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Research Article

Existence Results for Higher-Order Boundary Value Problems on Time Scales

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By using the fixed-point index theorem, we consider the existence of positive solutions for the following nonlinear higher-order four-point singular boundary value problem on time scales $u^{\Delta^n}(t)+g(t)f(u(t),u^{\Delta}(t),\dots,u^{\Delta^{n-2}}(t))=0, 0<t<T; u^{\Delta^i}(0)=0, 0\le i\le n-3; \alpha u^{\Delta^{n-2}}(0)-\beta u^{\Delta^{n-1}}(\xi)=0$, $n\ge 3$; $\gamma u^{\Delta^{n-2}}(T)+\delta u^{\Delta^{n-1}}(\eta)=0$, $n\ge 3$, where $\alpha>0$, $\beta\ge 0$, $\gamma>0$, $\delta\ge 0$, ξ, $\eta\in(0,T)$, $\xi<\eta$, and $g:(0,T)\to[0,+\infty)$ is rd-continuous.

1. Introduction

Time scales and time-scale notation are introduced well in the fundamental texts by Bohner and Peterson [1, 2], respectively, as important corollaries. In, the recent years, many authors have paid much attention to the study of boundary value problems on time scales (see, e.g., [3–17]). In particular, we would like to mention some results of Anderson et al. [3, 5, 6, 14, 16], DaCunha et al. [4], and Agarwal and O'Regan [7], which motivate us to consider our problem.

In [3], Anderson and Karaca discussed the dynamic equation on time scales

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t)=f\left(t,y^{\sigma}(t)\right)=0,\quad t\in(a,b),\\ \alpha_{i+1}y^{\Delta^{2i}}(\eta)+\beta_{i+1}y^{\Delta^{2i+1}}(a)=y^{\Delta^{2i}}(a),\gamma_{i+1}y^{\Delta^{2i}}(\eta)=y^{\Delta^{2i}}(\sigma(b)),\end{gathered}\tag{1.1}$$

and the eigenvalue problem

$$(-1)^n y^{\Delta^{2n}}(t)=\lambda f\left(t,y^{\sigma}(t)\right)=0,\quad t\in(a,b),\tag{1.2}$$

with the same boundary conditions where λ is a positive parameter. They obtained some results for the existence of positive solutions by using the Krasnoselskii, the Schauder, and the Avery-Henderson fixed-point theorem.

In [4], by using the Gatica-Oliker-Waltman fixed-point theorem, DaCunha, Davis, and Singh proved the existence of a positive solution for the three-point boundary value problem on a time scale $\mathbb{T}$ given by

$$\begin{aligned} y^{\Delta\Delta}(t) + f(x,y) = 0, \quad x \in [0,1]_{\mathbb{T}}, \\ y(0) = 0, \qquad y(p) = y\left(\sigma^2(1)\right), \end{aligned} \tag{1.3}$$

where $p \in (0,1) \cap T$ is fixed, and $f(x,y)$ is singular at $y = 0$ and possibly at $x = 0, y = \infty$.

Anderson et al. [5] gave a detailed presentation for the following higher-order self-adjoint boundary value problem on time scales:

$$\begin{aligned} Ly(t) =& \sum_{i=0}^{n}(-1)^{n-i}\left(p_i y^{\Delta^{n-i-1}\nabla}\right)^{\nabla^{n-i-1}\Delta}(t) = (-1)^n\left(p_0 y^{\Delta^{n-1}\nabla}\right)^{\nabla^{n-1}\Delta}(t) + \cdots \\ &- \left(p_{n-3}y^{\Delta^2\nabla}\right)^{\nabla^2\Delta}(t) + \left(p_{n-2}y^{\Delta\nabla}\right)^{\nabla\Delta}(t) - \left(p_{n-1}y^{\Delta}\right)^{\nabla}(t) + p_n(t)y(t), \end{aligned} \tag{1.4}$$

and got many excellent results.

In related papers, Sun [11] considered the following third-order two-point boundary value problem on time scales:

$$\begin{aligned} u^{\Delta\Delta\Delta}(t) + f\left(t, u(t), u^{\Delta\Delta}(t)\right) = 0, \quad t \in [a, \sigma(b)], \\ u(a) = A, \quad u\left(\sigma^b\right) = B, \quad u^{\Delta\Delta}(a) = C, \end{aligned} \tag{1.5}$$

where a, $b \in T$ and $a < b$. Some existence criteria of solution and positive solution are established by using the Leray-Schauder fixed point theorem.

In this paper, we consider the existence of positive solutions for the following higher-order four-point singular boundary value problem (BVP) on time scales

$$u^{\Delta^n}(t) + g(t)f\left(u(t), u^{\Delta}(t), \ldots, u^{\Delta^{n-2}}(t)\right) = 0, \quad 0 < t < T, \tag{1.6}$$

$$\begin{aligned} u^{\Delta^i}(0) = 0, \quad 0 \le i \le n-3, \\ \alpha u^{\Delta^{n-2}}(0) - \beta u^{\Delta^{n-1}}(\xi) = 0, \quad n \ge 3, \\ \gamma u^{\Delta^{n-2}}(T) + \delta u^{\Delta^{n-1}}(\eta) = 0, \quad n \ge 3, \end{aligned} \tag{1.7}$$

where $\alpha > 0, \beta \ge 0, \gamma > 0, \delta \ge 0, \xi, \eta \in (0,T), \xi < \eta$, and $g : (0,T) \to [0,+\infty)$ is rd-continuous. In the rest of the paper, we make the following assumptions:

(H_1) $f \in C([0,+\infty)^{n-1}, [0,+\infty))$

(H_2) $0 < \int_0^T g(t)\Delta t < +\infty$.

