weight function provides an orthogonality measure for the discrete basic Hermite polynomials of type II, related to (3.1) with $0 < q < 1$.

The question however remains whether all weight functions for the discrete basic Hermite polynomials of type II, being related to (3.1) with $0 < q < 1$ must fulfill a basic difference equation of type (3.3). We develop now an answer to this question which goes beyond the results known so far.

Let throughout the sequel $0 < q < 1$ and $\alpha > 0$. We first put forward the following definition.

Definition 3.1. By the moments of a given orthogonality measure, we understand—like in the previous sections—the numbers

$$\mu_m := \int_{-\infty}^{\infty} x^m \psi(x) dx, \quad m \in \mathbb{N}_0. \tag{3.4}$$

Let us now proceed to the main result of this section.

Theorem 3.2. *There exists a positive symmetric $C(\mathbb{R})$-solution ψ to the basic difference equation*

$$\psi\left(q^2x\right) = \left(1+\alpha(1-q)x^2\right)\left(1+\alpha(1-q)q^2x^2\right)\psi(x), \quad x \in \mathbb{R}, \tag{3.5}$$

not being a solution to

$$\psi(qx) = \left(1+\alpha(1-q)x^2\right)\psi(x), \quad x \in \mathbb{R}, \tag{3.6}$$

but generating the same moments and therefore yielding an orthogonality measure to the polynomials $(H_n^q)_{n\in\mathbb{N}_0}$ from (2.60) *in the following sense:*

$$\int_{-\infty}^{\infty} H_m^q(x)H_n^q(x)\psi(x)dx = v_n^q\delta_{mn}, \quad m,n \in \mathbb{N}_0. \tag{3.7}$$

The proof to establish will be a step beyond the already known orthogonality results for the polynomials under consideration.

Proof. Let us consider first the special basic difference equations:

$$\psi(qx) = \left(1+\alpha(1-q)x^2\right)\psi(x), \quad x \in \mathbb{R}, \tag{3.8}$$

$$\psi\left(q^2x\right) = \left(1+\alpha(1-q)\ x^2\right)\left(1+\alpha(1-q)\ q^2x^2\right)\psi(x), \quad x \in \mathbb{R}. \tag{3.9}$$

Obviously, any positive $C(\mathbb{R})$-solution ψ of (3.8) satisfies (3.9). Moreover, one can show that these $C(\mathbb{R})$-solutions of (3.8) are in $\mathcal{L}^1(\mathbb{R})$. Therefore, the set of positive symmetric solutions to (3.9) which are in $\mathcal{L}^1(\mathbb{R})\cap C(\mathbb{R})$ is nonempty. Let now $\psi \in \mathcal{L}^1(\mathbb{R})\cap C(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} x^{2m}\ \psi\left(q^2x\right)dx = \int_{-\infty}^{\infty} x^{2m}\left(1+(1-q)\alpha x^2\right)\left(1+(1-q)\alpha q^2x^2\right)\psi(x)dx, \quad m \in \mathbb{N}_0. \tag{3.10}$$

In the sequel, we are going to use the moments

$$\mu_m := \int_{-\infty}^{\infty} x^m\psi(x)dx, \quad m \in \mathbb{N}_0. \tag{3.11}$$

Remember that $\mu_{2m+1} = 0$ for $m \in \mathbb{N}_0$ as ψ was assumed to act symmetrically on the real axis. Equation (3.10) may therefore now be rewritten—in terms of the numbers μ_{2m}—as

$$q^{-2m-1}\mu_{2m} = \mu_{2m} + \alpha\left(1+q^2\right)(1-q)\mu_{2m+2} + \alpha^2q^2(1-q)^2\mu_{2m+4}, \quad m \in \mathbb{N}_0. \tag{3.12}$$

Let now $\varphi \in \mathcal{L}^1(\mathbb{R}) \cap C(\mathbb{R})$ be a positive symmetric solution to (3.8). Then, similar integration like in (3.10) shows that the corresponding moments

$$\nu_m := \int_{-\infty}^{\infty} x^m \varphi(x) dx, \quad m \in \mathbb{N}_0, \tag{3.13}$$

indeed satisfy (3.12), in particular they obey

$$\nu_{2m+2} = \frac{q^{-2m-1} - 1}{\alpha(1-q)} \nu_{2m}, \quad \nu_{2m+1} = 0, m \in \mathbb{N}_0, \tag{3.14}$$

Hence, φ provides an orthogonality measure to the discrete basic Hermite polynomials under consideration—see [1]. The main issue to address now is the following. we want to show that there are positive symmetric functions $\psi \in \mathcal{L}^1(\mathbb{R}) \cap C(\mathbb{R})$ such that ψ satisfies (3.9) but not (3.8) and such that the moments, given by (3.11) and (3.12), satisfy

$$\mu_{2m+2} = \frac{q^{-2m-1} - 1}{\alpha(1-q)} \mu_{2m}, \quad \mu_{2m+1} = 0, m \in \mathbb{N}_0. \tag{3.15}$$

In other words, we have to prove that there exist orthogonality measures to the discrete basic Hermite polynomials $(H_n^q)_{n \in \mathbb{N}_0}$ which stem from a solution to (3.9) but not from a solution to (3.8).

