

*Research Article*

# **Existence of Positive Solutions for Multipoint Boundary Value Problem with $p$ -Laplacian on Time Scales**

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We consider the existence of positive solutions for a class of second-order multi-point boundary value problem with  $p$ -Laplacian on time scales. By using the well-known Krasnosel'ski's fixed-point theorem, some new existence criteria for positive solutions of the boundary value problem are presented. As an application, an example is given to illustrate the main results.

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## **1. Introduction**

The theory of time scales has become a new important mathematical branch since it was introduced by Hilger [1]. Theoretically, the time scales approach not only unifies calculus of differential and difference equations, but also solves other problems that are a mix of stop start and continuous behavior. Practically, the time scales calculus has a tremendous potential for application, for example, Thomas believes that time scales calculus is the best way to understand Thomas models populations of mosquitoes that carry West Nile virus [2]. In addition, Spedding have used this theory to model how students suffering from the eating disorder bulimia are influenced by their college friends; with the theory on time scales, they can model how the number of sufferers changes during the continuous college term as well as during long breaks [2]. By using the theory on time scales we can also study insect population, biology, heat transfer, stock market, epidemic models [2–6], and so forth. At the same time, motivated by the wide application of boundary value problems in physical and applied mathematics, boundary value problems for dynamic equations with  $p$ -Laplacian on time scales have received lots of interest [7–16].

In [7], Anderson et al. considered the following three-point boundary value problem with  $p$ -Laplacian on time scales:

$$\begin{aligned} (\varphi_p(u^\Delta(t)))^\nabla + c(t)f(u(t)) &= 0, \quad t \in (a, b), \\ u(a) - B_0(u^\Delta(v)) &= 0, \quad u^\Delta(b) = 0, \end{aligned} \quad (1.1)$$

where  $v \in (a, b)$ ,  $f \in C_{\text{ld}}([0, \infty), [0, \infty))$ ,  $c \in C_{\text{ld}}([a, b], [0, \infty))$ , and  $K_m x \leq B_0(x) \leq K_M x$  for some positive constants  $K_m, K_M$ . They established the existence results for at least one positive solution by using a fixed point theorem of cone expansion and compression of functional type.

For the same boundary value problem, He in [8] using a new fixed point theorem due to Avery and Henderson obtained the existence results for at least two positive solutions.

In [9], Sun and Li studied the following one-dimensional  $p$ -Laplacian boundary value problem on time scales:

$$\begin{aligned} (\varphi_p(u^\Delta(t)))^\Delta + h(t)f(u^\sigma(t)) &= 0, \quad t \in [a, b], \\ u(a) - B_0(u^\Delta(a)) &= 0, \quad u^\Delta(\sigma(b)) = 0, \end{aligned} \quad (1.2)$$

where  $h(t)$  is a nonnegative rd-continuous function defined in  $[a, b]$  and satisfies that there exists  $t_0 \in [a, b]$  such that  $h(t_0) > 0$ ,  $f(u)$  is a nonnegative continuous function defined on  $[0, \infty)$ ,  $B_1 x \leq B_0(x) \leq B_2 x$  for some positive constants  $B_1, B_2$ . They established the existence results for at least single, twin, or triple positive solutions of the above problem by using Krasnosel'skii's fixed point theorem, new fixed point theorem due to Avery and Henderson and Leggett-Williams fixed point theorem.

For the Sturm-Liouville-like boundary value problem, in [17] Ji and Ge investigated a class of Sturm-Liouville-like four-point boundary value problem with  $p$ -Laplacian:

$$\begin{aligned} (\varphi_p(u'(t)))' + f(t, u(t)) &= 0, \quad t \in (0, 1), \\ u(0) - \alpha u'(\xi) &= 0, \quad u(1) + \beta u'(\eta) = 0, \end{aligned} \quad (1.3)$$

where  $\xi < \eta$ ,  $f \in C([0, 1] \times [0, \infty), [0, \infty))$ . By using fixed-point theorem for operators on a cone, they obtained some existence of at least three positive solutions for the above problem. However, to the best of our knowledge, there has not any results concerning the similar problems on time scales.

Motivated by the above works, in this paper we consider the following multi-point boundary value problem on time scales:

$$\begin{aligned} (\varphi_p(u^\Delta(t)))^\Delta + h(t)f(u(t)) &= 0, \quad t \in [a, b]_{\mathbb{T}}, \\ \alpha u(a) - \beta u^\Delta(\xi) &= 0, \quad \gamma u(\sigma^2(b)) + \delta u^\Delta(\eta) = 0, \quad u^\Delta(\theta) = 0, \end{aligned} \quad (1.4)$$

where  $\mathbb{T}$  is a time scale,  $\varphi_p(u) = |u|^{p-2}u$ ,  $p > 1$ ,  $\alpha > 0$ ,  $\beta \geq 0$ ,  $\gamma > 0$ ,  $\delta \geq 0$ ,  $a < \xi < \theta < \eta < b$ , and we denote  $(\varphi_p)^{-1} = \varphi_q$  with  $1/p + 1/q = 1$ .