In this paper, by constructing one integral equation which is equivalent to the BVP (1.6) and (1.7), we study the existence of positive solutions. Our main tool of this paper is the following fixed-point index theorem.

Theorem 1.1 ([18]). *Suppose E is a real Banach space, $K \subset E$ is a cone, let $\Omega_r = \{u \in K : \|u\| \le r\}$. Let operator $T : \Omega_r \to K$ be completely continuous and satisfy $Tx \ne x, \forall x \in \partial\Omega_r$. Then*

(i) *if $\|Tx\| \le \|x\|$, $\forall x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 1$*

(ii) *if $\|Tx\| \ge \|x\|$, $\forall x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 0$.*

The outline of the paper is as follows. In Section 2, for the convenience of the reader we give some definitions and theorems which can be found in the references, and we present some lemmas in order to prove our main results. Section 3 is developed in order to present and prove our main results. In Section 4 we present some examples to illustrate our results.

2. Preliminaries and Lemmas

For convenience, we list the following definitions which can be found in [1, 2, 9, 14, 17]. A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\begin{aligned} \sigma(t) &= \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \\ \rho(r) &= \sup\{\tau \in \mathbb{T} : \tau < r\} \in \mathbb{T}, \end{aligned} \tag{2.1}$$

for all $t, r \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^\kappa = \mathbb{T}$. In this general time-scale setting, Δ represents the delta (or Hilger) derivative [13, Definition 1.10],

$$z^\Delta(t) := \lim_{s \to t} \frac{z(\sigma(t)) - z(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{z^\sigma(t) - z(s)}{\sigma(t) - s}, \tag{2.2}$$

where $\sigma(t)$ is the forward jump operator, $\mu(t) := \sigma(t) - t$ is the forward graininess function, and $z \circ \sigma$ is abbreviated as z^σ. In particular, if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $x^\Delta = x'$, while if $\mathbb{T} = h\mathbb{Z}$ for any $h > 0$, then $\sigma(t) = t + h$ and

$$x^\Delta(t) = \frac{x(t + h) - x(t)}{h}. \tag{2.3}$$

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided that it is continuous at each right-dense point $t \in \mathbb{T}$ (a point where $\sigma(t) = t$) and has a left-sided limit at each left-dense point $t \in \mathbb{T}$. The set of right-dense continuous functions on $\mathbb{T}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. It can be shown that any right-dense continuous function f has an antiderivative (a function $\Phi : \mathbb{T} \to \mathbb{R}$ with the property $\Phi^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$). Then the Cauchy delta integral of f is defined by

$$\int_{t_0}^{t_1} f(t)\Delta t = \Phi(t_1) - \Phi(t_0), \tag{2.4}$$

where Φ is an antiderivative of f on $\mathbb{T}$. For example, if $\mathbb{T} = \mathbb{Z}$, then

$$\int_{t_0}^{t_1} f(t)\Delta t = \sum_{t=t_0}^{t_1-1} f(t), \tag{2.5}$$

and if $\mathbb{T} = \mathbb{R}$, then

$$\int_{t_0}^{t_1} f(t)\Delta t = \int_{t_0}^{t_1} f(t)dt. \tag{2.6}$$

Throughout we assume that $t_0 < t_1$ are points in $\mathbb{T}$, and define the time-scale interval $[t_0, t_1]_{\mathbb{T}} = \{t \in \mathbb{T} : t_0 \le t \le t_1\}$. In this paper, we also need the the following theorem which can be found in [1].

Theorem 2.1. *If $f \in C_{\mathrm{rd}}$ and $t \in \mathbb{T}^k$, then*

$$\int_{t}^{\sigma(t)} f(\tau)\Delta\tau = (\sigma(t) - t)f(t). \tag{2.7}$$

In this paper, let

$$E = \left\{ u \in C_{\mathrm{rd}}^{\Delta^{n-2}}[0,T] \ : u^{\Delta^i}(0) = 0,\ 0 \le i \le n-3 \right\}. \tag{2.8}$$

Then E is a Banach space with the norm $\|u\| = \max_{t\in[0,T]}|u^{\Delta^{n-2}}(t)|$. Define a cone K by

$$K = \left\{ u \in E \ : u^{\Delta^{n-2}}(t) \ge 0,\ u^{\Delta^{n}}(t) \le 0, t \in [0,T] \right\}. \tag{2.9}$$

Obviously, K is a cone in E. Set $K_r = \{u \in K : \|u\| \le r\}$. If $u^{\Delta\Delta} \le 0$ on $[0,T]$, then we say u is concave on $[0,T]$. We can get the following.

Lemma 2.2. *Suppose condition (H_2) holds. Then there exists a constant $\theta \in (0, T/2)$ satisfies*

$$0 < \int_{\theta}^{T-\theta} g(t)\Delta t < +\infty. \tag{2.10}$$

Furthermore, the function

$$A(t)=\int_{\theta}^{t}\left(\int_{s}^{t}g(s_1)\Delta s_1\right)\Delta s+\int_{t}^{T-\theta}\left(\int_{t}^{s}g(s_1)\Delta s_1\right)\Delta s,\quad t\in[\theta,T-\theta] \tag{2.11}$$

is a positive continuous function on $[\theta,T-\theta]$*, therefore* $A(t)$ *has minimum on* $[\theta,T-\theta]$*. Then there exists* $L>0$ *such that* $A(t)\geq L$*,* $t\in[\theta,T-\theta]$*.*

Lemma 2.3. *Let* $u\in K$ *and* $\theta\in(0,T/2)$ *in Lemma 2.2. Then*

$$u^{\Delta^{n-2}}(t)\geq\theta\|u\|,\quad t\in[\theta,T-\theta]. \tag{2.12}$$

Proof. Suppose $\tau=\inf\{\xi\in[0,T]:\sup_{t\in[0,T]}u^{\Delta^{n-2}}(t)=u^{\Delta^{n-2}}(\xi)\}$.