We proceed in a constructive way.

Let us denote first by V the $\mathbb{C}$-linear subspace of $\mathcal{L}^1(\mathbb{R}) \cap C(\mathbb{R})$ such that all $\psi \in V$ are in the kernel of the map $S : V \to C(\mathbb{R})$, given by

$$\psi \longmapsto (S\psi)(x) := \psi(qx) - \left(1 + (1-q)\ \alpha x^2\right)\psi(x), \quad x \in \mathbb{R}. \tag{3.16}$$

Let moreover W be the $\mathbb{C}$-linear subspace of $\mathcal{L}^1(\mathbb{R}) \cap C(\mathbb{R})$ containing all the functions which are in the kernel of the linear map $T : W \to C(\mathbb{R})$, given by

$$\psi \longmapsto (T\psi)(x) := \psi\left(q^2 x\right) - \left(1 + (1-q)\alpha x^2\right)\left(1 + (1-q)q^2 \alpha x^2\right)\psi(x), \quad x \in \mathbb{R}. \tag{3.17}$$

We will also make use of the $\mathbb{C}$-linear subspace $U \subseteq W$ which we choose as the maximal common domain on which the following two linear functionals are well defined:

$$\begin{aligned} s : U \longrightarrow \mathbb{C}, \quad \psi \longmapsto s(\psi) := \int_{-\infty}^{\infty} x^2 \psi(x) dx, \\ t : U \longrightarrow \mathbb{C}, \quad \psi \longmapsto t(\psi) := \int_{-\infty}^{\infty} \psi(x) dx. \end{aligned} \tag{3.18}$$

It is easy to see that s is continuous. to verify this, we consider the expression $|s(\psi)|/\|\psi\|_1$ in the sense of the $\mathcal{L}^1$-norm for any $\psi \in U$. We directly obtain

$$\frac{|s(\psi)|}{\|\psi\|_1} = \frac{\left|\int_{-\infty}^{\infty} x^2\psi(x)dx\right|}{\int_{-\infty}^{\infty}|\psi(x)|dx} \leq \frac{\int_{-\infty}^{\infty}(1+\alpha(1-q)x^2)(1+\alpha(1-q)q^2x^2)|\psi(x)|dx}{\alpha(1-q)\int_{-\infty}^{\infty}|\psi(x)|dx}. \tag{3.19}$$

But as ψ is assumed to be an element of U and hence of W, we may rewrite the expression in the denominator and give the following estimate (with a positive constant γ):

$$\frac{|s(\psi)|}{\|\psi\|_1} = \frac{\left|\int_{-\infty}^{\infty} x^2\psi(x)dx\right|}{\int_{-\infty}^{\infty}|\psi(x)|dx} \leq \gamma\frac{\int_{-\infty}^{\infty}|\psi(q^2x)|dx}{\int_{-\infty}^{\infty}|\psi(x)|dx} = q^{-2}\gamma\frac{\int_{-\infty}^{\infty}|\psi(x)|dx}{\int_{-\infty}^{\infty}|\psi(x)|dx} = q^{-2}\gamma. \tag{3.20}$$

Hence $s : U \to \mathbb{C}$ is a bounded linear map and therefore continuous. In the same way, we show that $t : U \to \mathbb{C}$ is continuous.

We now continue as follows.

Using the terminology of characteristic functions, we first look at

$$\begin{aligned} u_1 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto u_1(x) &:= \chi_{[q^2,q^{3/2}]}(x)\left(x-q^2\right)\left(q^{3/2}-x\right), \\ u_2 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto u_2(x) &:= \chi_{[q^{3/2},q]}(x)\left(x-q^{3/2}\right)(q-x). \end{aligned} \tag{3.21}$$

It is possible to choose continuous even $\mathcal{L}^1(\mathbb{R})$-functions $v_1, v_2 : \mathbb{R} \to \mathbb{R}$ being in U and fulfilling

$$\begin{gathered} Tv_1 = Tv_2 = 0, \\ v_1 \circ \chi_{[q^2,q^{3/2}]} = u_1, \qquad v_2 \circ \chi_{[q^{3/2},q]} = u_2. \end{gathered} \tag{3.22}$$

Let now $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1^2 + \lambda_2^2 > 0$. The function $v : \mathbb{R} \to \mathbb{R}$, given by

$$v := \lambda_1 v_1 + \lambda_2 v_2, \tag{3.23}$$

obeys by construction always $Tv = 0$ but never $Sv = 0$ as we have chosen λ_1, λ_2 such that $\lambda_1^2 + \lambda_2^2 > 0$ and as u_1, u_2 vanish by construction on $\mathbb{R} \setminus ([q^2, q^{3/2}] \cup [q^{3/2}, q])$.

