Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 134749, 36 pages
doi:10.1155/2009/134749

Research Article

Results and Conjectures about Order q Lyness' Difference Equation $u_{n+q}u_n = a + u_{n+q-1} + \cdots + u_{n+1}$ in $\mathbb{R}_*^+$, with a Particular Study of the Case $q = 3$

G. Bastien[1,2] and M. Rogalski[2,3]

[1] *Institut de Mathématiques de Jussieu, UPMC-Paris 06, UMR-CNRS 7586, 75251 Paris, France*
[2] *Equipe d'Analyse fonctionnelle, IMJ, 16 rue Clisson, 75013 Paris, France*
[3] *Laboratoire Paul Painlevé, USTL Lille 1, UMR-CNRS 8524, 59655 Villeneuve, France*

Correspondence should be addressed to G. Bastien, bastien@upmc.fr

Received 4 March 2009; Accepted 14 July 2009

Recommended by Istvan Gyori

We study order q Lyness' difference equation in $\mathbb{R}_*^+$: $u_{n+q}u_n = a + u_{n+q-1} + \cdots + u_{n+1}$, with $a > 0$ and the associated dynamical system F_a in $\mathbb{R}_*^{+q}$. We study its solutions (divergence, permanency, local stability of the equilibrium). We prove some results, about the first three invariant functions and the topological nature of the corresponding invariant sets, about the differential at the equilibrium, about the role of 2-periodic points when q is odd, about the nonexistence of some minimal periods, and so forth and discuss some problems, related to the search of common period to all solutions, or to the second and third invariants. We look at the case $q = 3$ with new methods using new invariants for the map F_a^2 and state some conjectures on the associated dynamical system in $\mathbb{R}_*^{+q}$ in more general cases.

1. Introduction

We will study the global behavior of solutions of the order q Lyness' difference equation for $q \geq 3$ and $a > 0$:

$$u_{n+q} = \frac{a + u_{n+q-1} + \cdots + u_{n+1}}{u_n}, \quad u_0, u_1, \ldots, u_{q-1} > 0. \tag{1.1}$$

The associated dynamical system in $\mathbb{R}_*^{+q}$ is given by the map F_a:

$$F_a : x = (x_1, \ldots, x_q) \longmapsto F_a(x) = \left(\frac{a + x_1 + \cdots + x_{q-1}}{x_q}, x_1, x_2, \ldots, x_{q-1} \right). \tag{1.2}$$

The orbit of $M_0 = (u_{q-1}, \ldots, u_1, u_0)$ under F_a is $\{M_0, M_1, \ldots, M_n, \ldots\}$ where the point M_n is $M_n = (u_{n+q-1}, \ldots, u_{n+1}, u_n)$.

For $q = 2$, we have so the ordinary order 2 Lyness' difference equation, which is studied in some previous papers; see, for example, [1, 7]. This equation was introduced in 1942 by Lyness, for the case $a = 1$, while he was working on a problem in number theory. This equation is also used in geometry and frieze patterns (see [3] for references). When $q = 3$, it is "Todd's equation," whose deep study for $a \neq 1$ is made in [4].

In Section 2 we study the elementary properties of the solutions of (1.1): divergence, permanent character, and local stability of the equilibrium. Then we look at the first invariant function for order q Lyness' equation, and prove that the invariant manifolds associated to it are homeomorphic to the sphere $\mathbb{S}^{q-1}$.

In Section 3 we look at the second invariant for order q Lyness' equation ($q \geq 3$), prove results about it, and make some conjectures, that we prove for $q = 3$, $q = 4$, and $q = 5$.

In Section 4 we study, for odd $q \geq 5$, the third invariant of order q Lyness' equation found in [5] and prove about it results analogous to these ones of Sections 2 and 3.

In Sections 5 and 6 we study the eigenvalues and the global nature of the differential dF_a at the equilibrium, give some elementary results about periods of solutions of (1.1), and study the possible common periods to all solutions of (1.1).

In Section 7 we study the particular case $q = 3$. We use another approach than in [4] (which solves almost completely the case $q = 3$), using new invariants of the map F_a^2.