In the following, we denote  $[a, b] := [a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$  for convenience. And we list the following hypotheses:

(C<sub>1</sub>)  $f(u)$  is a nonnegative continuous function defined on  $[0, \infty)$ ;

(C<sub>2</sub>)  $h : [a, \sigma^2(b)] \rightarrow [0, \infty)$  is rd-continuous with  $h \cdot f \neq 0$ .

## 2. Preliminaries

In this section, we provide some background material to facilitate analysis of problem (1.4).

Let the Banach space  $E = \{u : [a, \sigma^2(b)] \rightarrow \mathbb{R} \text{ is rd-continuous}\}$  be endowed with the norm  $\|u\| = \sup_{t \in [a, \sigma^2(b)]} |u(t)|$  and choose the cone  $P \subset E$  defined by

$$P = \left\{ u \in E : u(t) \geq 0, t \in [a, \sigma^2(b)], u^{\Delta\Delta}(t) \leq 0, t \in [a, b] \right\}. \tag{2.1}$$

It is easy to see that the solution of BVP (1.4) can be expressed as

$$u(t) = \begin{cases} \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^t \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s, & a \leq t \leq \theta, \\ \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_t^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s, & \theta \leq t \leq \sigma^2(b). \end{cases} \tag{2.2}$$

If  $V_1 = V_2$ , where

$$\begin{aligned} V_1 &= \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s, \\ V_2 &= \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s, \end{aligned} \tag{2.3}$$

we define the operator  $A : P \rightarrow E$  by

$$Au(t) = \begin{cases} \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^t \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s, & a \leq t \leq \theta, \\ \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_t^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s, & \theta \leq t \leq \sigma^2(b). \end{cases} \tag{2.4}$$

It is easy to see  $\|u\| = u(\theta)$ ,  $Au(t) \geq 0$  for  $t \in [a, \sigma^2(b)]$ , and if  $Au(t) = u(t)$ , then  $u(t)$  is the positive solution of BVP (1.4).

From the definition of  $A$ , for each  $u \in P$ , we have  $Au \in P$ , and  $\|Au\| = Au(\theta)$ .

In fact,

$$(Au)^\Delta(t) = \begin{cases} \varphi_q \left( \int_t^\theta h(r) f(u(r)) \Delta r \right) \geq 0, & a \leq t \leq \theta, \\ -\varphi_q \left( \int_\theta^t h(r) f(u(r)) \Delta r \right) \leq 0, & \theta \leq t \leq \sigma^2(b) \end{cases} \quad (2.5)$$

is continuous and nonincreasing in  $[a, \sigma^2(b)]$ . Moreover,  $\varphi_q(x)$  is a monotone increasing continuously differentiable function,

$$\left( \int_t^\theta h(s) f(u(s)) \Delta s \right)^\Delta = \left( -\int_\theta^t h(s) f(u(s)) \Delta s \right)^\Delta = -h(t) f(u(t)) \leq 0, \quad (2.6)$$

then by the chain rule on time scales, we obtain

$$(Au)^{\Delta\Delta}(t) \leq 0, \quad (2.7)$$

so,  $A : P \rightarrow P$ .

For the notational convenience, we denote

$$\begin{aligned} L_1 &= \left( \frac{\beta}{\alpha} + \theta - a \right) \varphi_q \left( \int_a^\theta h(r) \Delta r \right), \\ L_2 &= \left( \frac{\delta}{\gamma} + \sigma^2(b) - \theta \right) \varphi_q \left( \int_\theta^{\sigma^2(b)} h(r) \Delta r \right), \\ M_1 &= \frac{\beta}{\alpha} \varphi_q \left( \int_\xi^\theta h(r) \Delta r \right) + \int_\xi^\theta \varphi_q \left( \int_s^\theta h(r) \Delta r \right) \Delta s, \\ M_2 &= \frac{\delta}{\gamma} \varphi_q \left( \int_\theta^\eta h(r) \Delta r \right) + \int_\theta^\eta \varphi_q \left( \int_\theta^s h(r) \Delta r \right) \Delta s, \\ M_3 &= \min \left\{ \frac{\xi - a}{\theta - a}, \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} \right\}, \\ M_4 &= \max \left\{ \frac{\theta - a}{\xi - a}, \frac{\sigma^2(b) - \theta}{\sigma^2(b) - \eta} \right\}. \end{aligned} \quad (2.8)$$

**Lemma 2.1.**  $A : P \rightarrow P$  is completely continuous.

*Proof.* First, we show that  $A$  maps bounded set into bounded set.

