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Research Article

The Existence of Periodic Solutions for Non-Autonomous Differential Delay Equations via Minimax Methods

Rong Cheng[1, 2]

[1] *College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China*
[2] *Department of Mathematics, Southeast University, Nanjing 210096, China*

Correspondence should be addressed to Rong Cheng, mathchr@163.com

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By using variational methods directly, we establish the existence of periodic solutions for a class of nonautonomous differential delay equations which are superlinear both at zero and at infinity.

1. Introduction and Main Result

Many equations arising in nonlinear population growth models [1], communication systems [2], and even in ecology [3] can be written as the following differential delay equation:

$$x'(t) = -\alpha f(x(t-1)), \tag{1.1}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ is odd and α is parameter. Since Jone's work in [4], there has been a great deal of research on problems of existence, multiplicity, stability, bifurcation, uniqueness, density of periodic solutions to (1.1) by applying various approaches. See [2, 4–23]. But most of those results concern scalar equations (1.1) and generally slowly oscillating periodic solutions. A periodic solution $x(t)$ of (1.1) is called a "slowly oscillating periodic solution" if there exist numbers $p > 1$ and $q > p+1$ such that $x(t) > 0$ for $0 < t < p$, $x(t) < 0$ for $p < t < q$, and $x(t+q) = x(t)$ for all t.

In a recent paper [17], Guo and Yu applied variational methods directly to study the following vector equation:

$$x'(t) = -f(x(t-r)), \tag{1.2}$$

where $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is odd and $r > 0$ is a given constant. By using the pseudo index theory in [24], they established the existence and multiplicity of periodic solutions of (1.2) with f satisfying the following asymptotically linear conditions both at zero and at infinity:

$$\begin{aligned} f(x) &= B_0 x + o(|x|), \quad \text{as } |x| \longrightarrow 0, \\ f(x) &= B_\infty x + o(|x|), \quad \text{as } |x| \longrightarrow \infty, \end{aligned} \tag{1.3}$$

where B_0 and B_∞ are symmetric $n \times n$ constant matrices. Before Guo and Yu's work, many authors generally first use the reduction technique introduced by Kaplan and Yorke in [7] to reduce the search for periodic solutions of (1.2) with $n = 1$ and its similar ones to the problem of finding periodic solutions for a related system of ordinary differential equations. Then variational method was applied to study the related systems and the existence of periodic solutions of the equations is obtained.

The previous papers concern mainly autonomous differential delay equations. In this paper, we use minimax methods directly to study the following nonautonomous differential-delay equation:

$$x'(t) = -f(t, x(t - r)), \tag{1.4}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is odd with respect to x and satisfies the following superlinear conditions both at zero and at infinity

$$\begin{aligned} \lim_{|x| \to 0} \frac{|f(t,x)|}{|x|} &= 0, \quad \text{uniformly in } t, \\ \lim_{|x| \to \infty} \frac{|f(t,x)|}{|x|} &= \infty, \quad \text{uniformly in } t. \end{aligned} \tag{1.5}$$

When (1.2) satisfies (1.3), we can apply the twist condition between the zero and at infinity for f to establish the existence of periodic solutions of (1.2). Under the superlinear conditions (1.5), there is no twist condition for f, which brings difficulty to the study of the existence of periodic solutions of (1.4). But we can use minimax methods to consider the problem without twist condition for f.

Throughout this paper, we assume that the following conditions hold.

(H1) $f(t, x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is odd with respect to x and $2r$-periodic with respect to t.

(H2) write $f = (f_1, f_2, \ldots, f_n)$. There exist constants $\mu > 2$ and $R_1 > 0$ such that

$$0 < \mu \int_0^{x_i} f_i(t, x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) dy_i \le x_i f_i(t, x) \tag{1.6}$$

for all $x \in \mathbb{R}^n$ with $|x_i| > R_1$, for all $t \in [0, 2r]$ and $i = 1, 2, \ldots, n$.

(H3) there exist constants $c_1 > 0$, $R_2 > 0$ and $1 < \lambda < 2$ such that

$$|f_i(t, x)| < c_1 |x_i|^\lambda \tag{1.7}$$

for all $x \in \mathbb{R}^n$ with $|x_i| > R_2$, for all $t \in [0, 2r]$ and $i = 1, 2, \ldots, n$.

Then our main result can be read as follows.

Theorem 1.1. *Suppose that $f(t,x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies* (1.5) *and the conditions* $(H1)$–$(H3)$ *hold. Then* (1.4) *possesses a nontrivial $4r$-periodic solution.*

Remark 1.2. We shall use a minimax theorem in critical point theory in [25] to prove our main result. The ideas come from [25–27]. Theorem 1.1 will be proved in Section 2.