In Section 8 we make some general conjectures for the behaviour of the solutions of order q Lyness' difference equation.

We have recently received the nice preprint [6], which studies also order q Lyness' equation, with new and very interesting tools. Some points are common to this paper and to our one.

2. Elementary Properties, First Invariant Function, and Invariant Manifolds

We start with a classical result (see [3, 7, 8]).

Lemma 2.1. *The function on* $\mathbb{R}_*^{+q}$

$$G_a(x_1, x_2, \ldots, x_q) := \left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) \cdots \left(1 + \frac{1}{x_q}\right)(a + x_1 + x_2 + \cdots + x_q) \tag{2.1}$$

is invariant under the action of F_a. *So, the sequence*

$$G_a(M_n) = \left(1 + \frac{1}{u_{n+q-1}}\right)\left(1 + \frac{1}{u_{n+q-2}}\right) \cdots \left(1 + \frac{1}{u_n}\right)(a + u_{n+q-1} + u_{n+q-2} + \cdots + u_n) \tag{2.2}$$

is constant.

Some properties of function G_a *will be useful.*

Lemma 2.2. *(1) The quantity $G_a(x)$ tends to infinity when x tends to the point at infinity of $\mathbb{R}_*^{+q}$; that is, the sets $\{x \mid G_a(x) \le K\}$ are compact.*

(2) G_a has an unique critical point, at its absolute minimum, attained at the equilibrium of (1.1) *$L := (\ell, \ell, \dots, \ell)$, where ℓ is the unique positive solution of the equation*

$$X^2 - (q-1)X - a = 0. \tag{2.3}$$

One has $\ell > q-1$, and the minimum of G_a is

$$K(a) := \left(1 + \frac{1}{\ell}\right)^q (a + q\ell) = \frac{(\ell+1)^{q+1}}{\ell^{q-1}} > \left(q + \sqrt{a}\right)^2. \tag{2.4}$$

Proof. (1) On the closed sets $\{x \mid G_a(x) \le K\}$ of $\mathbb{R}_*^{+q}$ we have $\sum_i x_i \le K - a$ and $\forall i$ $1 + 1/x_i \le K/a$, and these estimates define a compact set in $\mathbb{R}_*^{+q}$.

(2) Substracting the two equations $\partial G_a/\partial x_i = 0$ and $\partial G_a/\partial x_j = 0$ we have $(x_i - x_j)(x_i + x_j + 1) = 0$, and so all the x_i are equal at a critical point. Hence, the equation $\partial G_a/\partial x_1 = 0$ gives $x_1^2 - (q-1)x_1 - a = 0$, and then for all i $x_i = \ell$. So point L is the unique critical point of G_a, and then G_a attains only its absolute minimum $K(a)$ at this point. Now, evaluation of this minimum is obvious. □

So we can apply [17, Proposition 2.1], to give a synthesis of new and older results about the dynamical system $(F_a, \mathbb{R}_*^{+q})$ (see also [8]).

Proposition 2.3. *(1) For $K > K(a)$, the sets*

$$\Sigma_a(K) := \{x \mid G_a(x) = K\}, \qquad \widetilde{\Sigma}_a(K) := \{x \mid G_a(x) \le K\} \tag{2.5}$$

are compact in $\mathbb{R}_*^{+q}$ and invariant by F_a. The sets $\Sigma_a(K)$ are the boundaries of the open sets $\{x \mid G_a(x) < K\}$, which are connected and constitute a fundamental system of neighborhoods of L.

If $G_a(M_0) = K$, then the orbit of M_0 remains on $\Sigma_a(K)$ and so is bounded.

(2) If $M_0 \ne L$, then the sequence (u_n) diverges, but for all n $a/(K-a) \le u_n \le K - aK = G_a(M_0)$, (u_n) is permanent.

(3) The equilibrium L of F_a is locally stable.

For $q = 2$, one knows that $\Sigma_a(K)$ is homeomorphic to a circle (see [1]). For $q \ge 3$ one can generalize this fact. First one proves a local result.