Assume that  $c > 0$  is a constant and  $u \in \overline{P}_c$ . Note that the continuity of  $f$  guarantees that there exists  $K > 0$  such that  $f(u) \leq \varphi_p(K)$ . So

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \int_a^{\theta} h(r) \varphi_p(K) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_a^{\theta} h(r) \varphi_p(K) \Delta r \right) \Delta s \\ &= K \left( \frac{\beta}{\alpha} + \theta - a \right) \varphi_q \left( \int_a^{\theta} h(r) \Delta r \right) \\ &= KL_1, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\delta}{\gamma} \varphi_q \left( \int_{\xi}^{\sigma^2(b)} h(r) \varphi_p(K) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \varphi_p(K) \Delta r \right) \Delta s \\ &= K \left( \frac{\delta}{\gamma} + \sigma^2(b) - \theta \right) \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \Delta r \right) \\ &= KL_2. \end{aligned}$$

That is,  $A\overline{P}_c$  is uniformly bounded. In addition, it is easy to see

$$|Au(t_1) - Au(t_2)| \leq \begin{cases} C|t_1 - t_2| \varphi_q \left( \int_a^{\theta} h(r) \Delta r \right), & t_1, t_2 \in [a, \theta], \\ C|t_1 - t_2| \varphi_q \left( \int_a^{\sigma^2(b)} h(r) \Delta r \right), & t_1 \in [a, \theta], t_2 \in [\theta, \sigma^2(b)] \\ & \text{or } t_2 \in [a, \theta], t_1 \in [\theta, \sigma^2(b)], \\ C|t_1 - t_2| \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \Delta r \right), & t_1, t_2 \in [\theta, \sigma^2(b)]. \end{cases} \tag{2.10}$$

So, by applying Arzela-Ascoli Theorem on time scales, we obtain that  $A\overline{P}_c$  is relatively compact.

Second, we will show that  $A : \overline{P}_c \rightarrow P$  is continuous. Suppose that  $\{u_n\}_{n=1}^\infty \subset \overline{P}_c$  and  $u_n(t)$  converges to  $u_0(t)$  uniformly on  $[a, \sigma^2(b)]$ . Hence,  $\{Au_n(t)\}_{n=1}^\infty$  is uniformly bounded and equicontinuous on  $[a, \sigma^2(b)]$ . The Arzela-Ascoli Theorem on time scales tells us that there exists uniformly convergent subsequence in  $\{Au_n(t)\}_{n=1}^\infty$ . Let  $\{Au_{n_l}(t)\}_{l=1}^\infty$  be a subsequence which converges to  $v(t)$  uniformly on  $[a, \sigma^2(b)]$ . In addition,

$$0 \leq Au_n(t) \leq \min\{KL_1, KL_2\}. \quad (2.11)$$

Observe that

$$Au_n(t) = \begin{cases} \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u_n(r)) \Delta r \right) + \int_a^t \varphi_q \left( \int_s^{\theta} h(r) f(u_n(r)) \Delta r \right) \Delta s, & a \leq t \leq \theta, \\ \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u_n(r)) \Delta r \right) + \int_t^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u_n(r)) \Delta r \right) \Delta s, & \theta \leq t \leq \sigma^2(b). \end{cases} \quad (2.12)$$

Inserting  $u_{n_l}$  into the above and then letting  $l \rightarrow \infty$ , we obtain

$$v(t) = \begin{cases} \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u_0(r)) \Delta r \right) + \int_a^t \varphi_q \left( \int_s^{\theta} h(r) f(u_0(r)) \Delta r \right) \Delta s, & a \leq t \leq \theta, \\ \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u_0(r)) \Delta r \right) + \int_t^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u_0(r)) \Delta r \right) \Delta s, & \theta \leq t \leq \sigma^2(b), \end{cases} \quad (2.13)$$

here we have used the Lebesgues dominated convergence theorem on time scales. From the definition of  $A$ , we know that  $v(t) = Au_0(t)$  on  $[a, \sigma^2(b)]$ . This shows that each subsequence of  $\{Au_n(t)\}_{n=1}^\infty$  uniformly converges to  $Au_0(t)$ . Therefore, the sequence  $\{Au_n(t)\}_{n=1}^\infty$  uniformly converges to  $Au_0(t)$ . This means that  $A$  is continuous at  $u_0 \in \overline{P}_c$ . So,  $A$  is continuous on  $\overline{P}_c$  since  $u_0$  is arbitrary. Thus,  $A$  is completely continuous.

The proof is complete.  $\square$

**Lemma 2.2.** Let  $u \in P$ , then  $u(t) \geq ((t - a)/(\theta - a))\|u\|$  for  $t \in [a, \theta]$ , and  $u(t) \geq ((\sigma^2(b) - t)/(\sigma^2(b) - \theta))\|u\|$  for  $t \in [\theta, \sigma^2(b)]$ .

*Proof.* Since  $u^{\Delta\Delta}(t) \leq 0$ , it follows that  $u^{\Delta}(t)$  is nonincreasing. Hence, for  $a < t < \theta$ ,

$$\begin{aligned} u(t) - u(a) &= \int_a^t u^{\Delta}(s) \Delta s \geq (t - a)u^{\Delta}(t), \\ u(\theta) - u(t) &= \int_t^{\theta} u^{\Delta}(s) \Delta s \leq (\theta - t)u^{\Delta}(t), \end{aligned} \quad (2.14)$$

from which we have

$$u(t) \geq \frac{u(a)(\theta - t) + (t - a)u(\theta)}{\theta - a} \geq \frac{t - a}{\theta - a}u(\theta) = \frac{t - a}{\theta - a}\|u\|. \tag{2.15}$$