2. Proof of the Main Result

First of all in this section, we introduce a minimax theorem which will be used in our discussion. Let E be a Hilbert space with $E = E_1 \oplus E_2$. Let P_1, P_2 be the projections of E onto E_1 and E_2, respectively.

Write

$$\Lambda = \{\varphi \in C([0,2r] \times E) \mid \varphi(0,u) = u \text{ and } P_2\varphi(t,u) = P_2 u - \Phi(t,u)\}, \tag{2.1}$$

where $\Phi : [0,2r] \times E \to E_2$ is compact.

Definition 2.1. Let $S, Q \subset E$, and Q be boundary. One calls S and ∂Q link if whenever $\varphi \in \Lambda$ and $\varphi(t,\partial Q) \cap S = \emptyset$ for all t, then $\varphi(t,Q) \cap S \neq \emptyset$.

Definition 2.2. A functional $\phi \in C^1(E,\mathbb{R})$ satisfies (PS) condition, if every sequence that $\{x_m\} \subset E$, $\phi'(x_m) \to 0$ and $\phi(x_m)$ being bounded, possesses a convergent subsequence.

Then [25, Theorem 5.29] can be stated as follows.

Theorem A. *Let E be a real Hilbert space with $E = E_1 \oplus E_2$, $E_2 = E_1^{\perp}$ and inner product $\langle \cdot,\cdot \rangle$. Suppose $\phi \in C^1(E,\mathbb{R})$ satisfies (PS) condition,*

(I_1) *$\phi(x) = (1/2)\langle Ax, x\rangle + \psi(x)$, where $A(z) = A_1P_1x + A_2P_2x$ and $A_i : E_i \to E_i$ is bounded and selfadjoint, $i = 1,2$,*

(I_2) *ψ' is compact, and*

(I_3) *there exists a subspace $\widetilde{E} \subset E$ and sets $S \subset E$, $Q \subset \widetilde{E}$ and constants $\alpha > \omega$ such that*

(i) *$S \subset E_1$ and $\phi|_S \geq \alpha$,*

(ii) *Q is bounded and $\phi|_{\partial Q} \leq \omega$,*

(iii) *S and ∂Q link.*

Then ϕ possesses a critical value $c \geq \alpha$.

Let

$$F(t,x) = \int_0^{x_1} f_1(t,y_1,x_2,\ldots,x_n)\,dy_1 + \cdots + \int_0^{x_n} f_n(t,x_1,\ldots,x_{n-1},y_n)\,dy_n. \tag{2.2}$$

Then $F(t,0) = 0$ and $F'(t,x) = (f_1, f_2, \ldots, f_n)$, where F' denotes the gradient of F with respect to x. We have the following lemma.

Lemma 2.3. *Under the conditions of Theorem 1.1, the function F satisfies the following.*

(i) $F(t,x) \in C^1([0,2r] \times \mathbb{R}^n, \mathbb{R})$ *is $2r$-periodic with respect to t and $F(t,x) \geq 0$ for all $(t,x) \in [0,2r] \times \mathbb{R}^n$,*

(ii)

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = 0, \quad \textit{uniformly in } t, \tag{2.3}$$

$$\lim_{|x|\to \infty} \frac{F(t,x)}{|x|^2} = \infty, \quad \textit{uniformly in } t. \tag{2.4}$$

(iii) *There exist constants c_2, $L > 0$, and $R > 0$ such that for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $|x| > L$ and $|x_i| \geq R$, $i = 1, 2, \ldots, n$, and $t \in [0,2r]$*

$$0 < \mu F(t,x) \leq \left(x, F'(t,x)\right), \tag{2.5}$$

$$\left|F'(t,x)\right| \leq c_2 |x|^\lambda, \tag{2.6}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^n$.

Proof. The definition of F implies (i) directly. We prove case (ii) and case (iii).

Case (ii). Let

$$\begin{gathered} x_1 = r\sin\theta_1, \\ x_2 = r\sin\theta_1 \operatorname{con}\theta_2, \\ x_3 = r\sin\theta_1\sin\theta_2 \operatorname{con}\theta_3, \\ \cdots \\ x_{n-1} = r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ x_n = r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-2}\sin\theta_{n-1}. \end{gathered} \tag{2.7}$$

Then $|x|^2 = r^2$ and $|x| \to 0$ or $|x| \to \infty$ is equivalent to $r \to 0$ or $r \to \infty$, respectively.

From (1.5) and L'Hospital rules, we have (2.3) by a direct computation.

Case (iii). By (H2), we have a constant $L_1 = \sqrt{n}R_1$ such that $0 < \mu F(t,x) \leq (x, F'(t,x))$ for $|x| > L_1$ with $|x_i| \geq R_1$.