We will discuss it from three perspectives.

(i) $\tau\in[0,\theta]$. It follows from the concavity of $u^{\Delta^{n-2}}(t)$ that

$$u^{\Delta^{n-2}}(t)\geq u^{\Delta^{n-2}}(\tau)+\frac{u^{\Delta^{n-2}}(T)-u^{\Delta^{n-2}}(\tau)}{T-\tau}(t-\tau),\quad t\in[\theta,T-\theta], \tag{2.13}$$

then

$$\begin{aligned}u^{\Delta^{n-2}}(t)&\geq\min_{t\in[\theta,T-\theta]}\left[u^{\Delta^{n-2}}(\tau)+\frac{u^{\Delta^{n-2}}(T)-u^{\Delta^{n-2}}(\tau)}{T-\tau}(t-\tau)\right]\\&=u^{\Delta^{n-2}}(\tau)+\frac{u^{\Delta^{n-2}}(T)-u^{\Delta^{n-2}}(\tau)}{T-\tau}(T-\theta-\tau)\\&=\frac{T-\theta-\tau}{T-\tau}u^{\Delta^{n-2}}(T)+\frac{\theta}{T-\tau}u^{\Delta^{n-2}}(\tau)\geq\theta u(\tau),\end{aligned} \tag{2.14}$$

which means $u^{\Delta^{n-2}}(t)\geq\theta\|u\|$, $t\in[\theta,T-\theta]$.

(ii) $\tau\in[\theta,T-\theta]$. If $t\in[\theta,\tau]$, we have

$$u^{\Delta^{n-2}}(t)\geq u^{\Delta^{n-2}}(\tau)+\frac{u^{\Delta^{n-2}}(\tau)-u^{\Delta^{n-2}}(0)}{\tau}(t-\tau),\quad t\in[\theta,\tau], \tag{2.15}$$

then

$$\begin{aligned}u^{\Delta^{n-2}}(t)&\geq\min_{t\in[\theta,T-\theta]}\left[u^{\Delta^{n-2}}(\tau)+\frac{u^{\Delta^{n-2}}(\tau)-u^{\Delta^{n-2}}(0)}{\tau}(t-\tau)\right]\\&=\frac{\theta}{\tau}u^{\Delta^{n-2}}(\tau)+\frac{\tau-\theta}{\tau}u^{\Delta^{n-2}}(0)\geq\theta u^{\Delta^{n-2}}(\tau),\end{aligned} \tag{2.16}$$

If $t \in [\tau, T-\theta]$, we have

$$u^{\Delta^{n-2}}(t) \geq u^{\Delta^{n-2}}(\tau) + \frac{u^{\Delta^{n-2}}(T) - u^{\Delta^{n-2}}(\tau)}{T-\tau}(t-\tau), \quad t \in [\tau, T-\theta], \tag{2.17}$$

then

$$\begin{aligned} u^{\Delta^{n-2}}(t) &\geq \min_{t\in[\theta,T-\theta]}\left[u^{\Delta^{n-2}}(\tau) + \frac{u^{\Delta^{n-2}}(T) - u^{\Delta^{n-2}}(\tau)}{T-\tau}(t-\tau)\right] \\ &= \frac{\theta}{T-\tau}u^{\Delta^{n-2}}(\tau) + \frac{T-\theta-\tau}{T-\tau}u^{\Delta^{n-2}}(T) \\ &\geq \theta u^{\Delta^{n-2}}(\tau), \end{aligned} \tag{2.18}$$

and this means $u^{\Delta^{n-2}}(t) \geq \theta\|u\|$, $t \in [\theta, T-\theta]$.

(iii) $\tau \in [T-\theta, T]$. Similarly, we have

$$u^{\Delta^{n-2}}(t) \geq u^{\Delta^{n-2}}(\tau) + \frac{u^{\Delta^{n-2}}(\tau) - u^{\Delta^{n-2}}(0)}{\tau}(t-\tau), \quad t \in [\theta, T-\theta], \tag{2.19}$$

then

$$\begin{aligned} u^{\Delta^{n-2}}(t) &\geq \min_{t\in[\theta,T-\theta]}\left[u(\tau) + \frac{u^{\Delta^{n-2}}(\tau) - u^{\Delta^{n-2}}(0)}{\tau}(t-\tau)\right] \\ &= \frac{\theta}{\tau}u^{\Delta^{n-2}}(\tau) + \frac{\tau-\theta}{\tau}u^{\Delta^{n-2}}(0) \\ &\geq \theta u^{\Delta^{n-2}}(\tau), \end{aligned} \tag{2.20}$$

which means $u^{\Delta^{n-2}}(t) \geq \theta\|u\|$, $t \in [\theta, T-\theta]$.