We now choose an intermediate value argumentation. According to the continuity of s resp. t, we can choose parameters $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1^2 + \lambda_2^2 > 0$ such that v fulfills the moment property

$$s(v) = \mu_2, \qquad t(v) = \mu_0. \tag{3.24}$$

Let now $\varphi \in C(\mathbb{R})$ be a positive and even function such that $S\varphi = 0$. Note that we have in particular $\varphi \in \mathcal{L}^1(\mathbb{R})$ as well as $s(\varphi) = \mu_2$ and $t(\varphi) = \mu_0$. The function (3.23) is by construction also in $\mathcal{L}^1(\mathbb{R})$. According to the construction of the function v, we now may choose $n \in \mathbb{N}$ sufficiently large such that the positive continuous even function

$$\psi := n\varphi + v \tag{3.25}$$

finally fulfills the required properties, namely,

$$S\psi = S(n\varphi + v) = Sv \neq 0, \qquad T\psi = T(n\varphi + v) = n, T\varphi + Tv = 0 \tag{3.26}$$

as well as the moment conditions

$$s(\psi) = (n+1)\mu_2, \qquad t(\psi) = (n+1)\mu_0. \tag{3.27}$$

From (3.27) and by integrating

$$x^{2m}\psi\left(q^2x\right) = x^{2m}\left(1 + (1-q)\alpha x^2\right)\left(1 + (1-q)\alpha q^2x^2\right)\psi(x), \quad x \in \mathbb{R}, m \in \mathbb{N}_0 \tag{3.28}$$

now it follows for $m \in \mathbb{N}_0$ that the moments $\mu_m := \int_{-\infty}^{\infty} x^m\varphi(x)dx$ satisfy (3.15) and therefore also (3.12). In particular, the function ψ yields therefore an orthogonality measure to the discrete basic Hermite polynomials, given by (2.60). Note that by construction, the function ψ now fulfills all the assertions of Theorem 3.2. Hence, Theorem 3.2 holds in total. □

References

[1] K. Ey and A. Ruffing, "The moment problem of discrete q-Hermite polynomials on different time scales," *Dynamic Systems and Applications*, vol. 13, no. 3-4, pp. 409–418, 2004.

[2] L. Biedenharn, "The quantum group SU(2) and a q analog of the boson operators," *Journal of Physics A*, vol. 26, p. L873, 1993.

[3] V. V. Dodonov, ""Nonclassical" states in quantum optics: a "squeezed" review of the first 75 years," *Journal of Optics B*, vol. 4, no. 1, pp. R1–R33, 2002.

[4] X.-M. Liu and C. Quesne, "Even and odd q-deformed charge coherent states and their nonclassical properties," *Physics Letters A*, vol. 317, no. 3-4, pp. 210–222, 2003.

[5] K. A. Penson and A. I. Solomon, "New generalized coherent states," *Journal of Mathematical Physics*, vol. 40, no. 5, pp. 2354–2363, 1999.

[6] M. Arik and D. D. Coon, "Hilbert spaces of analytic functions and generalized coherent states," *Journal of Mathematical Physics*, vol. 17, no. 4, pp. 524–527, 1976.

[7] N. Garbers and A. Ruffing, "Using supermodels in quantum optics," *Advances in Difference Equations*, vol. 2006, Article ID 72668, 14 pages, 2006.

[8] C. Quesne, K. A. Penson, and V. M. Tkachuk, "Maths-type q-deformed coherent states for $q > 1$," *Physics Letters A*, vol. 313, no. 1-2, pp. 29–36, 2003.

[9] A. Ruffing, J. Lorenz, and K. Ziegler, "Difference ladder operators for a harmonic Schrödinger oscillator using unitary linear lattices," *Journal of Computational and Applied Mathematics*, vol. 153, no. 1-2, pp. 395–410, 2003.

[10] B. Simon, "The classical moment problem as a self-adjoint finite difference operator," *Advances in Mathematics*, vol. 137, no. 1, pp. 82–203, 1998.

[11] M. Simon and A. Ruffing, "Power series techniques for a special Schrödinger operator and related difference equations," *Advances in Difference Equations*, vol. 2005, no. 2, pp. 109–118, 2005.

[12] S. K. Suslov, *An Introduction to Basic Fourier Series*, vol. 9 of *Developments in Mathematics*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.