Now we prove $|F'(t,x)| \leq c_2|x|^\lambda$ for $|x| > L_2 = \sqrt{n}R_2$ with $|x_i| \geq R_2$, that is,

$$f_1^2(t,x_1) + \cdots + f_n^2(t,x_n) \leq c_2^2\left(x_1^2 + \cdots + x_n^2\right)^\lambda. \tag{2.8}$$

Firstly, it follows from $|f_1(t,x_1)| \leq c_1|x_1|^\lambda$ that $f_1^2(t,x_1) \leq c_1^2|x_1|^{2\lambda}$.

Now we show $f_1^2(t,x_1) + f_2^2(t,x_2) \leq c_1^2(|x_1|^{2\lambda} + |x_2|^{2\lambda})$. Let $|x_1|^\lambda = \tau\cos\theta$, $|x_2|^\lambda = \tau\sin\theta$. By $1 < \lambda < 2$, $1 - \sin^2\theta \geq (1 - \sin^{2/\lambda})^\lambda$, that is, $(\cos^{2/\lambda} + \sin^{2/\lambda})^\lambda \geq 1$. Then

$$\left(x_1^2 + x_2^2\right)^\lambda = \tau^2\left(\cos^{2/\lambda}\theta + \sin^{2/\lambda}\theta\right) \geq \tau^2 = |x_1|^{2\lambda} + |x_2|^{2\lambda}. \tag{2.9}$$

By reducing method, we have

$$\left(x_1^{2\lambda} + \cdots + x_n^{2\lambda}\right) \leq \left(x_1^2 + \cdots + x_n^2\right)^{\lambda} = |x|^{2\lambda}. \tag{2.10}$$

Thus, the inequality $|F'(t,x)| \leq c_2|x|^{\lambda}$ for $|x_i| \geq R_2$ holds.

Take $L = \max\{L_1, L_2\}$ and $R = \max\{R_1, R_2\}$. Then (2.5) and (2.6) hold with $|x| > L$ and $|x_i| > R$. □

Below we will construct a variational functional of (1.4) defined on a suitable Hilbert space such that finding $4r$-periodic solutions of (1.4) is equivalent to seeking critical points of the functional.

Firstly, we make the change of variable

$$t \longmapsto \frac{\pi}{2r}t = \nu^{-1}t. \tag{2.11}$$

Then (1.4) can be changed to

$$x'(t) = -\nu f\left(t, x\left(t - \frac{\pi}{2}\right)\right), \tag{2.12}$$

where f is π-periodic with respect to t. Therefore we only seek 2π-periodic solution of (2.12) which corresponds to the $4r$-periodic solution of (1.4).

We work in the Sobolev space $H = W^{1/2,2}(S^1, \mathbb{R}^{2N})$. The simplest way to introduce this space seems as follows. Every function $x \in L^2(S^1, \mathbb{R}^n)$ has a Fourier expansion:

$$x(t) = a_0 + \sum_{m=1}^{+\infty}(a_m \cos mt + b_m \sin mt), \tag{2.13}$$

where a_m, b_m are n-vectors. H is the set of such functions that

$$\|x\|^2 = |a_0|^2 + \sum_{m=1}^{+\infty} m\left(|a_m|^2 + |b_m|^2\right) < +\infty. \tag{2.14}$$

With this norm $\|\cdot\|$, H is a Hilbert space induced by the inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, y\rangle = 2\pi(a_0, a_0') + \pi\sum_{m=1}^{\infty} m\left((a_m, a_m') + (b_m, b_m')\right), \tag{2.15}$$

where $y = a_0' + \sum_{m=1}^{+\infty}(a_m' \cos mt + b_m' \sin mt)$.

We define a functional $\phi : H \to \mathbb{R}$ by

$$\phi(x) = \int_0^{2\pi} \frac{1}{2}\left(x'\left(t + \frac{\pi}{2}\right), x(t)\right)dt + \nu\int_0^{2\pi} F(t, x(t))dt. \tag{2.16}$$

By Riesz representation theorem, H identifies with its dual space H∗. Then we define an operator A:H→H∗=H by extending the bilinear form:

$$\langle Ax, y\rangle = \int_0^{2\pi} \left(x'\left(t+\frac{\pi}{2}\right), y(t)\right) dt, \quad \forall x, y \in H. \tag{2.17}$$

It is not difficult to see that A is a bounded linear operator on H and $\ker A = \mathbb{R}^n$.