From the above, we know $u^{\Delta^{n-2}}(t) \geq \theta\|u\|$, $t \in [\theta, T-\theta]$. The proof is complete. □

Lemma 2.4. *Suppose that conditions* (H_1), (H_2) *hold, then* $u(t)$ *is a solution of boundary value problem* (1.6), (1.7) *if and only if* $u(t) \in E$ *is a solution of the following integral equation:*

$$u(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2})\Delta s_{n-2}\Delta s_{n-3}\cdots\Delta s_1, \tag{2.21}$$

where

$$w(t)=\begin{cases}\dfrac{\beta}{\alpha}\displaystyle\int_{\xi}^{\delta}g(s)f\left(u(s),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\right)\Delta s\\ \quad+\displaystyle\int_{0}^{t}\int_{s}^{\delta}g(r)f\left(u(r),u^{\Delta}(r),\ldots,u^{\Delta^{n-2}}(r)\right)\Delta r\Delta s, \quad 0\le t\le\delta,\\ \dfrac{\delta}{\gamma}\displaystyle\int_{\delta}^{\eta}g(s)f\left(u(s),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\right)\Delta s\\ \quad+\displaystyle\int_{t}^{1}\int_{\delta}^{s}g(r)f\left(u(r),u^{\Delta}(r),\ldots,u^{\Delta^{n-2}}(r)\right)\Delta r\Delta s, \quad \delta\le t\le T.\end{cases} \tag{2.22}$$

Proof. Necessity. By the equation of the boundary condition, we see that $u^{\Delta^{n-1}}(\xi)\ge 0, u^{\Delta^{n-1}}(\eta)\le 0$, then there exists a constant $\delta\in[\xi,\eta]\subset(0,T)$ such that $u^{\Delta^{n-1}}(\delta)=0$. Firstly, by delta integrating the equation of the problems (1.6) on (δ,t), we have

$$u^{\Delta^{n-1}}(t)=u^{\Delta^{n-1}}(\delta)-\int_{\delta}^{t}g(s)f\left(u(s),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\right)\Delta s, \tag{2.23}$$

thus

$$u^{\Delta^{n-2}}(t)=u^{\Delta^{n-2}}(\delta)-\int_{\delta}^{t}\left(\int_{\delta}^{s}g(r)f\left(u(r),u^{\Delta}(r),\ldots,u^{\Delta^{n-2}}(r)\right)\Delta r\right)\Delta s. \tag{2.24}$$

By $u^{\Delta^{n-1}}(\delta)=0$ and the boundary condition (1.7), let $t=\eta$ on (2.23), we have

$$u^{\Delta^{n-1}}(\eta)=-\int_{\delta}^{\eta}g(s)f\left(u(s),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\right)\Delta s. \tag{2.25}$$

By the equation of the boundary condition (1.7), we get

$$u^{\Delta^{n-2}}(T)=-\frac{\delta}{\gamma}\left(u^{\Delta^{n-1}}(\eta)\right), \tag{2.26}$$

then

$$u^{\Delta^{n-2}}(T)=\frac{\delta}{\gamma}\int_{\delta}^{\eta}g(s)f\left(u(s),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\Delta s\right). \tag{2.27}$$

Secondly, by (2.24) and let $t=T$ on (2.24), we have

$$\begin{aligned}u^{\Delta^{n-2}}(\delta)=&\frac{\delta}{\gamma}\int_{\delta}^{\eta}g(s)f\left(u(s),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\right)\Delta s\\&+\int_{\delta}^{T}\left(\int_{\delta}^{s}g(r)f\left(u(r),u^{\Delta}(r),\ldots,u^{\Delta^{n-2}}(r)\right)\Delta r\right)\Delta s.\end{aligned} \tag{2.28}$$

Then

$$u^{\Delta^{n-2}}(t) = \frac{\delta}{\gamma}\int_{\delta}^{\eta} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\Delta s + \int_{t}^{T}\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\Delta r\right)\Delta s. \tag{2.29}$$

Then by delta integrating (2.29) for $n-2$ times on $(0,T)$, we have

$$\begin{aligned} u(t) = {} & \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\left(\frac{\delta}{\gamma}\int_{\delta}^{\eta} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\Delta s\right)\Delta s_{n-2}\cdots\Delta s_2\Delta s_1 \\ & + \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\left(\int_{s_{n-2}}^{T}\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\Delta r\right)\Delta s\right)\Delta s_{n-2}\cdots\Delta s_2\Delta s_1. \end{aligned} \tag{2.30}$$

Similarly, for $t \in (0,\delta)$, by delta integrating the equation of problems (1.6) on $(0,\delta)$, we have

$$\begin{aligned} u(t) = {} & \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\left(\frac{\delta}{\gamma}\int_{\xi}^{\delta} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\Delta s\right)\Delta s_{n-2}\cdots\Delta s_2\Delta s_1 \\ & + \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\left(\int_{0}^{s_{n-2}}\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\Delta r\right)\Delta s\right)\Delta s_{n-2}\cdots\Delta s_2\Delta s_1. \end{aligned} \tag{2.31}$$

Therefore, for any $t \in [0,T]$, $u(t)$ can be expressed as the equation

$$u(t) = \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}} w(s_{n-2})\Delta s_{n-2}\Delta s_{n-3}\cdots\Delta s_1, \tag{2.32}$$

where $w(t)$ is expressed as (2.22).