Proposition 2.4. *If $K > K(a)$ is sufficiently near to $K(a)$, then the set $\widetilde{\Sigma}_a(K)$ is starlike with respect to the point L, and a ray from L cuts $\Sigma_a(K)$ at exactly one point. Hence $\widetilde{\Sigma}_a(K)$ is homeomorphic to an euclidean ball, and $\Sigma_a(K)$ is homeomorphic to a sphere, if K is sufficiently near to $K(a)$.*

Proof. Put, for $\rho > 0$, $\rho < \ell$, and $\overrightarrow{u}$ an unitary vector of $\mathbb{R}^q$, $\delta_{\overrightarrow{u}}(\rho) := \ln G_a(L + \rho\overrightarrow{u})$. It is sufficient to prove the relation $\delta'_{\overrightarrow{u}}(\rho) = \rho C(\overrightarrow{u}) + o(\rho)$, uniformly with respect to $\overrightarrow{u}$, with $C(\overrightarrow{u}) \geq D > 0$. In fact, this assertion will prove that $\delta'_{\overrightarrow{u}}(\rho) > 0$ if ρ is sufficiently small, with $\delta'_{\overrightarrow{u}}(0) = 0$. Hence δ will be increasing on a neighborhood of 0 independent of $\overrightarrow{u}$, and the equation $\ln G_a(L + \rho\overrightarrow{u}) = \ln K$ will have a unique solution ρ if K is sufficiently near to $K(a)$, because the sets $\{x \mid G_a(x) < K\}$ are fundamental neighborhoods of L.

But easy calculations give

$$\begin{aligned}
\delta'_{\overrightarrow{u}}(\rho) &= \sum_j u_j \left\{ \frac{1}{a + q\ell + \rho \sum_i u_i} - \frac{1}{(\ell + \rho u_j)^2 + \ell + \rho u_j} \right\} \\
&= \frac{1}{a + q\ell} \sum_j u_j \left\{ \left(1 - \frac{\rho \sum_i u_i}{a + q\ell} + o(\rho)\right) - \left(1 - \frac{\rho(2\ell + 1)u_j}{a + q\ell} + o(\rho)\right) \right\} \\
&= \frac{1}{(a + q\ell)^2} \left(2\ell + 1 - \left(\sum_i u_i\right)^2 \right) \rho + o(\rho),
\end{aligned} \tag{2.6}$$

where it is easy to see that $o(\rho)$ is uniform with respect to $\overrightarrow{u}$. The vector $\overrightarrow{u}$ is unitary, so $|\sum_i u_i| \leq \sqrt{q}$, and then $2\ell + 1 - (\sum_i u_i)^2 \geq q - 1 > 0$, and we can take $D = (q-1)/(a + q\ell)^2$. Hence this proves the proposition. □

In fact, the homeomorphism of $\Sigma_a(K)$ with a sphere is true for every value of $K > K(a)$.

Theorem 2.5. *If $K > K(a)$, then the invariant set $\Sigma_a(K)$ is homeomorphic to the $(q-1)$-dimensional sphere $\mathbb{S}^{q-1}$.*

Proof. First, we remark that the set $\Sigma_a(K)$ is a compact $(q-1)$-manifold in $\mathbb{R}_*^{+q}$, because the function G_a has no critical point in $\mathbb{R}_*^{+q} \setminus \{L\}$.

Now consider the function $s(x) := \sum_{i=1}^q x_i$. It is a C^∞ function, and has exactly 2 critical points on $\Sigma_a(K)$. In fact, if we write the equation $ds = \lambda dG_a$, we find, with $p := \prod_{j=1}^q (1 + 1/x_j)$, the relations $x_i^2 + x_i = (a + s(x))/(1 - 1/\lambda p)$. By substracting, we get $(x_i - x_j)(x_i + x_j + 1) = 0$, and so at a critical point we would have for all i $x_i = t$, for some $t > 0$. Of course, at a point $(t, t, \ldots, t)$ on $\Sigma_a(K)$ one has, by symmetry, $ds = \lambda dG_a$. So we have to find the intersections of $\Sigma_a(K)$ with the diagonal, *id est* to solve the equation $\phi_a(t) := (1 + 1/t)^q (a + qt) = K$. But we have $\phi'_a(t) = (q/t^2)(1 + 1/t)^{q-1}[t^2 - (q-1)t - a]$. So ϕ_a is decreasing on $]0, \ell]$ from $+\infty$ to $K(a)$ and increasing on $[\ell, +\infty[$ from $K(a)$ to $+\infty$. Hence, for $K > K(a)$, there are exactly two numbers t_1 and t_2, with $0 < t_1 < l < t_2$, such that $\phi_a(t_i) = K$. Then, the two points $T_1 = (t_1, t_1, \ldots, t_1)$ and $T_2 = (t_2, t_2, \ldots, t_2)$ are the two critical points of the function s on $\Sigma_a(K)$.

