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*Research Article*

# Symmetry Properties of Higher-Order Bernoulli Polynomials

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We investigate properties of identities and some interesting identities of symmetry for the Bernoulli polynomials of higher order using the multivariate $p$-adic invariant integral on $\mathbb{Z}_p$.



## 1. Introduction

Let $p$ be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. For $x \in \mathbb{C}_p$, we use the notation $[x]_q = (1-q^x)/(1-q)$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$, and let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, the $q$-Volkenborn integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad f \in \mathrm{UD}(\mathbb{Z}_p) \tag{1.1}$$

(see [1, 2]). The ordinary $p$-adic invariant integral on $\mathbb{Z}_p$ is given by

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)\,dx \tag{1.2}$$

(see [1–15]). Let $f'(0) = (df(x)/dx)|_{x=0}$. Then we easily see that

$$I_1(f_1) = I_1(f) + f'(0), \quad \text{where } f_1(x) = f(x+1). \tag{1.3}$$

From (1.3), we can derive

$$\int_{\mathbb{Z}_p} e^{xt} dx = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{1.4}$$

(see [2, 8–10]), where $B_n$ are the $n$th Bernoulli numbers.

By (1.2) and (1.3), we easily see that

$$\begin{aligned}\frac{n\int_{\mathbb{Z}_p} e^{xt} dx}{\int_{\mathbb{Z}_p} e^{nxt} dx} &= \frac{1}{t}\left(\int_{\mathbb{Z}_p} e^{(x+n)t} dx - \int_{\mathbb{Z}_p} e^{xt} dx\right) \\ &= \sum_{i=0}^{n-1} e^{it} = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n-1} i^k\right)\frac{t^k}{k!} = \sum_{k=0}^{\infty} S_k(n-1)\frac{t^k}{k!},\end{aligned} \tag{1.5}$$

where $S_k(n) = 0^k + 1^k + \cdots + n^k$ for $k \in \mathbb{Z}_+$.

It is known that the Bernoulli polynomials are defined by

$$\int_{\mathbb{Z}_p} e^{(x+y)t} dx = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.6}$$

where $B_n(x)$ are called the $n$th Bernoulli polynomials. The Bernoulli polynomials of order $k$, denoted $B_n^k(x)$, are defined as

$$e^{xt}\left(\frac{t}{e^t - 1}\right)^k = \left(\frac{t}{e^t - 1}\right) \times \cdots \times \left(\frac{t}{e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \tag{1.7}$$

(see [3–6]). Then the values of $B_n^{(k)}(x)$ at $x = 0$ are called the Bernoulli numbers of order $k$. When $k = 1$, the polynomials or numbers are called the Bernoulli polynomials or numbers. The purpose of this paper is to investigate some interesting properties of symmetry for the multivariate $p$-adic invariant integral on $\mathbb{Z}_p$. From the properties of symmetry for the multivariate $p$-adic invariant integral on $\mathbb{Z}_p$, we derive some interesting identities of symmetry for the Bernoulli polynomials of higher order.

## 2. Symmetry Properties of Higher-Order Bernoulli Polynomials

Let $w_1, w_2 \in \mathbb{N}$. Then we define

$$D^{(m)}(w_1, w_2) = \left(\frac{w_1 t}{e^{w_1 t} - 1}\right)^m e^{w_1 w_2 t x}\left(e^{w_1 w_2 t} - 1\right)\left(\frac{w_2 t}{e^{w_2 t} - 1}\right)^m \frac{e^{w_1 w_2 y t}}{w_1 w_2 t}. \tag{2.1}$$

From (2.1), we note that

$$D^{(m)}(w_1, w_2) = \frac{\int_{\mathbb{Z}_p^m} e^{w_1(x_1+x_2+\cdots+x_m+w_2x)t} dx_1 \cdots dx_m \int_{\mathbb{Z}_p^m} e^{w_2(x_1+x_2+\cdots+x_m+w_1y)t} dx_1 \cdots dx_m}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} dx}, \tag{2.2}$$

where $\int_{\mathbb{Z}_p^m} f(x_1,\ldots,x_m) dx_1 \cdots dx_m = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} f(x_1,\ldots,x_m) dx_1 \cdots dx_m$.

In (2.1), we note that $D^{(m)}(w_1, w_2)$ is symmetric in $w_1$, $w_2$. By (2.1), we see that

$$\begin{aligned} D^{(m)}(w_1, w_2) &= \left( \int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m)t} dx_1 \cdots dx_m \right) e^{w_1w_2xt} \left( \frac{\int_{\mathbb{Z}_p} e^{w_2x_mt} dx_m}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} dx} \right) \\ &\quad \times \left( \int_{\mathbb{Z}_p^{m-1}} e^{w_2(x_1+\cdots+x_{m-1})t} dx_1 \cdots dx_{m-1} \right) e^{w_1w_2yt}. \end{aligned} \tag{2.3}$$

It is easy to see that

$$e^{w_1w_2xt} \int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m)t} dx_1 \cdots dx_m = \left( \frac{w_1t}{e^{w_1t}-1} \right)^m e^{w_1w_2xt} = \sum_{n=0}^{\infty} B_n^{(m)}(w_2x) w_1^n \frac{t^n}{n!}. \tag{2.4}$$