Sufficiency. Suppose that

$$u(t) = \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}} w(s_{n-2})\Delta s_{n-2}\Delta s_{n-3}\cdots\Delta s_1, \tag{2.33}$$

then by (2.22), we have

$$u^{\Delta^{n-1}}(t) = \begin{cases} \displaystyle\int_t^{\delta} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\Delta s \ge 0, & 0 \le t \le \delta, \\ \displaystyle -\int_{\delta}^{t} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\Delta s \le 0, & \delta \le t \le T, \end{cases} \tag{2.34}$$

So,

$$u^{\Delta^n}(t) + g(t)f\left(u(t), u^{\Delta}(t), \ldots, u^{\Delta^{n-2}}(t)\right) = 0, \quad 0 < t < T, \tag{2.35}$$

which imply that (1.6) holds. Furthermore, by letting $t = 0$ and $t = T$ on (2.22) and (2.34), we can obtain the boundary value equations of (1.7). The proof is complete. □

Now, we define a mapping $T : K \rightarrow C_{\mathrm{rd}}^{\Delta^{n-1}}[0,T]$ given by

$$(Tu)(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2})\Delta s_{n-2}\Delta s_{n-3} \cdots \Delta s_1, \tag{2.36}$$

where $w(t)$ is given by (2.22).

Lemma 2.5. *Suppose that conditions* (H_1), (H_2) *hold, the solution* $u(t)$ *of problem* (1.6), (1.7) *satisfies*

$$u(t) \leq Tu^{\Delta}(t) \leq \cdots \leq T^{n-3}u^{\Delta^{n-3}}(t), \quad t \in [0,T], \tag{2.37}$$

and for $\theta \in (0, T/2)$ *in Lemma 2.2, one has*

$$u^{\Delta^{n-3}}(t) \leq \frac{T}{\theta}u^{\Delta^{n-2}}(t), \quad t \in [\theta, T-\theta]. \tag{2.38}$$

Proof. If $u(t)$ is the solution of (1.6), (1.7), then $u^{\Delta^{n-1}}(t)$ is a concave function, and $u^i(t) \geq 0$, $i = 0, 1, \ldots, n-2$, $t \in [0,T]$, thus we have

$$u^{\Delta^i}(t) = \int_0^t u^{\Delta^{i+1}}(s)\Delta s \leq tu^{\Delta^{i+1}}(t) \leq Tu^{\Delta^{i+1}}(t), \quad i = 0, 1, \ldots, n-4, \tag{2.39}$$

that is,

$$u(t) \leq Tu^{\Delta}(t) \leq \cdots \leq T^{n-3}u^{\Delta^{n-3}}(t), \quad t \in [0,T]. \tag{2.40}$$

By Lemma 2.3, for $t \in [\theta, T-\theta]$, we have

$$u^{\Delta^{n-2}}(t) \geq \theta\|u\|, \tag{2.41}$$

then $u^{\Delta^{n-3}}(t) = \int_0^t u^{\Delta^{n-2}}(s)\Delta s \leq tu^{\Delta^{n-2}}(t) \leq T\|u\| \leq (T/\theta)u^{\Delta^{n-2}}(t)$.The proof is complete. □

Lemma 2.6. *$T : K \to K$ is completely continuous.*

Proof. Because

$$(Tu)^{\Delta^{n-1}}(t) = w^{\Delta}(t) = \begin{cases} \int_t^{\delta} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\Delta s \geq 0, & 0 \leq t \leq \delta, \\ -\int_{\delta}^{t} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\Delta s \leq 0, & \delta \leq t \leq T \end{cases} \tag{2.42}$$

is continuous, decreasing on $[0, T]$, and satisfies $(Tu)^{\Delta^{n-1}}(\delta) = 0$. Then, $Tu \in K$ for each $u \in K$ and $(Tu)^{\Delta^{n-2}}(\delta) = \max_{t\in[0,T]}(Tu)^{\Delta^{n-2}}(t)$. This shows that $TK \subset K$. Furthermore, it is easy to check that $T : K \to K$ is completely continuous by Arzela-ascoli Theorem.

For convenience, we set

$$\theta^* = \frac{2}{L}, \qquad \theta_* = \frac{1}{(1 + (\beta/\alpha))\left(\int_0^1 g(r)\Delta r\right)}, \tag{2.43}$$

where L is the constant from Lemma 2.2. By Lemma 2.5, we can also set

$$f_0 = \lim_{u_{n-1}\to 0}\ \max_{0\leq u_1\leq Tu_2\leq\cdots\leq T^{n-2}u_{n-2}\leq(T/\theta)u_{n-1}} \frac{f(u_1, u_2, \ldots, u_{n-1})}{u_{n-1}},$$
$$f_\infty = \lim_{u_{n-1}\to \infty}\ \min_{0\leq u_1\leq Tu_2\leq\cdots\leq T^{n-2}u_{n-2}\leq(T/\theta)u_{n-1}} \frac{f(u_1, u_2, \ldots, u_{n-1})}{u_{n-1}}. \tag{2.44}$$

□

3. The Existence of Positive Solution

Theorem 3.1. *Suppose that conditions (H_1), (H_2) hold. Assume that f also satisfies*

- (A_1) $f(u_1, u_2, \ldots, u_{n-1}) \geq mr$, *for* $\theta r \leq u_{n-1} \leq r, 0 \leq u_1 \leq Tu_2 \leq \cdots \leq T^{n-2}u_{n-2} \leq (T/\theta)u_{n-1}$,
- (A_2) $f(u_1, u_2, \ldots, u_{n-1}) \leq MR$, *for* $0 \leq u_{n-1} \leq R$, $0 \leq u_1 \leq Tu_2 \leq \cdots \leq T^{n-2}u_{n-2} \leq (T/\theta)u_{n-1}$,

where $m \in (\theta^*, +\infty)$, $M \in (0, \theta_*)$.

Then, the boundary value problem (1.6), (1.7) *has a solution* u *such that* $\|u\|$ *lies between* r *and* R.