Define a mapping $\psi : H \to \mathbb{R}$ as

$$\psi(x) = \nu \int_0^{2\pi} F(t, x(t)) dt, \tag{2.18}$$

Then the functional ϕ can be rewritten as

$$\phi(x) = \frac{1}{2}\langle Ax, x\rangle + \psi(x), \quad \forall x \in H. \tag{2.19}$$

According to a standard argument in [24], one has for any $x, y \in H$,

$$\langle \phi'(x), y\rangle = \int_0^{2\pi} \frac{1}{2}\left(x'\left(t+\frac{\pi}{2}\right) - x'\left(t-\frac{\pi}{2}\right), y(t)\right) dt + \nu \int_0^{2\pi} (f(t, x(t)), y(t)) dt. \tag{2.20}$$

Moreover according to [28], $\psi' : H \to H$ is a compact operator defined by

$$\langle \psi'(x), y\rangle = \nu \int_0^{2\pi} (f(t, x(t)), y(t)) dt. \tag{2.21}$$

Our aim is to reduce the existence of periodic solutions of (2.12) to the existence of critical points of ϕ. For this we introduce a shift operator $\Gamma : H \to H$ defined by

$$\Gamma x(t) = x\left(t+\frac{\pi}{2}\right). \tag{2.22}$$

It is easy to compute that Γ is bounded and linear. Moreover Γ is isometric, that is, $\|\Gamma x\| = \|x\|$ and $\Gamma^4 = id$, where id denotes the identity mapping on H.

Write

$$E = \left\{x \in H : \Gamma^2 x(t) = -x(t)\right\}. \tag{2.23}$$

Lemma 2.4. *Critical points of $\phi|_E$ over E are critical points of ϕ on H, where $\phi|_E$ is the restriction of ϕ over E.*

Proof. Note that any $x \in E$ is 2π-periodic and f is odd with respect to x. It is enough for us to prove $\langle \phi'(x), y\rangle = 0$ for any $y \in H$ and x being a critical point of ϕ in E.

For any $y \in H$, we have

$$\begin{aligned}
\left\langle \Gamma^2 \phi'(x), y \right\rangle &= \left\langle \Gamma^2 Ax, y \right\rangle + \left\langle \Gamma^2 \psi'(x), y \right\rangle = \left\langle Ax, \Gamma^{-2} y \right\rangle + \left\langle \psi'(x), \Gamma^{-2} y \right\rangle \\
&= \int_0^{2\pi} \left(x'\left(t + \frac{\pi}{2}\right), y(t-\pi) \right) dt + \nu \int_0^{2\pi} (f(t, x(t)), y(t-\pi)) dt \\
&= \int_0^{2\pi} \left(x'\left(t + \frac{\pi}{2} + \pi\right), y(t) \right) dt + \nu \int_0^{2\pi} (f(t+\pi, x(t+\pi)), y(t)) dt \\
&= \int_0^{2\pi} \left(-x'\left(t + \frac{\pi}{2}\right), y(t) \right) dt + \nu \int_0^{2\pi} (f(t, -x(t)), y(t)) dt \\
&= -\int_0^{2\pi} \left(x'\left(t + \frac{\pi}{2}\right), y(t) \right) dt - \nu \int_0^{2\pi} (f(t, x(t)), y(t)) dt \\
&= \langle -\phi'(x), y \rangle.
\end{aligned} \tag{2.24}$$

This yields $\Gamma^2 \phi'(x) = -\phi'(x)$, that is, $\phi'(x) \in E$.

Suppose that x is a critical point of ϕ in E. We only need to show that $\langle \phi'(x), y \rangle = 0$ for any $y \in H$. Writing $y = y_1 \oplus y_2$ with $y_1 \in E,\ y_2 \in E^{\perp}$ and noting $\phi'(x) \in E$, one has

$$\langle \phi'(x), y \rangle = \langle \phi'(x), y_1 \rangle + \langle \phi'(x), y_2 \rangle = 0. \tag{2.25}$$

The proof is complete. □

Remark 2.5. By Lemma 2.4, we only need to find critical points of $\phi|_E$ over E. Therefore in the following ϕ will be assumed on E.

For $x \in E$, $x(t+\pi) = -x(t)$ yields that $a_0 = 0$, where a_0 is in the Fourier expansion of x. Thus $\ker A|_E = \{0\}$. Moreover for any $x, y \in E$,

$$\begin{aligned}
\langle Ax, y \rangle &= \int_0^{2\pi} \left(x'\left(t + \frac{\pi}{2}\right), y(t) \right) dt = -\int_0^{2\pi} \left(x\left(t + \frac{\pi}{2}\right), y'(t) \right) dt \\
&= -\int_0^{2\pi} \left(x(t), y'\left(t - \frac{\pi}{2}\right) \right) dt = \int_0^{2\pi} \left(x(t), y'\left(t + \frac{\pi}{2}\right) \right) dt \\
&= \langle x, Ay \rangle.
\end{aligned} \tag{2.26}$$

Hence A is self-adjoint on E.