For  $\theta \leq t \leq \sigma^2(b)$ ,

$$\begin{aligned} u(\sigma^2(b)) - u(t) &= \int_t^{\sigma^2(b)} u^\Delta(s) \Delta s \leq (\sigma^2(b) - t)u^\Delta(t), \\ u(t) - u(\theta) &= \int_\theta^t u^\Delta(s) \Delta s \geq (t - \theta)u^\Delta(t), \end{aligned} \tag{2.16}$$

we know

$$u(t) \geq \frac{(\sigma^2(b) - t)u(\theta) + (t - \theta)u(\sigma^2(b))}{\sigma^2(b) - \theta} \geq \frac{\sigma^2(b) - t}{\sigma^2(b) - \theta}u(\theta) = \frac{\sigma^2(b) - t}{\sigma^2(b) - \theta}\|u\|. \tag{2.17}$$

The proof is complete. □

**Lemma 2.3** ([18]). *Let  $P$  be a cone in a Banach space  $E$ . Assume that  $\Omega_1, \Omega_2$  are open subsets of  $E$  with  $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$ . If*

$$A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow P \tag{2.18}$$

*is a completely continuous operator such that either*

- (i)  $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$  and  $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$ , or
- (ii)  $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$  and  $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$ .

*Then  $A$  has a fixed point in  $P \cap (\overline{\Omega_2} \setminus \Omega_1)$ .*

### 3. Main Results

In this section, we present our main results with respect to BVP (1.4).

For the sake of convenience, we define  $f_0 = \lim_{u \rightarrow 0^+} (f(u)/\varphi_p(u))$ ,  $f_\infty = \lim_{u \rightarrow \infty} (f(u)/\varphi_p(u))$ ,  $i_0 =$  number of zeros in the set  $\{f_0, f_\infty\}$ , and  $i_\infty =$  number of  $\infty$  in the set  $\{f_0, f_\infty\}$ .

Clearly,  $i_0, i_\infty = 0, 1$ , or  $2$  and there are six possible cases:

- (i)  $i_0 = 0$  and  $i_\infty = 0$ ;
- (ii)  $i_0 = 0$  and  $i_\infty = 1$ ;
- (iii)  $i_0 = 0$  and  $i_\infty = 2$ ;

(iv)  $i_0 = 1$  and  $i_\infty = 0$ ;

(v)  $i_0 = 1$  and  $i_\infty = 1$ ;

(vi)  $i_0 = 2$  and  $i_\infty = 0$ .

**Theorem 3.1.** *BVP (1.4) has at least one positive solution in the case  $i_0 = 1$  and  $i_\infty = 1$ .*

*Proof.* First, we consider the case  $f_0 = 0$  and  $f_\infty = \infty$ . Since  $f_0 = 0$ , then there exists  $H_1 > 0$  such that  $f(u) \leq \varphi_p(\varepsilon)\varphi_p(u) = \varphi_p(\varepsilon u)$ , for  $0 < u \leq H_1$ , where  $\varepsilon$  satisfies

$$\max\{\varepsilon L_1, \varepsilon L_2\} \leq 1. \quad (3.1)$$

If  $u \in P$ , with  $\|u\| = H_1$ , then

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \int_a^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_a^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \int_a^{\theta} h(r) \varphi_p(\varepsilon \|u\|) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_a^{\theta} h(r) \varphi_p(\varepsilon \|u\|) \Delta r \right) \Delta s \\ &= \|u\| \varepsilon L_1 \\ &\leq \|u\|, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) f(u(r)) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \varphi_p(\varepsilon \|u\|) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \varphi_p(\varepsilon \|u\|) \Delta r \right) \Delta s \\ &= \|u\| \varepsilon L_2 \\ &\leq \|u\|. \end{aligned}$$

It follows that if  $\Omega_{H_1} = \{u \in E : \|u\| < H_1\}$ , then  $\|Au\| \leq \|u\|$  for  $u \in P \cap \partial\Omega_{H_1}$ .

Since  $f_\infty = \infty$ , then there exists  $H'_2 > 0$  such that  $f(u) \geq \varphi_p(k)\varphi_p(u) = \varphi_p(ku)$ , for  $u \geq H'_2$ , where  $k > 0$  is chosen such that

$$\min \left\{ k \frac{\xi - a}{\theta - a} M_1, k \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} M_2 \right\} \geq 1. \tag{3.3}$$

Set  $H_2 = \max\{2H_1, ((\theta - a)/(\xi - a))H'_2, ((\sigma^2(b) - \theta)/(\sigma^2(b) - \eta))H'_2\}$ , and  $\Omega_{H_2} = \{u \in E : \|u\| < H_2\}$ .