Theorem 3.2. *Suppose that conditions (H_1), (H_2) hold. Assume that f also satisfies*

- (A_3) $f_0 = \varphi \in [0, \theta_*/4)$
- (A_4) $f_\infty = \lambda \in (2\theta^*/\theta, +\infty)$.

Then, the boundary value problem (1.6), (1.7) *has a solution* u *such that* $\|u\|$ *lies between* r *and* R.

Theorem 3.3. *Suppose that conditions (H_1), (H_2) hold. Assume that f also satisfies*

(A_5) $f_\infty = \lambda \in [0, \theta_*/4)$

(A_6) $f_0 = \varphi \in (2\theta^*/\theta, +\infty)$.

Then, the boundary value problem (1.6), (1.7) *has a solution u such that $\|u\|$ lies between r and R.*

Proof of Theorem 3.1. Without loss of generality, we suppose that $r < R$. For any $u \in K$, by Lemma 2.3, we have

$$u^{\Delta^{n-2}}(t) \geq \theta\|u\|, \quad t \in [\theta, T-\theta]. \tag{3.1}$$

We define two open subsets Ω_1 and Ω_2 of E:

$$\Omega_1 = \{u \in K : \|u\| < r\}, \qquad \Omega_2 = \{u \in K : \|u\| < R\}. \tag{3.2}$$

For any $u \in \partial\Omega_1$, by (3.1) we have

$$r = \|u\| \geq u^{\Delta^{n-2}}(t) \geq \theta\|u\| = \theta r, \quad t \in [\theta, T-\theta]. \tag{3.3}$$

For $t \in [\theta, T-\theta]$ and $u \in \partial\Omega_1$, we will discuss it from three perspectives.

(i) If $\delta \in [\theta, T-\theta]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$\begin{aligned}
2\|Tu\| &= 2(Tu)^{\Delta^{n-2}}(\delta) \\
&\geq \int_0^\delta \left(\int_s^\delta g(r) f\left(u(r), u^\Delta(r), \ldots, u^{\Delta^{n-1}}(r)\right) \Delta r \right) \Delta s \\
&\quad + \int_\delta^T \left(\int_\delta^s g(r) f\left(u(r), u^\Delta(r), \ldots, u^{\Delta^{n-1}}(r)\right) \Delta r \right) \Delta s \\
&\geq \int_\theta^\delta \left(\int_s^\delta g(r) f\left(u(r), u^\Delta(r), \ldots, u^{\Delta^{n-1}}(r)\right) \Delta r \right) \Delta s \\
&\quad + \int_\delta^{T-\theta} \left(\int_\delta^s g(r) f\left(u(r), u^\Delta(r), \ldots, u^{\Delta^{n-1}}(r)\right) \Delta r \right) \Delta s \\
&\geq mrA(\delta) \geq mrL > 2r = 2\|u\|.
\end{aligned} \tag{3.4}$$

(ii) If $\delta \in (T-\theta, T]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{\Delta^{n-2}}(\delta) \\
&\geq \frac{\beta}{\alpha}\int_{\xi}^{\delta} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-1}}(s)\right)\Delta s \\
&\quad + \int_{0}^{\delta}\int_{s}^{\delta} g(r) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-1}}(s)\right)\Delta r \Delta s \\
&\geq \int_{\theta}^{T-\theta}\left(\int_{s}^{T-\theta} g(r) f\left(u(r), u^{\Delta} r, \ldots, u^{\Delta^{n-1}}(r)\right)\Delta r\right)\Delta s \\
&\geq mrA(T-\theta) \geq mrL > 2r > r = \|u\|.
\end{aligned} \tag{3.5}$$

(iii) If $\delta \in (0, \theta)$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{\Delta^{n-2}}(\delta) \\
&\geq \frac{\delta}{\gamma}\int_{\delta}^{\eta} g(s) f\left(u(s), u^{\Delta} s, \ldots, u^{\Delta^{n-1}}(s)\right)\Delta s \\
&\quad + \int_{\delta}^{1}\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta} r, \ldots, u^{\Delta^{n-1}}(r)\Delta r\right)\Delta s \\
&\geq \int_{\theta}^{T-\theta}\left(\int_{\theta}^{s} g(r) f\left(u(r), u^{\Delta} r, \ldots, u^{\Delta^{n-1}}(r)\right)\Delta r\right)\Delta s \\
&\geq mrA(\theta) \geq mrL > 2r > r = \|u\|.
\end{aligned} \tag{3.6}$$

Therefore, no matter under which condition, we all have

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_1. \tag{3.7}$$

Then by Theorem 2.1, we have

$$i(T, \Omega_1, K) = 0. \tag{3.8}$$

On the other hand, for $u \in \partial\Omega_2$, we have $u(t) \leq \|u\| = R$; by (A_2) we know

$$\begin{aligned}\|Tu\| &= (Tu)^{\Delta^{n-1}}(\delta)\\ &\leq \frac{\beta}{\alpha}\int_{\xi}^{\delta} g(s) f\left(u(s), u^{\Delta}(s), \ldots, u^{\Delta^{n-1}}(s)\right)\Delta s\\ &\quad + \int_0^1\int_s^{\delta} g(r) f\left(u(r), u^{\Delta}(r), \ldots, u^{\Delta^{n-1}}(r)\Delta r\right)\Delta s\\ &\leq \left(1+\frac{\beta}{\alpha}\right) MR\left(\int_0^1 g(r)\Delta r\right) \leq R = \|u\|.\end{aligned} \tag{3.9}$$

thus

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_2. \tag{3.10}$$

Then, by Theorem 2.1, we get

$$i(T, \Omega_2, K) = 1. \tag{3.11}$$

Therefore, by (3.8), (3.11), $r < R$, we have

$$i\left(T, \Omega_2 \setminus \overline{\Omega}_1, K\right) = 1. \tag{3.12}$$

Then operator T has a fixed point $u \in (\Omega_1 \setminus \overline{\Omega}_2)$, and $r \leq \|u\| \leq R$. Then the proof of Theorem 3.1 is complete . □