Let E^+ and E^- denote the positive definite and negative definite subspace of A in E, respectively. Then $E = E^+ \oplus E^-$. Letting $E_1 = E^+$, $E_2 = E^-$, we see that (I_1) of Theorem A holds. Since ψ' is compact, (I_2) of Theorem A holds. Now we establish (I_3) of Theorem A by the following three lemmas.

Lemma 2.6. *Under the assumptions of Theorem 1.1, (i) of (I_3) holds for ϕ.*

Proof. From the assumptions of Theorem 1.1 and Lemma 2.3, one has

$$F(t, x) \le c_3 + c_4 |x|^{\lambda+1}, \quad \forall (t, x) \in [0, \pi] \times \mathbb{R}^n. \tag{2.27}$$

By (2.3), for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$F(t,x) \le \varepsilon |x|^2, \quad \forall t \in [0,\pi],\ |x| \le \delta. \tag{2.28}$$

Therefore, there is an $M = M(\varepsilon) > 0$ such that

$$F(t,x) \le \varepsilon |x|^2 + M|x|^{\lambda+1}, \quad \forall (t,x) \in [0,\pi] \times \mathbb{R}^n. \tag{2.29}$$

Since E is compactly embedded in $L^s(S^1,\mathbb{R}^n)$ for all $s \ge 1$ and by (2.29), we have

$$\int_0^{2\pi} F(t,x)dt \le \varepsilon \|x\|_{L^2}^2 + M\|x\|_{L^{\lambda+1}}^{\lambda+1} \le \left(\varepsilon c_5 + M c_6 \|x\|^{\lambda-1}\right)\|x\|^2. \tag{2.30}$$

Consequently, for $x \in E_1 = E^+$,

$$\phi(x) \ge \|x\|^2 - \nu\left(\varepsilon c_5 + M c_6 \|x\|^{\lambda-1}\right)\|x\|^2. \tag{2.31}$$

Choose $\varepsilon = (3\nu c_5)^{-1}$ and ρ so that $3\nu M c_6 \rho^{\lambda-1} = 1$. Then for any $x \in \partial B_\rho \cap E_1$,

$$\phi(x) \ge \frac{1}{3}\rho^2. \tag{2.32}$$

Thus ϕ satisfies (i) of (I_3) with $S = \partial B_\rho \cap E_1$ and $\alpha = (1/3)\rho^2$. □

Lemma 2.7. *Under the assumptions of Theorem 1.1, ϕ satisfies (ii) of (I_3).*

Proof. Set $e \in S = \partial B_\rho \cap E_1$ and let

$$Q = \{se : 0 \le s \le 2s_1\} \oplus B_{2s_1} \cap E_2, \tag{2.33}$$

where s_1 is free for the moment.

Let $\widetilde{E} = E^- \oplus \operatorname{span}\{e\}$. Write

$$K = \left\{x \in \widetilde{E} : \|x\| = 1\right\}, \qquad \lambda^- = \inf_{x \in E^-, \|x\|=1} \left|\langle Ax^-, x^-\rangle\right|, \qquad \lambda^+ = \sup_{x \in E^+, \|x\|=1} |\langle Ax^+, x^+\rangle|. \tag{2.34}$$

Case (1). If $\|x^-\| > \gamma\|x^+\|$ with $\gamma = \sqrt{\lambda^+/\lambda^-}$, one has

$$\begin{aligned} \phi(sx) &= \frac{1}{2}\langle Asx^+, sx^+\rangle + \frac{1}{2}\langle Asx^-, sx^-\rangle - \nu\int_0^{2\pi} F(t,sx)dt \\ &\le -\frac{1}{2}\lambda^- s^2 \|x^-\|^2 + \frac{1}{2}\lambda^+ s^2 \|x^+\|^2 \le 0. \end{aligned} \tag{2.35}$$

Case (2). If $\|x^-\| \le \gamma\|x^+\|$, we have

$$1 = \|x\|^2 = \|x^+\|^2 + \left\|x^-\right\|^2 \le \left(1+\gamma^2\right)\|x^+\|^2. \tag{2.36}$$

That is

$$\|x^+\|^2 \ge \frac{1}{1+\gamma^2} > 0. \tag{2.37}$$

Denote $\widetilde{K} = \{x \in K : \|x^-\| \le \gamma\|x^+\|\}$. By appendix, there exists $\varepsilon_1 > 0$ such that $\forall u \in \widetilde{K}$,

$$\operatorname{meas}\{t \in [0,\pi] : |u(t)| \ge \varepsilon_1\} \ge \varepsilon_1. \tag{2.38}$$