If  $u \in P$  with  $\|u\| = H_2$ , then

$$\begin{aligned} \min_{t \in [\xi, \theta]} u(t) = u(\xi) &\geq \frac{\xi - a}{\theta - a} \|u\| \geq H'_2, \\ \min_{t \in [\theta, \eta]} u(t) = u(\eta) &\geq \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} \|u\| \geq H'_2. \end{aligned} \tag{3.4}$$

So that

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \int_\xi^\theta h(r) f(u(r)) \Delta r \right) + \int_a^\theta \varphi_q \left( \int_s^\theta h(r) f(u(r)) \Delta r \right) \Delta s \\ &\geq \frac{\beta}{\alpha} \varphi_q \left( \int_\xi^\theta h(r) \varphi_p(ku) \Delta r \right) + \int_\xi^\theta \varphi_q \left( \int_s^\theta h(r) \varphi_p(ku) \Delta r \right) \Delta s \\ &\geq \frac{\beta}{\alpha} \varphi_q \left( \int_\xi^\theta h(r) \varphi_p \left( k \frac{\xi - a}{\theta - a} \|u\| \right) \Delta r \right) + \int_\xi^\theta \varphi_q \left( \int_s^\theta h(r) \varphi_p \left( k \frac{\xi - a}{\theta - a} \|u\| \right) \Delta r \right) \Delta s \\ &= \|u\| k \frac{\xi - a}{\theta - a} M_1 \\ &\geq \|u\|, \\ \|Au\| &= Au(\theta) = \frac{\delta}{\gamma} \varphi_q \left( \int_\theta^\eta h(r) f(u(r)) \Delta r \right) + \int_\theta^{\sigma^2(b)} \varphi_q \left( \int_\theta^s h(r) f(u(r)) \Delta r \right) \Delta s \\ &\geq \frac{\delta}{\gamma} \varphi_q \left( \int_\theta^\eta h(r) \varphi_p \left( k \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} \|u\| \right) \Delta r \right) + \int_\theta^\eta \varphi_q \left( \int_\theta^s h(r) \varphi_p \left( k \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} \|u\| \right) \Delta r \right) \Delta s \\ &= \|u\| k \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} M_2 \\ &\geq \|u\|. \end{aligned} \tag{3.5}$$

In other words, if  $u \in P \cap \partial\Omega_{H_2}$ , then  $\|Au\| \geq \|u\|$ . Thus by (i) of Lemma 2.3, it follows that  $A$  has a fixed point in  $P \cap (\overline{\Omega_{H_2}} \setminus \Omega_{H_1})$  with  $H_1 \leq \|u\| \leq H_2$ .

Now we consider the case  $f_0 = \infty$  and  $f_\infty = 0$ . Since  $f_0 = \infty$ , there exists  $H_3 > 0$ , such that  $f(u) \geq \varphi_p(m)\varphi_p(u) = \varphi_p(mu)$  for  $0 < u \leq H_3$ , where  $m$  is such that

$$\min \left\{ mM_1 \frac{\xi - a}{\theta - a}, mM_2 \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} \right\} \geq 1. \quad (3.6)$$

If  $u \in P$  with  $\|u\| = H_3$ , then we have

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s \\ &\geq \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) \varphi_p \left( m \frac{\xi - a}{\theta - a} \|u\| \right) \Delta r \right) + \int_{\xi}^{\theta} \varphi_q \left( \int_s^{\theta} h(r) \varphi_p \left( m \frac{\xi - a}{\theta - a} \|u\| \right) \Delta r \right) \Delta s \\ &= \|u\| m \frac{\xi - a}{\theta - a} M_1 \\ &\geq \|u\|, \\ \|Au\| &= Au(\theta) \\ &= \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s \\ &\geq \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) \varphi_p \left( m \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} \|u\| \right) \Delta r \right) + \int_{\theta}^{\eta} \varphi_q \left( \int_{\theta}^s h(r) \varphi_p \left( m \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} \|u\| \right) \Delta r \right) \Delta s \\ &= \|u\| m \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} M_2 \\ &\geq \|u\|. \end{aligned} \quad (3.7)$$

Thus, we let  $\Omega_{H_3} = \{u \in E : \|u\| < H_3\}$ , so that  $\|Au\| \geq \|u\|$  for  $u \in P \cap \partial\Omega_{H_3}$ .

Next consider  $f_\infty = 0$ . By definition, there exists  $H'_4 > 0$  such that  $f(u) \leq \varphi_p(\varepsilon)\varphi_p(u) = \varphi_p(\varepsilon u)$  for  $u \geq H'_4$ , where  $\varepsilon > 0$  satisfies

$$\max\{\varepsilon L_1, \varepsilon L_2\} \leq 1. \quad (3.8)$$

Suppose  $f$  is bounded, then  $f(u) \leq \varphi_p(K)$  for all  $u \in [0, \infty)$ , pick

$$H_4 = \max\{2H_3, KL_1, KL_2\}. \tag{3.9}$$

If  $u \in P$  with  $\|u\| = H_4$ , then

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \int_a^{\theta} h(r) \varphi_p(K) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_a^{\theta} h(r) \varphi_p(K) \Delta r \right) \Delta s \\ &= KL_1 \\ &\leq H_4 \\ &= \|u\|, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \varphi_p(K) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \varphi_p(K) \Delta r \right) \Delta s \\ &= KL_2 \\ &\leq H_4 \\ &= \|u\|. \end{aligned}$$