Now, a classical result of differential topology due to G. Reeb says that if on a compact manifold there is a smooth function which has exactly two critical points, then this manifold is homeomorphic to a sphere (see [10, page 116], or [2, page 25]), and so $\Sigma_a(K) \approx \mathbb{S}^{q-1}$. □

3. The Second Invariant of Order q Lyness' Equation

3.1. General Results

In [5] the authors found new invariants for order q Lyness' equation.

Proposition 3.1 ([5]). *If $q \geq 3$, the following function H_a on $\mathbb{R}_*^{+q}$,*

$$H_a(x) = \frac{(1 + x_1 + x_2)(1 + x_2 + x_3)\cdots(1 + x_{q-1} + x_q)\left(a + x_1 x_q + x_1 + x_2 + \cdots + x_q\right)}{x_1 \cdots x_q} \tag{3.1}$$

is invariant under the action of F_a.

It is proved in [5] that H_a and G_a are independent.

Proposition 3.2. *(1) The quantity $H_a(x)$ tends to infinity when x tends to the point at infinity of $\mathbb{R}_*^{+q}$; that is, the sets $\{H_a(x) \leq M\}$ are compact.*

(2) H_a has a strict minimum at the point $L = (\ell, \ldots, \ell)$, whose value is

$$M(a) := (2\ell + 1)^q \ell^{1-q}. \tag{3.2}$$

Proof. (1) We have to prove that for $M > 0$ the closed set $\widetilde{S}_a(M) := \{H_a(x) \leq M\}$ is compact. Let h_a be the numerator in H_a. Every product $x_1 x_2 \cdots x_{i-1} x_i^2 x_{i+1} \cdots x_q$, for $i = 1, 2, \ldots, q$, appears in the left-hand member of relation $h_a(x_1, \ldots, x_q) \leq M x_1 \cdots x_q$. So we have $x_i \leq M$, and $\widetilde{S}_a(M)$ is bounded. One see that every product of $q-1$ variables x_i appears in the left-hand member of the inequality $h_a(x) \leq M x_1 \cdots x_q$, with coefficient a. So we have $x_i \geq a/M$, and these inequalities imply the compactness of $\widetilde{S}_a(M)$.

(2) We have not succeed in proving that the only critical point of H_a is L (see Conjecture 1), which would give an easy proof of the second point of Proposition 3.2 (as for G_a), and so we give here a direct proof of this point, with inequalities.

(i) First, we have the inequality, with equality only if $x_1 + x_2 = x_2 + x_3 = \cdots = x_{q-1} + x_q$:

$$\prod_1^{q-1} (1 + x_j + x_{j+1}) \geq \left\{ \prod_1^{q-1} (x_j + x_{j+1})^{1/(q-1)} + 1 \right\}^{q-1}. \tag{3.3}$$

This results from the inequality $\prod_1^n (u_j + 1)^{1/n} \geq \left(\prod_1^n u_j\right)^{1/n} + 1$, with equality only when the $u_j > 0$ are all equal. This comes from the strict convexity of the function $v \mapsto \ln(e^v + 1)$.

(ii) Now we write $a + x_1 x_q + x_1 + \cdots + x_q = a + x_1 x_q + (x_1 + x_q)/2 + (1/2) \sum_{j=1}^{q-1} (x_j + x_{j+1})$. So we have

$$a + x_1 x_q + x_1 + \cdots + x_q \geq a + x_1 x_q + \frac{x_1 + x_q}{2} + \frac{q-1}{2} \left[\prod_1^{q-1} (x_j + x_{j+1}) \right]^{1/(q-1)} \tag{3.4}$$

with equality only if $x_1 + x_2 = \cdots = x_{q-1} + x_q$ (arithmetic-geometric inequality).

