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Research Article

Asymptotic Behavior of Equilibrium Point for a Class of Nonlinear Difference Equation

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We study the asymptotic behavior of the solutions for the following nonlinear difference equation $x_{n+1} = \sum_{i=1}^{s} A_{k_i} x_{n-k_i} / (B_0 + \sum_{j=1}^{t} B_{l_j} x_{n-l_j}), n = 0, 1, \ldots,$ where the initial conditions $x_{-r}, x_{-r+1}, \ldots, x_1, x_0$ are arbitrary nonnegative real numbers, $k_1, \ldots, k_s, l_1, \ldots, l_t$ are nonnegative integers, $r = \max\{k_1, \ldots, k_s, l_1, \ldots, l_t\}$, and $A_{k_1}, \ldots, A_{k_s}, B_0, B_{l_1}, \ldots, B_{l_t}$ are positive constants. Moreover, some numerical simulations to the equation are given to illustrate our results.

1. Introduction

Difference equations appear naturally as discrete analogues and in the numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, and so forth [1]. The study of nonlinear difference equations is of paramount importance not only in their own field but in understanding the behavior of their differential counterparts. There has been a lot of work concerning the globally asymptotic behavior of solutions of rational difference equations [2–6]. In particular, Elabbasy et al. [7] investigated the global stability and periodicity of the solution for the following recursive sequence:

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0, 1, \ldots. \tag{1.1}$$

In [8] Elabbasy et al. investigated the global stability, boundedness, and the periodicity of solutions of the difference equation:

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{A x_n + B x_{n-1} + C x_{n-2}}, \quad n = 0, 1, \dots. \tag{1.2}$$

Yang et al. [9] investigated the global attractivity of equilibrium points and the asymptotic behavior of the solutions of the recursive sequence:

$$x_{n+1} = \frac{a x_{n-1} + b x_{n-2}}{c + d x_{n-1} x_{n-2}}, \quad n = 0, 1, \dots. \tag{1.3}$$

The purpose of this paper is to investigate the global attractivity of the equilibrium point, and the asymptotic behavior of the solutions of the following difference equation

$$x_{n+1} = \frac{\sum_{i=1}^{s} A_{k_i} x_{n-k_i}}{B_0 + \sum_{j=1}^{t} B_{l_j} x_{n-l_j}}, \quad n = 0, 1, \dots, \tag{1.4}$$

where the initial conditions $x_{-r}, x_{-r+1}, \dots, x_1, x_0$ are arbitrary nonnegative real numbers, $k_1, \dots, k_s, l_1, \dots, l_t$ are nonnegative integers, $r = \max\{k_1, \dots, k_s, l_1, \dots, l_t\}$,and $A_{k_1}, \dots, A_{k_s}, B_0$, $B_{l_1}, \dots, B_{l_t}$ are positive constants. Moreover, some numerical simulations to the equation are given to illustrate our results.

This paper is arranged as follows. In Section 2, we give some definitions and preliminary results. The main results and their proofs are given in Section 3. Finally, some numerical simulations are given to illustrate our theoretical analysis.

2. Some Preliminary Results

To prove the main results in this paper we first give some definitions and preliminary results [10, 11] which are basically used throughout this paper.

Lemma 2.1. *Let I be some interval of real numbers and let*

$$f : I^{k+1} \longrightarrow I \tag{2.1}$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots \tag{2.2}$$

has a unique solution $\{x_n\}_{n=-k}^{+\infty}$.

Definition 2.2. A point $\overline{x} \in I$ is called an equilibrium point of (2.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \dots, \overline{x}). \tag{2.3}$$

That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of (2.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2.3. Let p, q be two nonnegative integers such that $p + q = n$. Splitting $x = (x_1, x_2, \ldots, x_n)$ into $x = ([x]_p, [x]_q)$, where $[x]_\sigma$ denotes a vector with σ-components of x, we say that the function $f(x_1, x_2, \ldots, x_n)$ possesses a mixed monotone property in subsets I^n of R^n if $f([x]_p, [x]_q)$ is monotone nondecreasing in each component of $[x]_p$ and is monotone nonincreasing in each component of $[x]_q$ for $x \in I^n$. In particular, if $q = 0$, then it is said to be monotone nondecreasing in I^n.

Definition 2.4. Let $\overline{x}$ be an equilibrium point of (2.1).