Now suppose  $f$  is unbounded. From condition  $(C_1)$ , it is easy to know that there exists  $H_4 \geq \max\{2H_3, H_4\}$  such that  $f(u) \leq f(H_4)$  for  $0 \leq u \leq H_4$ . If  $u \in P$  with  $\|u\| = H_4$ , then by using (3.8) we have

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \int_a^{\theta} h(r) f(H_4) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_a^{\theta} h(r) f(H_4) \Delta r \right) \Delta s \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\beta}{\alpha} \varphi_q \left( \int_a^\theta h(r) \varphi_p(\varepsilon H_4) \Delta r \right) + \int_a^\theta \varphi_q \left( \int_a^\theta h(r) \varphi_p(\varepsilon H_4) \Delta r \right) \Delta s \\
&= H_4 \varepsilon L_1 \\
&\leq H_4 \\
&= \|u\|, \\
\|Au\| &= Au(\theta) \\
&= \frac{\delta}{\gamma} \varphi_q \left( \int_\theta^\eta h(r) f(u(r)) \Delta r \right) + \int_\theta^{\sigma^2(b)} \varphi_q \left( \int_\theta^s h(r) f(u(r)) \Delta r \right) \Delta s \\
&\leq \frac{\delta}{\gamma} \varphi_q \left( \int_\theta^{\sigma^2(b)} h(r) f(H_4) \Delta r \right) + \int_\theta^{\sigma^2(b)} \varphi_q \left( \int_\theta^{\sigma^2(b)} h(r) f(H_4) \Delta r \right) \Delta s \\
&\leq \frac{\delta}{\gamma} \varphi_q \left( \int_\theta^{\sigma^2(b)} h(r) \varphi_p(\varepsilon H_4) \Delta r \right) + \int_\theta^{\sigma^2(b)} \varphi_q \left( \int_\theta^{\sigma^2(b)} h(r) \varphi_p(\varepsilon H_4) \Delta r \right) \Delta s \\
&= H_4 \varepsilon L_2 \\
&\leq H_4 \\
&= \|u\|.
\end{aligned} \tag{3.11}$$

Consequently, in either case we take

$$\Omega_{H_4} = \{u \in E : \|u\| < H_4\}, \tag{3.12}$$

so that for  $u \in P \cap \partial\Omega_{H_4}$ , we have  $\|Au\| \geq \|u\|$ . Thus by (ii) of Lemma 2.3, it follows that  $A$  has a fixed point  $u$  in  $P \cap (\overline{\Omega_{H_4}} \setminus \Omega_{H_3})$  with  $H_3 \leq \|u\| \leq H_4$ .

The proof is complete.  $\square$

**Theorem 3.2.** *Suppose  $i_0 = 0$ ,  $i_\infty = 1$ , and the following conditions hold,*

(C<sub>3</sub>): *there exists constant  $p' > 0$  such that  $f(u) \leq \varphi_p(p' A_1)$  for  $0 \leq u \leq p'$ , where*

$$A_1 = \min\{L_1^{-1}, L_2^{-1}\}, \tag{3.13}$$

(C<sub>4</sub>): there exists constant  $q' > 0$  such that  $f(u) \geq \varphi_p(q' A_2)$  for  $u \in [M_3 q', M_3]$ , where

$$A_2 = \max\{M_1^{-1}, M_2^{-1}\}, \tag{3.14}$$

furthermore,  $p' \neq q'$ . Then BVP (1.4) has at least one positive solution  $u$ , such that  $\|u\|$  lies between  $p'$  and  $q'$ .

*Proof.* Without loss of generality, we may assume that  $p' < q'$ .

Let  $\Omega_{p'} = \{u \in E : \|u\| < p'\}$ , for any  $u \in P \cap \partial\Omega_{p'}$ . In view of (C<sub>3</sub>) we have

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \int_a^{\theta} h(r) \varphi_p(p' A_1) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_a^{\theta} h(r) \varphi_p(p' A_1) \Delta r \right) \Delta s \\ &= p' A_1 L_1 \\ &\leq p', \end{aligned} \tag{3.15}$$

$$\begin{aligned} \|Au\| &= Au(\theta) \\ &= \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s \\ &\leq \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \varphi_p(p' A_1) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^{\sigma^2(b)} h(r) \varphi_p(p' A_1) \Delta r \right) \Delta s \\ &= p' A_1 L_2 \\ &\leq p', \end{aligned}$$

which yields

$$\|Au\| \leq \|u\| \quad \text{for } u \in P \cap \partial\Omega_{p'}. \tag{3.16}$$

Now set  $\Omega_{q'} = \{u \in E : \|u\| < q'\}$  for  $u \in P \cap \partial\Omega_{q'}$ , we have

$$\begin{aligned} \frac{\xi - a}{\theta - a} q' \leq u(t) \leq q' \quad \text{for } t \in [\xi, \theta], \\ \frac{\sigma^2(b) - \eta}{\sigma^2(b) - \theta} q' \leq u(t) \leq q' \quad \text{for } t \in [\theta, \eta]. \end{aligned} \tag{3.17}$$