Now for $x = x^+ + x^- \in \widetilde{K}$, set $\Omega_x = \{t \in [0,\pi] : |x(t)| \ge \varepsilon_1\}$. By (2.4), for a constant $M_0 = \|A\|/\nu\varepsilon_1^3 > 0$, there is an $L_3 > 0$ such that

$$F(t,z) \ge M_0|x|^2, \quad \forall|x| \ge L_3 \text{ uniformly in } t. \tag{2.39}$$

Choosing $s_1 \ge L_3/\varepsilon_1$, for $s \ge s_1$,

$$F(t,sx(t)) \ge M_0|sx(t)|^2 \ge M_0 s^2\varepsilon_1^2, \quad \forall t \in \Omega_x. \tag{2.40}$$

For $s \ge s_1$, we have

$$\begin{aligned}\phi(sx) &= \frac{1}{2}s^2\langle Ax^+,x^+\rangle + \frac{1}{2}s^2\langle Ax^-,x^-\rangle - \nu\int_0^{2\pi} F(t,sx)dt \\ &\le \frac{1}{2}\|A\|s^2 - \nu\int_{\Omega_x} F(t,sx)dt \\ &\le \frac{1}{2}\|A\|s^2 - M_0 s^2\varepsilon_1^2\operatorname{meas}(\Omega_x) \\ &\le \frac{1}{2}\|A\|s^2 - M_0 s^2\varepsilon_1^3 = -\frac{1}{2}\|A\|s^2 < 0.\end{aligned} \tag{2.41}$$

Henceforth, $\phi(sx) \le 0$ for any $x \in K$ and $s \ge s_1$, that is, $\phi|_{\partial Q} \le 0$. Then (ii) of (I_3) holds. □

Lemma 2.8. *S and ∂Q link.*

Proof. Suppose $\varphi \in \Lambda$ and $\varphi(\partial Q) \cap S = \emptyset$ for all $t \in [0,\pi]$. Then we claim that for each $t \in [0,\pi]$, there is a $w(t) \in Q$ such that $\phi(t,w(t)) \in S$, that is,

$$P\varphi(w(t)) = 0, \quad \|w(t)\| = \rho, \tag{2.42}$$

where $P : E \to E^-$ is a projection. Define

$$G : [0,\pi] \times Q \longrightarrow E^- \times \mathbb{R}e \tag{2.43}$$

as follows:

$$G(t, u + se) = \left[(1 - t)u + P\varphi(u + se)\right] + \left[(1 - t)s + t\left\|(I - P)\varphi(u + se)\right\| - \rho\right]e. \tag{2.44}$$

It is easy to see that

$$G(t, u + se) = u + (s - \rho)e \neq 0, \quad \text{as } u + se \in \partial Q. \tag{2.45}$$

However,

$$\begin{gathered} G(1, u + se) = P\varphi(u + se) + \left[\left\|(I - P)\varphi(u + se)\right\| - \rho\right]e \\ G(0, u + se) = u + (s - \rho)e. \end{gathered} \tag{2.46}$$

According to topological degree theory in [29], we have

$$\begin{aligned} \deg(G(1, \cdot); Q, 0) &= \deg(G(0, \cdot); Q, 0) \\ &= \deg\left(id_{E^-}; E^- \cap B_{2s_1}, 0\right) \deg\left(s - \rho, (0, 2s_1), 0\right) = 1 \end{aligned} \tag{2.47}$$

since $\rho \in (0, 2s_1)$. Therefore S and ∂Q link. □

Now it remains to verify that ϕ satisfies (PS)-condition.

Lemma 2.9. *Under the assumptions of Theorem 1.1, ϕ satisfies (PS)-condition.*

Proof. Suppose that

$$\left|\phi(x_m)\right| \leq M', \quad \phi'(x_m) \longrightarrow 0, \quad \text{as } m \longrightarrow \infty. \tag{2.48}$$

We first show that $\{x_m\}$ is bounded. If $\{x_m\}$ is not bounded, then by passing to a subsequence if necessary, let $\|x_m\| \to +\infty$ as $m \to +\infty$.