(i) $\overline{x}$ is stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial conditions $(x_{-k}, x_{-k+1}, \ldots, x_0) \in I^{k+1}$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta$, $|x_n - \overline{x}| < \varepsilon$ hold for $n = 1, 2, \ldots$.

(ii) $\overline{x}$ is a local attractor if there exists $\gamma > 0$ such that $x_n \to \overline{x}$ holds for any initial conditions $(x_{-k}, x_{-k+1}, \ldots, x_0) \in I^{k+1}$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma$.

(iii) $\overline{x}$ is locally asymptotically stable if it is stable and is a local attractor.

(iv) $\overline{x}$ is a global attractor if $x_n \to \overline{x}$ holds for any initial conditions $(x_{-k}, x_{-k+1}, \ldots, x_0) \in I^{k+1}$.

(v) $\overline{x}$ is globally asymptotically stable if it is stable and is a global attractor.

(vi) $\overline{x}$ is unstable if it is not locally stable.

Lemma 2.5. *Assume that* $s_1, s_2, \ldots, s_k \in R$ *and* $k \in \{0, 1, 2, \ldots\}$. *Then*

$$|s_1| + |s_2| + \cdots + |s_k| < 1 \tag{2.4}$$

is a sufficient condition for the local stability of the difference equation:

$$x_{n+k} + s_1 x_{n+k-1} + \cdots + s_k x_n = 0, \quad n = 0, 1, \ldots. \tag{2.5}$$

3. The Main Results and Their Proofs

In this section we investigate the globally asymptotic stability of the equilibrium point of (1.4).

Let $f : (0, \infty)^r \to (0, \infty)$ be a function defined by

$$f(x_{n-k_1}, \ldots, x_{n-k_s}, x_{n-l_1}, \ldots, x_{n-l_t}) = \frac{\sum_{i=1}^{s} A_{k_i} x_{n-k_i}}{B_0 + \sum_{j=1}^{t} B_{l_j} x_{n-l_j}}, \quad n = 0, 1, \ldots. \tag{3.1}$$

If $x_{n-k_1} \neq x_{n-k_2} \neq \cdots \neq x_{n-k_s} \neq x_{n-l_1} \neq x_{n-l_2} \cdots \neq x_{n-l_t}$, then it follows that

$$\begin{aligned} f_{x_{n-k_i}}(x_{n-k_1}, \ldots, x_{n-k_s}, x_{n-l_1}, \ldots, x_{n-l_t}) &= \frac{A_{k_i}}{B_0 + \sum_{j=1}^{t} B_{l_j} x_{n-l_j}}, \quad i = 1, 2, \ldots, s, \\ f_{x_{n-l_j}}(x_{n-k_1}, \ldots, x_{n-k_s}, x_{n-l_1}, \ldots, x_{n-l_t}) &= -\frac{B_{l_j} \sum_{i=1}^{t} A_{k_i} x_{n-k_i}}{\left(B_0 + \sum_{j=1}^{t} B_{l_j} x_{n-l_j}\right)^2}, \quad j = 1, 2, \ldots, t. \end{aligned} \tag{3.2}$$

Let $\overline{x}, \overline{\overline{x}}$ be the equilibrium points of (1.4), then we have

$$\overline{x} = 0, \qquad \overline{\overline{x}} = \frac{\sum_{i=1}^{t} A_{k_i} - B_0}{\sum_{j=1}^{t} B_{l_j}}. \tag{3.3}$$

Moreover, we have that

$$\begin{aligned} f_{x_{n-k_i}}(\overline{x}, \dots, \overline{x}, \overline{x}, \dots, \overline{x}) &= \frac{A_{k_i}}{B_0}, \quad i = 1, 2, \dots, s, \\ f_{x_{n-l_j}}(\overline{x}, \dots, \overline{x}, \overline{x}, \dots, \overline{x}) &= 0, \quad j = 1, 2, \dots, t. \end{aligned} \tag{3.4}$$

Thus, the linearized equation of (1.4) about $\overline{x}$ is

$$z_{n+1} + \frac{A_{k_1}}{B_0} z_{n-k_1} + \cdots + \frac{A_{k_s}}{B_0} z_{n-k_s} = 0. \tag{3.5}$$

Theorem 3.1. *If* $\sum_{i=1}^{s} A_{k_i} < B_0$ *and* $x_{n-k_1} \neq x_{n-k_2} \neq \cdots \neq x_{n-k_s} \neq x_{n-l_1} \neq x_{n-l_2} \cdots \neq x_{n-l_t}$, *then the equilibrium point* $\overline{x} = 0$ *of* (1.4) *is locally stable.*