(iii) But for each $j \in \{1,2,\dots q-1\}$, we have $x_j + x_{j+1} \geq 2(x_j\ x_{j+1})^{1/2}$ with equality only if $x_j = x_{j+1}$. So relations (3.3) and (3.4) give easily

$$\prod_{1}^{q-1}(1 + x_j + x_{j+1}) \geq \left[\left(\frac{2^{q-1}}{\sqrt{x_1 x_q}}\prod_{1}^{q} x_j\right)^{1/(q-1)} + 1\right]^{q-1} \tag{3.5}$$

with equality only if all the x_j are equal, and

$$a + x_1 x_q + x_1 + \cdots + x_q \geq a + x_1 x_q + \sqrt{x_1 x_q} + \frac{q-1}{2}\left[\frac{2^{q-1}}{\sqrt{x_1 x_q}}\prod_{1}^{q} x_j\right]^{1/(q-1)} \tag{3.6}$$

with equality only if all the x_j are equal.

(iv) Now we put $\sqrt{x_1 x_q} = u$ and $x_2 x_3 \cdots x_{q-1} = t^{q-1}$. From (3.5) and (3.6) we obtain

$$H_a(x_1,\dots,x_q) \geq \varphi(t,u) := \left(2t\ u^{1/(q-1)} + 1\right)^{q-1}\left(u^2 + u + a + (q-1)t\ u^{1/(q-1)}\right)t^{1-q}u^{-2} \tag{3.7}$$

with equality only if all x_j are equal.

(v) We have to find the minimum of $\varphi(t,u)$ on $\mathbb{R}_*^{+2}$. We put $s = t\ u^{1/(q-1)}$ and have

$$\varphi(t,u) = \tilde{\varphi}(s,u) := \left(2 + \frac{1}{s}\right)^{q-1}\left(u + \frac{a + (q-1)s}{u} + 1\right). \tag{3.8}$$

First, it is easy to see that if, for $C > 0$, $\tilde{\varphi}(s,u) \leq C$, then one has $0 < \alpha \leq s \leq \alpha'$ and $0 < \beta \leq u \leq \beta'$ for some positive constants: $\tilde{\varphi}$ tends to infinity at the infinity point of $\mathbb{R}_*^{+2}$. So it has a minimum, which is a critical point. At such a point, we have

$$u^2 = a + (q-1)s, \qquad s(2s+1) - \left(u^2 + u + a + (q-1)s\right) = 0. \tag{3.9}$$

So we obtain $u(2u+1) = s(2s+1)$, and then $u = s$ and $u^2 - (q-1)u - a = 0$, that is, $u = s = \ell$. At this point, we have $\tilde{\varphi}(s,u) = (2\ell+1)^q/\ell^{q-1} = H_a(\ell,\dots,\ell)$. This value is then the minimum of H_a, and it can be attained only when the x_j are all equal, and then all equal to number ℓ. □

3.2. *Conjectures about the Second Invariant*

As we noticed before, the proof of Proposition 3.2 will be dramatically simpler if the following conjecture would be true.

Conjecture 1. The point L is the only critical point for H_a.

We have a partial result in this direction, for a general type of invariants.

Definition 3.3. A differentiable numerical function R on $\mathbb{R}_*^{+q}$, invariant under the action of F_a, is said "diagonal" if it has the two properties:

(i) every critical point of R with form $(t, t, t, \ldots, t)$ is the point L;

(ii) if q is odd, every critical point of R with form $(t, u, t, u, t, \ldots, u, t)$ is the point L.

Proposition 3.4. *Let R be a differentiable diagonal invariant, and suppose $q \geq 3$.*

(a) If it exists some $j \in \{1, \ldots, q-1\}$ such that every critical point of R satisfies $x_j = x_{j+1}$, then R has a unique critical point which is the point L.