By (2.4), there exists a constant $M'' > 0$ such that $F(t, x) > c_7|x|^2$ as $|x| > M''$. By (2.5), one has

$$\begin{aligned} 2\phi(x_m) - \left\langle \phi'(x_m), x_m \right\rangle &= \int_0^{2\pi} \left(\left(x_m, \nu F'(t, x_m)\right) - 2\nu F(t, x_m)\right) dt \\ &\geq \int_0^{2\pi} \nu(\mu - 2) F(t, x_m) dt \\ &\geq c_7 \nu(\mu - 2) \int_0^{2\pi} |x_m|^2 dt. \end{aligned} \tag{2.49}$$

This yields

$$\frac{\int_0^{2\pi} |x_m|^2 dt}{\|x_m\|} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{2.50}$$

Write $\kappa = 1/2(\lambda - 1)$. By (2.6), there is a constant $c_9 > 0$ such that

$$\left|F'(t,x)\right|^{\kappa} \le c_2^{\kappa}|x|^{\lambda\kappa} + c_8, \quad \forall (t,x) \in [0,\pi] \times \mathbb{R}^n. \tag{2.51}$$

Therefore,

$$\begin{aligned}\int_0^{2\pi} \left|F'(t,x_m)\right|^{k} dt &\le \int_0^{2\pi} \left(c_2^{\kappa}|x_m|^{\lambda\kappa} + c_8\right) dt \\ &\le c_9 \left(\int_0^{2\pi} \|x_m\|^2 dt\right)^{1/2} \left(\int_0^{2\pi} \|x_m\|^{2(\kappa\lambda-1)} dt\right)^{1/2} + c_{10} \\ &\le c_{11} \left(\int_0^{2\pi} \|x_m\|^2 dt\right)^{1/2} \|x_m\|^{\kappa\lambda-1} + c_{12}.\end{aligned} \tag{2.52}$$

This inequality and (2.50) imply that

$$\left(\frac{\left(\int_0^{2\pi}|F'(t,x_m)|^{\kappa}dt\right)^{1/\kappa}}{\|x_m\|^{\kappa}}\right)^{1/\kappa} \le \frac{c_{11}\left(\int_0^{2\pi}\|x_m\|^2 dt\right)^{1/2}}{\|x_m\|^{1/2}} \frac{\|x_m\|^{\kappa\lambda-1}}{\|x_m\|^{\kappa-1/2}} + \frac{c_{12}}{\|x_m\|^{\kappa}} \longrightarrow 0 \tag{2.53}$$

as $m \to \infty$, since $\kappa > 1$.

Denote $x_m = x_m^+ + x_m^- \in E^+ \oplus E^-$. We have

$$\begin{aligned}\left\langle \phi'(x_m), x_m^- \right\rangle &= \left\langle Ax_m^-, x_m^- \right\rangle - \int_0^{2\pi} \left(x_m^-, F'(t,x_m)\right) dt \\ &\ge \left\langle Ax_m^-, x_m^- \right\rangle - \int_0^{2\pi} \left|x_m^-\right|\left|F'(t,x_m)\right| dt \\ &\ge \left\langle Ax_m^-, x_m^- \right\rangle - \left(\int_0^{2\pi} \left|F'(t,x_m)\right|^{\kappa} dt\right)^{1/\kappa} C_{\kappa}\left\|x_m^-\right\|,\end{aligned} \tag{2.54}$$

where $C_\kappa > 0$ is a constant independent of m.

By the above inequality, one has

$$\frac{\langle Ax_m^-, x_m^- \rangle}{\|x_m\|\|x_m^-\|} \le \frac{\left\|\phi'(x_m)\right\|\|x_m^-\|}{\|x_m\|\|x_m^-\|} + \frac{\left(\int_0^{2\pi}|F'(t,x_m)|^{\kappa}dt\right)^{1/\kappa}}{\|x_m\|} \frac{C_\kappa\|x_m^-\|}{\|x_m^-\|} \longrightarrow 0 \tag{2.55}$$

as $m \to \infty$. This yields

$$\frac{\|x_m^-\|}{\|x_m\|} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{2.56}$$

Similarly, we have

$$\frac{\|x_m^+\|}{\|x_m\|} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{2.57}$$

Thus it follows from (2.56) and (2.57) that

$$1 = \frac{\|x_m\|}{\|x_m\|} \leq \frac{\|x_m^+\| + \|x_m^-\|}{\|x_m\|} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{2.58}$$

which is a contradiction. Hence $\{x_m\}$ is bounded.

Below we show that $\{x_m\}$ has a convergent subsequence. Notice that $\ker A|_E = \{0\}$ and $\psi' : H \to H$ is compact. Since $\{x_m\}$ is bounded, we may suppose that

$$\psi'(x_m) \longrightarrow y \quad \text{as } m \longrightarrow \infty. \tag{2.59}$$

Since A has continuous inverse A^{-1} in E, it follows from

$$Ax_m = \phi'(x_m) + \psi'(x_m) \tag{2.60}$$

that

$$x_m = A^{-1}\left(\phi'(x_m) + \psi'(x_m)\right) \longrightarrow A^{-1}y \quad \text{as } m \longrightarrow \infty. \tag{2.61}$$

Henceforth $\{x_m\}$ has a convergent subsequence. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It is obviously that Theorem 1.1 holds from Lemmas 2.3, 2.4, 2.6, 2.7, 2.8, and 2.9 and Theorem A. □

Appendix

The purpose of this appendix is to prove the following lemma. The main idea of the proof comes from [26].