From (2.1), (2.3), and the above formula, we can derive

$$\begin{aligned} D^{(m)}(w_1, w_2) &= \left( \sum_{\ell=0}^{\infty} B_\ell^{(m)}(w_2x) w_1^\ell \frac{t^\ell}{\ell!} \right) \left( \sum_{k=0}^{\infty} S_k(w_1-1) \frac{w_2^k}{k!} t^k \right) \left( \sum_{i=0}^{\infty} B_i^{(m-1)}(w_1y) \frac{w_2^i}{i!} t^i \right) \frac{1}{w_1} \\ &= \left( \sum_{\ell=0}^{\infty} B_\ell^{(m)}(w_2x) w_1^{\ell-1} \frac{t^\ell}{\ell!} \right) \left( \sum_{j=0}^{\infty} \left( \sum_{k=0}^{j} S_k(w_1-1) w_2^k w_2^{j-k} \frac{B_{j-k}^{(m-1)}(w_1y)}{k!(j-k)!} j! \right) \frac{t^j}{j!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j}^{(m)}(w_2x) \sum_{k=0}^{j} S_k(w_1-1) \binom{j}{k} B_{j-k}^{(m-1)}(w_1y) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

By the symmetry of $D^{(m)}(w_1, w_2)$ in $w_1$ and $w_2$, we see that

$$D^{(m)}(w_1, w_2) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j}^{(m)}(w_1x) \sum_{k=0}^{j} \binom{j}{k} S_k(w_2-1) B_{j-k}^{(m-1)}(w_2y) \right) \frac{t^n}{n!}. \tag{2.6}$$

By comparing the coefficients on both sides of (2.5) and (2.6), we obtain the following theorem.

**Theorem 2.1.** *For $w_1, w_2 \in \mathbb{N},\ n \geq 0,\ m \geq 1$, one has*

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}w_2^{j}w_1^{n-j-1}B_{n-j}^{(m)}(w_2x)\sum_{k=0}^{j}S_k(w_1-1)\binom{j}{k}B_{j-k}^{(m-1)}(w_1y)\\&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^{j}w_2^{n-j-1}B_{n-j}^{(m)}(w_1x)\sum_{k=0}^{j}\binom{j}{k}S_k(w_2-1)B_{j-k}^{(m-1)}(w_2y).\end{aligned}\tag{2.7}$$

Let $y = 0$ and $m = 1$ in (2.7). Then we have the following corollary.

**Corollary 2.2.** *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}w_1^{n-j-1}w_2^{j}B_{n-j}(w_2x)S_j(w_1-1)\\&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^{j}w_2^{n-j-1}B_{n-j}(w_1x)S_j(w_2-1).\end{aligned}\tag{2.8}$$

If we take $w_2 = 1$ in (2.8), then we also obtain the following corollary.

**Corollary 2.3.** *For $w_1 \in \mathbb{N}$, one has*

$$B_n(w_1x)=\sum_{i=0}^{n}\binom{n}{i}w_1^{i-1}B_i(x)S_{n-i}(w_1-1).\tag{2.9}$$

By the definition of $D^{(m)}(w_1, w_2)$, we easily see that

$$\begin{aligned}D^{(m)}(w_1,w_2)&=\left(\frac{w_1t}{e^{w_1t}-1}\right)^{m}e^{xw_1w_2t}\frac{e^{w_1w_2t}-1}{e^{w_2t}-1}\left(\frac{w_2t}{e^{w_2t}-1}\right)^{m-1}e^{yw_1w_2t}\frac{1}{w_1}\\&=\frac{1}{w_1}\left(\sum_{i=0}^{w_1-1}\sum_{k=0}^{\infty}B_k^{(m)}\left(w_2x+\frac{w_2}{w_1}i\right)w_1^{k}\frac{t^k}{k!}\right)\left(\sum_{\ell=0}^{\infty}B_{\ell}^{(m-1)}(w_1y)w_2^{\ell}\frac{t^{\ell}}{\ell!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\sum_{i=0}^{w_1-1}B_k^{(m)}\left(w_2x+\frac{w_2}{w_1}i\right)\right)\frac{w_1^{k-1}}{k!}B_{n-k}^{(m-1)}(w_1y)\frac{w_2^{n-k}}{(n-k)!}n!\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}w_1^{k-1}w_2^{n-k}B_{n-k}^{(m-1)}(w_1y)\sum_{i=0}^{w_1-1}B_k^{(m)}\left(w_2x+\frac{w_2}{w_1}i\right)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.10}$$

From the symmetric property of $D^{(m)}(w_1, w_2)$ in $w_1$, $w_2$, we note that

$$D^{(m)}(w_1, w_2) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k}^{(m-1)}(w_2 y)\sum_{i=0}^{w_2-1} B_k^{(m)}\left(w_1 x + \frac{w_1}{w_2} i\right)\right)\frac{t^n}{n!}. \tag{2.11}$$

By comparing the coefficients on both sides of (2.10) and (2.11), we obtain the following theorem.

**Theorem 2.4.** *For $w_1, w_2 \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$, one has*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k}^{(m-1)}(w_1 y)\sum_{i=0}^{w_1-1} B_k^{(m)}\left(w_2 x + \frac{w_2}{w_1} i\right)\\ = \sum_{k=0}^{n}\binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k}^{(m-1)}(w_2 y)\sum_{i=0}^{w_2-1} B_k^{(m)}\left(w_1 x + \frac{w_1}{w_2} i\right).\end{aligned} \tag{2.12}$$

Let $y = 0$ and $m = 1$ in (2.12). Then we obtain the following Corollary 2.5.

**Corollary 2.5.** *For $w_1, w_2 \in \mathbb{N}$, one has*

$$w_1^{n-1}\sum_{i=0}^{w_1-1} B_n\left(w_2 x + \frac{w_2}{w_1} i\right) = w_2^{n-1}\sum_{i=0}^{w_2-1} B_n\left(w_1 x + \frac{w_1}{w_2} i\right). \tag{2.13}$$

From (2.12), we can get the well-known result due to Raabe:

$$\sum_{i=0}^{w_1-1} B_n\left(x + \frac{1}{w_1} i\right) = w_1^{1-n} B_n(w_1 x). \tag{2.14}$$

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