Hence by condition (C<sub>4</sub>), we can get

$$\begin{aligned}
\|Au\| &= Au(\theta) \\
&= \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) f(u(r)) \Delta r \right) + \int_a^{\theta} \varphi_q \left( \int_s^{\theta} h(r) f(u(r)) \Delta r \right) \Delta s \\
&\geq \frac{\beta}{\alpha} \varphi_q \left( \int_{\xi}^{\theta} h(r) \varphi_p(q' A_2) \Delta r \right) + \int_{\xi}^{\theta} \varphi_q \left( \int_s^{\theta} h(r) \varphi_p(q' A_2) \Delta r \right) \Delta s \\
&= q' A_2 M_1 \\
&\geq q', \\
\|Au\| &= Au(\theta) \\
&= \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) f(u(r)) \Delta r \right) + \int_{\theta}^{\sigma^2(b)} \varphi_q \left( \int_{\theta}^s h(r) f(u(r)) \Delta r \right) \Delta s \\
&\geq \frac{\delta}{\gamma} \varphi_q \left( \int_{\theta}^{\eta} h(r) \varphi_p(q' A_2) \Delta r \right) + \int_{\theta}^{\eta} \varphi_q \left( \int_{\theta}^s h(r) \varphi_p(q' A_2) \right) \Delta s \\
&= q' A_2 M_2 \\
&\geq q'.
\end{aligned} \tag{3.18}$$

So if we take  $\Omega_{q'} = \{u \in E : \|u\| < q'\}$ , then

$$\|Au\| \geq \|u\|, \quad u \in P \cap \partial\Omega_{q'}. \tag{3.19}$$

Consequently, in view of  $p' < q'$ , (3.16), and (3.19), it follows from Lemma 2.3 that  $A$  has a fixed point  $u$  in  $P \cap (\overline{\Omega_{q'}} \setminus \Omega_{p'})$ . Moreover, it is a positive solution of (1.4) and  $p' < u < q'$ .

The proof is complete.  $\square$

For the case  $i_0 = 1, i_{\infty} = 0$  or  $i_0 = 0, i_{\infty} = 1$  we have the following results.

**Theorem 3.3.** *Suppose that  $f_0 \in [0, \varphi_p(A_1))$  and  $f_{\infty} \in (\varphi_p(M_4 A_2), \infty)$  hold. Then BVP (1.4) has at least one positive solution.*

*Proof.* It is easy to see that under the assumptions, the conditions (C<sub>3</sub>) and (C<sub>4</sub>) in Theorem 3.2 are satisfied. So the proof is easy and we omit it here.  $\square$

**Theorem 3.4.** *Suppose that  $f_0 \in (\varphi_p(M_4 A_2), \infty)$  and  $f_{\infty} \in [0, \varphi_p(A_1))$  hold. Then BVP (1.4) has at least one positive solution.*

*Proof.* Since  $f_0 \in (\varphi_p(M_4 A_2), \infty)$ , for  $\varepsilon = f_0 - \varphi_p((\theta - a) / (\xi - a) A_2)$ , there exists a sufficiently small  $q'_1$  such that

$$\frac{f(u)}{\varphi_p(u)} \geq f_0 - \varepsilon = \varphi_p\left(\frac{\theta - a}{\xi - a} A_2\right), \quad u \in (0, q'_1]. \tag{3.20}$$

Thus, if  $u \in (((\xi - a)/(\theta - a))q'_1, q'_1]$ , then we have

$$f(u) \geq \varphi_p(u)\varphi_p\left(\frac{\theta - a}{\xi - a}A_2\right) \geq \varphi_p(q'_1A_2); \tag{3.21}$$

by the similar method, one can get if  $u \in [((\sigma^2(b) - \eta)/(\sigma^2(b) - \theta))q'_2, q'_2]$ , then

$$f(u) \geq \varphi_p(u)\varphi_p\left(\frac{\sigma^2(b) - \theta}{\sigma^2(b) - \eta}A_2\right) \geq \varphi_p(q'_2A_2). \tag{3.22}$$

So, if we choose  $q' = \min\{q'_1, q'_2\}$ , then for  $u \in [M_3q', q']$ , we have  $f(u) \geq \varphi_p(q'A_2)$ , which yields condition  $(C_4)$  in Theorem 3.2.

Next, by  $f_\infty \in [0, \varphi_p(A_1))$ , for  $\varepsilon = \varphi_p(A_1) - f_\infty$ , there exists a sufficiently large  $p'' (> q')$  such that

$$\frac{f(u)}{\varphi_p(u)} \leq f_\infty + \varepsilon = \varphi_p(A_1), \quad u \in [p'', \infty), \tag{3.23}$$

where we consider two cases.

*Case 1.* Suppose that  $f$  is bounded, say

$$f(u) \leq \varphi_p(K), \quad u \in [0, \infty). \tag{3.24}$$

In this case, take sufficiently large  $p'$  such that  $p' \geq \max\{K/A_1, p''\}$ , then from (3.24), we know  $f(u) \leq \varphi_p(K) \leq \varphi_p(A_1p')$  for  $u \in (0, p']$ , which yields condition  $(C_3)$  in Theorem 3.2.