Proof of Theorem 3.2. First, by $f_0 = \varphi \in [0, \theta_*/4)$, for $\epsilon = (\theta_*/4) - \varphi$, there exists an adequately small positive number ρ, as $0 \leq u_{n-1} \leq \rho$, $u_{n-1} \neq 0$, we have

$$f(u_1, u_2, \ldots, u_{n-1}) \leq (\varphi + \epsilon)(u_{n-1}) \leq \left(\frac{\theta_*}{4}\right)\rho = \frac{\theta_*}{4}\rho. \tag{3.13}$$

Then let $R = \rho$, $M = \theta_*/4 \in (0, \theta_*)$, thus by (3.13)

$$f(u_1, u_2, \ldots, u_{n-1}) \leq MR, \quad 0 \leq u_{n-1} \leq R. \tag{3.14}$$

So condition (A_2) holds. Next, by condition (A_4), $f_\infty = \lambda \in ((2\theta^*/\theta), +\infty)$, then for $\epsilon = \lambda - (2\theta^*/\theta)$, there exists an appropriately big positive number $r \neq R$, as $u_{n-1} \geq \theta r$, we have

$$f(u_1, u_2, \ldots, u_{n-1}) \geq (\lambda - \epsilon)(u_{n-1}) \geq \left(\frac{2\theta^*}{\theta}\right)(\theta r) = (2\theta^* r). \tag{3.15}$$

Let $m = 2\theta^* > \theta^*$, thus by (3.15), condition (A_1) holds. Therefore by Theorem 3.1 we know that the results of Theorem 3.2 hold. The proof of Theorem 3.2 is complete. □

Proof of Theorem 3.3. Firstly, by condition (A_6), $f_0 = \varphi \in ((2\theta^*/\theta), +\infty)$, then for $\epsilon = \varphi - (2\theta^*/\theta)$, there exists an adequately small positive number r, as $0 \le u_{n-1} \le r, u_{n-1} \neq 0$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \ge (\varphi - \epsilon) u_{n-1} = \frac{2\theta^*}{\theta} u_{n-1}, \tag{3.16}$$

thus when $\theta r \le u_{n-1} \le r$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \ge \frac{2\theta^*}{\theta} \theta r = 2\theta^* r. \tag{3.17}$$

Let $m = 2\theta^* > \theta^*$, so by (3.17), condition (A_1) holds.

Secondly, by condition (A_5), $f_\infty = \lambda \in [0, \theta_*/4)$, then for $\epsilon = (\theta_*/4) - \lambda$, there exists a suitably big positive number $\rho \neq r$, as $u_{n-1} \ge \rho$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \le (\lambda + \epsilon)(u_{n-1}) \le \frac{\theta_*}{4} u_{n-1}. \tag{3.18}$$

If f is unbounded, by the continuity of f on $[0, T] \times [0, +\infty)^{n-1}$, then there exist a constant $R\ \ (\neq r) \ge \rho$, and a point $(\widehat{u}_1, \widehat{u}_2, \dots, \widehat{u}_{n-1}) \in [0, T] \times [0, +\infty)^{n-1}$ such that

$$\begin{gathered} \rho \le \widehat{u}_{n-1} \le R, \\ f(u_1, u_2, \dots, u_{n-1}) \le f(\widehat{u}_1, \widehat{u}_2, \dots, \widehat{u}_{n-1}), \quad 0 \le u_{n-1} \le R. \end{gathered} \tag{3.19}$$

Thus, by $\rho \le u_{0n-1} \le R$, we know

$$f(u_1, u_2, \dots, u_{n-1}) \le f(\widehat{u}_1, \widehat{u}_2, \dots, \widehat{u}_{n-1}) \le \frac{\theta_*}{4} \widehat{u}_{n-1} \le \frac{\theta_*}{4} R. \tag{3.20}$$

Choose $M = \theta_*/4 \in (0, \theta_*)$. Then, we have

$$f(u_1, u_2, \dots, u_{n-1}) \le MR, \quad 0 \le u_{n-1} \le R. \tag{3.21}$$

If f is bounded, we suppose $f(u_1, u_2, \dots, u_{n-1}) \le \overline{M}$, $u_{n-1} \in [0, +\infty)$, $\overline{M} \in R_+$, there exists an appropriately big positive number $R > 4/\theta_* \overline{M}$, then choose $M = \theta_*/4 \in (0, \theta_*)$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \le \overline{M} \le \frac{\theta_*}{4} R = MR, \quad 0 \le u_{n-1} \le R. \tag{3.22}$$

Therefore, condition (A_2) holds. Thus, by Theorem 3.1, we know that the result of Theorem 3.3 holds. The proof of Theorem 3.3 is complete. □

4. Application

In this section, in order to illustrate our results, we consider the following examples.