Proof. It is obvious by Lemma 2.5 that (3.5) is locally stable if

$$\left| \frac{A_{k_1}}{B_0} \right| + \cdots + \left| \frac{A_{k_s}}{B_0} \right| < 1, \tag{3.6}$$

that is

$$\sum_{i=1}^{s} A_{k_i} < B_0, \tag{3.7}$$

from which the result follows. □

Theorem 3.2. *Let* $[a, b]$ *be an interval of real numbers and assume that* $f : [a, b]^{k+1} \to R$ *is a continuous function satisfying the mixed monotone property. If there exists*

$$m_0 \leq \min\{x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0\} \leq \max\{x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0\} \leq M_0 \tag{3.8}$$

such that

$$m_0 \leq f\Big([m_0]_p, [M_0]_q\Big) \leq f\Big([M_0]_p, [m_0]_q\Big) \leq M_0, \tag{3.9}$$

then there exist $(m, M) \in [m_0, M_0]^2$ *satisfying*

$$M = f\Big([M]_p, [m]_q\Big), \qquad m = f\Big([m]_p, [M]_q\Big). \tag{3.10}$$

Moreover, if $m = M$, *then* (2.2) *has a unique equilibrium point* $\overline{x} \in [m_0, M_0]$ *and every solution of* (2.2) *converges to* $\overline{x}$.

Proof. Using m_0 and M_0 as a couple of initial iteration conditions we construct two sequences $\{m_i\}$ and $\{M_i\}$ $(i = 1, 2, \ldots)$ from the equation

$$m_i = f\left([m_{i-1}]_p, [M_{i-1}]_q\right), \qquad M_i = f\left([M_{i-1}]_p, [m_{i-1}]_q\right). \tag{3.11}$$

It is obvious from the mixed monotone property of f that the sequences $\{m_i\}$ and $\{M_i\}$ possess the following monotone property:

$$m_0 \le m_1 \le \cdots \le m_i \le \cdots \le M_i \le \cdots \le M_1 \le M_0, \tag{3.12}$$

where $i = 0, 1, 2, \ldots$, and

$$m_i \le x_l \le M_i \quad \text{for } l \ge (k+1)i + 1. \tag{3.13}$$

Set

$$m = \lim_{i \to \infty} m_i, \qquad M = \lim_{i \to \infty} M_i, \tag{3.14}$$

then

$$m \le \lim_{i \to \infty} \inf x_i \le \lim_{i \to \infty} \sup x_i \le M. \tag{3.15}$$

By the continuity of f we have

$$M = f\left([M]_p, [m]_q\right), \qquad m = f\left([m]_p, [M]_q\right). \tag{3.16}$$

Moreover, if $m = M$, then $m = M = \lim_{i \to \infty} x_i = \overline{x}$, and then the proof is complete. □

Theorem 3.3. *If there exists*

$$\overline{\overline{x}} \le m_0 \le \min\{x_{-r}, x_{-r+1}, \ldots, x_{-1}, x_0\} \le \max\{x_{-r}, x_{-r+1}, \ldots, x_{-1}, x_0\} \le M_0 \tag{3.17}$$

such that

$$m_0 \le \frac{\sum_{i=1}^{s} A_{k_i} m_0}{B_0 + \sum_{j=1}^{t} B_{l_j} M_0} \le \frac{\sum_{i=1}^{s} A_{k_i} M_0}{B_0 + \sum_{j=1}^{t} B_{l_j} m_0} \le M_0, \tag{3.18}$$

then the equilibrium point $\overline{x} = 0$ *of* (1.4) *is global attractor when* $\sum_{i=1}^{s} A_k - B_0 < 0$.

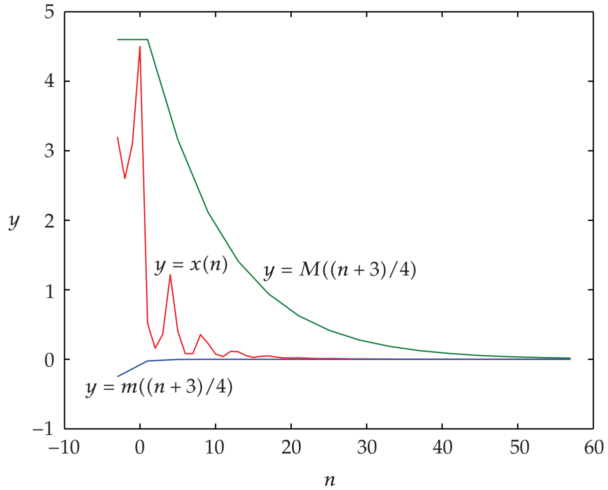

Figure 1: Chart of (4.1) with $x(-3) = 3.2, x(-2) = 2.6, x(-1) = 3.1, x(0) = 4.5$.