*Case 2.* Suppose that  $f$  is unbounded. it is easy to know that there is  $p' > p''$  such that

$$f(u) \leq f(p'), \quad u \in [0, p']. \tag{3.25}$$

Since  $p' > p''$  then from (3.23) and (3.25), we get

$$f(u) \leq f(p') \leq \varphi_p(p'A_1), \quad u \in [0, p']. \tag{3.26}$$

Thus, the condition  $(C_3)$  of Theorem 3.2 is satisfied.

Hence, from Theorem 3.2, BVP (1.4) has at least one positive solution.

The proof is complete. □

From Theorems 3.3 and 3.4, we have the following two results.

**Corollary 3.5.** *Suppose that  $f_0 = 0$  and the condition  $(C_4)$  in Theorem 3.2 hold. Then BVP (1.4) has at least one positive solution.*

**Corollary 3.6.** *Suppose that  $f_\infty = 0$  and the condition  $(C_4)$  in Theorem 3.2 hold. Then BVP (1.4) has at least one positive solution.*

**Theorem 3.7.** *Suppose that  $f_0 \in (0, \varphi_p(A_1))$  and  $f_\infty = \infty$  hold. Then BVP (1.4) has at least one positive solution.*

*Proof.* In view of  $f_\infty = \infty$ , similar to the first part of Theorem 3.1, we have

$$\|Au\| \geq \|u\|, \quad u \in P \cap \partial\Omega_{H_2}. \quad (3.27)$$

Since  $f_0 \in (0, \varphi_p(A_1))$ , for  $\varepsilon = \varphi_p(A_1) - f_0 > 0$ , there exists a sufficiently small  $p' \in (0, H_2)$  such that

$$f(u) \leq (f_0 + \varepsilon)\varphi_p(u) = \varphi_p(A_1u) \leq \varphi_p(A_1p'), \quad u \in [0, p']. \quad (3.28)$$

Similar to the proof of Theorem 3.2, we obtain

$$\|Au\| \leq \|u\|, \quad u \in P \cap \partial\Omega_{p'}. \quad (3.29)$$

The result is obtained, and the proof is complete.  $\square$

**Theorem 3.8.** *Suppose that  $f_\infty \in (0, \varphi_p(A_1))$  and  $f_0 = \infty$  hold. Then BVP (1.4) has at least one positive solution.*

*Proof.* Since  $f_0 = \infty$ , similar to the second part of Theorem 3.1, we have  $\|Au\| \geq \|u\|$  for  $u \in P \cap \partial\Omega_{H_3}$ .

By  $f_\infty \in (0, \varphi_p(A_1))$ , similar to the second part of proof of Theorem 3.4, we have  $\|Au\| \leq \|u\|$  for  $u \in P \cap \partial\Omega_{p'}$ , where  $p' > H_3$ . Thus BVP (1.4) has at least one positive solution.

The proof is complete.  $\square$

From Theorems 3.7 and 3.8, we can get the following corollaries.

**Corollary 3.9.** *Suppose that  $f_\infty = \infty$  and the condition  $(C_3)$  in Theorem 3.2 hold. Then BVP (1.4) has at least one positive solution.*

**Corollary 3.10.** *Suppose that  $f_0 = \infty$  and the condition  $(C_3)$  in Theorem 3.2 hold. Then BVP (1.4) has at least one positive solution.*

**Theorem 3.11.** *Suppose that  $i_0 = 0$ ,  $i_\infty = 2$ , and the condition  $(C_3)$  of Theorem 3.2 hold. Then BVP (1.4) has at least two positive solutions  $u_1, u_2 \in P$  such that  $0 < \|u_1\| < p' < \|u_2\|$ .*

*Proof.* By using the method of proving Theorems 3.1 and 3.2, we can deduce the conclusion easily, so we omit it here.  $\square$

**Theorem 3.12.** *Suppose that  $i_0 = 2$ ,  $i_\infty = 0$ , and the condition  $(C_4)$  of Theorem 3.2 hold. Then BVP (1.4) has at least two positive solutions  $u_1, u_2 \in P$  such that  $0 < \|u_1\| < q' < \|u_2\|$ .*

*Proof.* Combining the proofs of Theorems 3.1 and 3.2, the conclusion is easy to see, and we omit it here.  $\square$

## 4. Applications and Examples

In this section, we present a simple example to explain our result. When  $\mathbb{T} = \mathbb{R}$ ,

$$\begin{aligned} u'|u'| &= (1-t)(4 - \arctan u), \quad 0 < t < 1, \\ u(0) &= u'\left(\frac{1}{4}\right), \quad u(1) = -u'\left(\frac{1}{2}\right), \end{aligned} \quad (4.1)$$

where,  $p = 3$ ,  $\alpha = \beta = \gamma = \delta = 1$ ,  $h(t) = 1 - t$ ,  $f(u) = 4 - \arctan u$ .

It is easy to see that the condition  $(C_1)$  and  $(C_2)$  are satisfied and

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{\varphi_p(u)} = \infty, \quad f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{\varphi_p(u)} = 0. \quad (4.2)$$

So, by Theorem 3.1, the BVP (4.1) has at least one positive solution.

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