Example 4.1. Consider the following boundary value problem on the specific time scale $\mathbb{T} = [0,1/3] \cup \{1/2,2/3,1\}$:

$$\begin{gathered} u^{\Delta\Delta\Delta}(t) + tu^{\Delta}\left[\frac{((16/L)+1)e^{2u^{\Delta}} - (16/L)}{u + 5e^{u^{\Delta}} + e^{2u^{\Delta}}}\right] = 0, \quad t \in [0,1]_{\mathbb{T}}, \\ u(0) = 0, \\ u^{\Delta}(0) - u^{\Delta\Delta}\left(\frac{1}{4}\right) = 0, \qquad u^{\Delta}(1) + \delta u^{\Delta\Delta}\left(\frac{1}{2}\right) = 0, \end{gathered} \tag{4.1}$$

where

$$\alpha = \gamma = 1, \quad \beta = 1, \quad \delta \geq 0, \quad \xi = \frac{1}{4}, \quad \eta = \frac{1}{2}, \quad \theta = \frac{1}{4}, \quad T = 1, \tag{4.2}$$

and L is the constant defined in Lemma 2.2,

$$g(t) = t, \qquad f\left(u, u^{\Delta}\right) = u^{\Delta}\left[\frac{((16/L)+1)e^{2u^{\Delta}} - (16/L)}{u + 5e^{u^{\Delta}} + e^{2u^{\Delta}}}\right]. \tag{4.3}$$

Then obviously

$$\begin{gathered} f_0 = \varphi = \lim_{u^{\Delta} \to 0^+} \max_{0 \leq u \leq 4u^{\Delta}} \frac{f(u,u^{\Delta})}{u^{\Delta}} = \frac{1}{6}, \\ f_{\infty} = \lambda = \lim_{u^{\Delta} \to \infty} \min_{0 \leq u \leq 4u^{\Delta}} \frac{f(u,u^{\Delta})}{u^{\Delta}} = \frac{16}{L} + 1, \end{gathered} \tag{4.4}$$

By Theorem 2.1, we have

$$\int_0^1 g(t)\Delta t = \int_0^{1/3} g(t)dt + \int_{1/3}^{\sigma(1/3)} g(t)\Delta t + \int_{1/2}^{\sigma(1/2)} g(t)\Delta t + \int_{2/3}^{\sigma(2/3)} g(t)\Delta t = \frac{5}{12}, \tag{4.5}$$

so conditions (H_1), (H_2) hold.

By simple calculations, we have

$$\theta_* = \frac{1}{(1+(\beta/\alpha))\left(\int_0^1 g(r)\Delta r\right)} = \frac{6}{5}, \tag{4.6}$$

then $\theta_*/4 = 3/10$, that is, $\varphi \in [0, \theta_*/4)$, so condition (A_3) holds.

For $\theta = 1/4$, it is easy to see that

$$\lambda \in \left(\frac{2\theta^*}{\theta}, +\infty\right), \tag{4.7}$$

so condition (A_4) holds. Then by Theorem 3.2, BVP (4.1) has at least one positive solution.

Example 4.2. Consider the following boundary value problem on the specific time scale $\mathbb{T} = [0,1/3] \cup [1/2,1]$.

$$\begin{gathered} u^{\Delta\Delta\Delta}(t) + tu^{\Delta}\left[\frac{(1/4)e^{u^{\Delta}} + \sin u^{\Delta} + 16/L}{u + e^{u^{\Delta}}}\right] = 0, \quad t \in [0,1]_{\mathbb{T}}, \\ u(0) = 0, \\ u^{\Delta}(0) - u^{\Delta\Delta}\left(\frac{1}{4}\right) = 0, \; u^{\Delta}(1) + \delta u^{\Delta\Delta}\left(\frac{1}{2}\right) = 0, \end{gathered} \tag{4.8}$$

where

$$\alpha = \gamma = 1, \quad \beta = 1, \quad \delta \geq 0, \quad \xi = \frac{1}{4}, \quad \eta = \frac{1}{2}, \quad \theta = \frac{1}{4}, \quad T = 1, \tag{4.9}$$

and L is the constant from Lemma 2.2,

$$g(t) = t, \qquad f\left(u, u^{\Delta}\right) = u^{\Delta}\left[\frac{(1/4)e^{u^{\Delta}} + \sin u^{\Delta} + 16/L}{u + e^{u^{\Delta}}}\right]. \tag{4.10}$$

Then obviously

$$\begin{gathered} f_0 = \varphi = \lim_{u^{\Delta} \to 0^+} \max_{0 \leq u \leq 4u^{\Delta}} \frac{f(u, u^{\Delta})}{u^{\Delta}} = \frac{16}{L} + \frac{1}{4}, \\ f_{\infty} = \lambda = \lim_{u^{\Delta} \to \infty} \min_{0 \leq u \leq 4u^{\Delta}} \frac{f(u, u^{\Delta})}{u^{\Delta}} = \frac{1}{4}, \end{gathered} \tag{4.11}$$

By Theorem 2.1, we have

$$\int_0^1 g(t)\Delta t = \int_0^{1/3} g(t)dt + \int_{1/3}^{\sigma(1/3)} g(t)\Delta t + \int_{1/2}^1 g(t)dt = \frac{35}{72}, \tag{4.12}$$

so conditions (H_1), (H_2) hold. By simple calculations, we have

$$\theta_* = \frac{1}{(1 + (\beta/\alpha))\left(\int_0^1 g(r)dr\right)} = \frac{36}{35}, \tag{4.13}$$

then $\theta_*/4 = 9/35$, that is, $\lambda \in [0, \theta_*/4)$, so condition (A_5) holds.

For $\theta = 1/4$, it is easy to see that

$$\varphi \in \left(\frac{2\theta^*}{\theta}, +\infty\right), \tag{4.14}$$

then condition (A_6) holds. Thus by Theorem 3.3, BVP (4.8) has at least one positive solution.

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