Proof. We can easily see that the function $f(x_{n-k_1}, \ldots, x_{n-k_s}, x_{n-l_1}, \ldots, x_{n-l_t})$ defined by (3.1) is nondecreasing in $x_{n-k_1}, \ldots, x_{n-k_s}$ and nonincreasing in $x_{n-l_1}, \ldots, x_{n-l_t}$. Then from (1.4) and Theorem 3.2, there exist $(m, M) \in [m_0, M_0]^2$ satisfying

$$m = \frac{\sum_{i=1}^{s} A_{k_i} m}{B_0 + \sum_{j=1}^{t} B_{l_j} M}, \qquad M = \frac{\sum_{i=1}^{s} A_{k_i} M}{B_0 + \sum_{j=1}^{t} B_{l_j} m}, \tag{3.19}$$

thus

$$\left(\sum_{i=1}^{s} A_k - B_0\right)(M - m) = 0. \tag{3.20}$$

In view of $\sum_{i=1}^{s} A_k - B_0 < 0$, we have

$$M = m. \tag{3.21}$$

It follows by Theorem 3.2 that the equilibrium point $\overline{x} = 0$ of (1.4) is global attractor. The proof is therefore complete. □

4. Numerical Simulations

In this section, we give numerical simulations supporting our theoretical analysis. As examples, we consider the following difference equations:

$$x_{n+1} = \frac{x_n + x_{n-3}}{3 + x_{n-2} + 3x_{n-1}}, \quad n = 0, 1, \ldots, \tag{4.1}$$

$$x_{n+1} = \frac{x_n + 2x_{n-1} + x_{n-2} + x_{n-3}}{6 + x_n + x_{n-1} + x_{n-2} + x_{n-3}}, \quad n = 0, 1, \ldots. \tag{4.2}$$

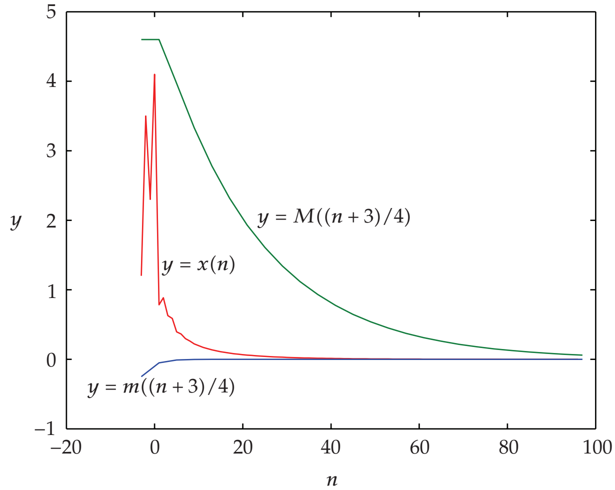

Figure 2: Chart of (4.2) with $x(-3) = 1.2, x(-2) = 3.5, x(-1) = 2.3, x(0) = 3.1$.

Let $m_0 = -1/4, M_0 = 4.6$. It is obvious that (4.1) and (4.2) satisfy the conditions of Theorem 3.3 when the initial conditions are $(x_{-3}, x_{-2}, x_{-1}, x_0) \in [0, 4.6]^4$.

Figure 1 shows the numerical solution of (4.1) with $x_{-3} = 3.2, x_{-2} = 2.6, x_{-1} = 3.1, x_0 = 4.5$ and the relations that $m_i \le x_l \le M_i$ when $l \ge (k+1)i + 1, i = 0, 1, 2, \ldots$.

Figure 2 shows the numerical solution of (4.2) with $x_{-3} = 1.2, x_{-2} = 3.5, x_{-1} = 2.3, x_0 = 3.1$ and the relations that $m_i \le x_l \le M_i$ when $l \ge (k+1)i + 1, i = 0, 1, 2, \ldots$.

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