Lemma A.1. *There exists $\varepsilon_1 > 0$ such that, for all $u \in \widetilde{K}$,*

$$\operatorname{meas}\{t \in [0,\pi] : |u(t)| \geq \varepsilon_1\} \geq \varepsilon_1. \tag{A.1}$$

Proof. If (A.1) is not true, $\forall k > 0$, there exists $u_k \in \widetilde{K}$ such that

$$\operatorname{meas}\left\{t \in [0,\pi] : |u(t)| \geq \frac{1}{k}\right\} \leq \frac{1}{k}. \tag{A.2}$$

Write $u_k = u_k^- + u_k^+ \in \widetilde{E}$. Notice that $\dim\operatorname{span}\{e\} < \infty$ and $\|u_k^+\| \le 1$. In the sense of subsequence, we have

$$u_k^+ \longrightarrow u_0^+ \in \operatorname{span}\{e\} \quad \text{as } k \longrightarrow +\infty. \tag{A.3}$$

Then (2.37) implies that

$$\left\|u_0^+\right\|^2 \ge \frac{1}{1+\gamma^2} > 0. \tag{A.4}$$

Note that $\|u_k^-\| \le 1$, in the sense of subsequence $u_k^- \rightharpoonup u_0^- \in E^-$ as $k \to +\infty$. Thus in the sense of subsequence,

$$u_k \rightharpoonup u_0 = u_0^- + u_0^+ \quad \text{as } k \longrightarrow +\infty. \tag{A.5}$$

This means that $u_k \to u_0$ in L^2, that is,

$$\int_0^\pi |u_k - u_0|^2 dt \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty. \tag{A.6}$$

By (A.4) we know that $\|u_0\| > 0$. Therefore, $\int_0^\pi |u_0|^2 dt > 0$. Then there exist $\delta_1 > 0, \delta_2 > 0$ such that

$$\operatorname{meas}\{t \in [0,\pi] : |u_0(t)| \ge \delta_1\} \ge \delta_2. \tag{A.7}$$

Otherwise, for all $n > 0$, we must have

$$\operatorname{meas}\left\{t \in [0,\pi] : |u_0(t)| \ge \frac{1}{n}\right\} = 0, \tag{A.8}$$

that is, $\operatorname{meas}\{t \in [0,\pi] : |u_0(t)| \le 1/n\} = 1$, $0 < \int_0^\pi |u_0|^2 dt < 1/n^2 \to 0$ *as* $n \to +\infty$. We get a contradiction. Thus (A.7) holds. Let $\Omega_0 = \{t \in [0,\pi] : |u_0(t)| \ge \delta_1\}$, $\Omega_k = \{t \in [0,\pi] : |u_0(t)| \le 1/k\}$, and $\Omega_k^\perp = [0,\pi] \setminus \Omega_k$. By (A.2), we have

$$\operatorname{meas}(\Omega_k \cap \Omega_0) = \operatorname{meas}\left(\Omega_0 - \Omega_0 \cap \Omega_k^\perp\right) \ge \operatorname{meas}(\Omega_0) - \operatorname{meas}\left(\Omega_0 \cap \Omega_k^\perp\right) \ge \delta_2 - \frac{1}{k}. \tag{A.9}$$

Let k be large enough such that $\delta_2 - 1/k \ge (1/2)\delta_2$ and $\delta_1 - 1/k \ge (1/2)\delta_1$. Then we have

$$|u_k - u_0|^2 \ge \left(\delta_1 - \frac{1}{k}\right)^2 \ge \left(\frac{1}{2}\delta_1\right)^2, \quad \forall t \in \Omega_k \cap \Omega_0. \tag{A.10}$$

This implies that

$$\begin{aligned}\int_0^{\pi}|u_k-u_0|^2dt &\geq \int_{\Omega_k\cap\Omega_0}|u_k-u_0|^2dt \geq \left(\frac{1}{2}\delta_1\right)^2\cdot \operatorname{meas}(\Omega_k\cap\Omega_0)\\ &\geq \left(\frac{1}{2}\delta_1\right)^2\cdot\left(\delta_2-\frac{1}{k}\right)\geq\left(\frac{1}{2}\delta_1\right)^2\left(\frac{1}{2}\delta_2\right)>0.\end{aligned} \tag{A.11}$$

This is a contradiction to (A.6). Therefore the lemma is true and (A.1) holds. □

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