(b) If the equalities $x_1 = x_q$ and $x_2 = x_{q-1}$ are true for every critical point of R, then R has only one critical point which is the point L.

Proof. We will use the following easy lemma.

Lemma 3.5. *If M is a critical point of a differentiable invariant R, all the points $F_a^p(M)$ are also critical points of R for $p \in \mathbb{Z}$.*

This lemma results from the relations $R[F_a(M)] = R(M)$ and from the fact that $dF_a(M)$ and $dF_a^{-1}(M)$ are invertible, which is easy to see from (1.2).

Now, the maps F_a and F_a^{-1} act on last or first coordinates of M as right or left shift ($j-1$ time for F_a and $q-j-1$ times for F_a^{-1}), and the equality $x_j = x_{j+1}$ spreads right and left. Then, such a critical point is $(t, t, \ldots, t)$ for some $t > 0$. Because R is diagonal, this point is L; this gives assertion (a) of the proposition.

In order to prove assertion (b), we suppose that for every critical point $M = (x_1, \ldots, x_q)$ we have $x_1 = x_q$ and $x_2 = x_{q-1}$.

We write $F_a(M) = M' = (x'_1, \ldots, x'_q) = ((a + x_1 + \cdots + x_{q-1})/x_q, x_1, x_2, \ldots, x_{q-2}, x_{q-1})$,

$$F_a^{-1}(M) = M'' = \left(x''_1, \ldots x''_q\right) = \left(x_2, x_3, \ldots, x_{q-1}, x_q, \frac{a + x_2 + \cdots + x_q}{x_1}\right). \tag{3.10}$$

So, by Lemma 3.5 we have the implications, if $q \geq 4$,$\{x'_2 = x'_{q-1}\} \Rightarrow \{x_1 = x_{q-2}\}$, $\{x''_2 = x''_{q-1}\} \Rightarrow \{x_3 = x_q\}$, then, $\{x'_{q-2} = x'_q\} \Rightarrow \{x_{q-3} = x_{q-1}\}$. And then, $\{x''_1 = x''_{q-2}\} \Rightarrow \{x_2 = x_{q-1}\}$; also $\{x''_1 = x''_3\} \Rightarrow \{x_2 = x_4\}$; $\{x'_{q-1} = x'_{q-3}\} \Rightarrow \{x_{q-2} = x_{q-4}\}$.

At this stage, we have $x_1 = x_3 = x_{q-2} = x_q$ and $x_2 = x_4 = x_{q-3} = x_{q-1}$. If $q = 4$, all the x_i have the same value. If $q > 4$, we can continue in this way and obtain by induction the following:

($*$) if q is even, all the x_i have the same value s;

($**$) if q is odd, for all i $x_{2i+1} = t$ and for all j $x_{2j} = u$ for some positive t and u.

If $q = 3$, the hypothesis said already that the critical point has form (t, u, t).

In all cases, the diagonal character of R gives the result. □

Of course, for proving Conjecture 1 from Proposition 3.4 it suffices to see that H_a is diagonal, and to prove the hypothesis $x_j = x_{j+1}$ or $\{x_1 = x_q$ and $x_2 = x_{q-1}\}$ for *one* critical point from the relations $\partial H_a/\partial x_k = 0$ *only*, because every critical point satisfies these equalities. In fact, this will be useful for all usual invariants.

Lemma 3.6. *Invariants G_a, H_a (for $q \geq 3$), and J_a (for odd $q \geq 5$) are diagonal.*

The third invariant J_a is defined in Section 4.

Proof. It is very easy for G_a, and we will not use this result. For H_a notice that for the critical point $(t, t, \ldots, t)$ the relation $\partial H_a/\partial x_1 = 0$ gives the equation $t(t+1)(2t+1) = (a+t^2+qt)(t+1)$, that is, $t = \ell$: this is property (i). If the critical point has the form $(t, u, t, u, \ldots, u, t)$, the same relation gives $(1+u)/t(1+t+u) = (1+t)/(a+t^2+(m+1)t+mu)$, where $q = 2m+1$. But the relation $\partial H_a/\partial x_2 = 0$ gives $2/(1+t+u) - 1/u + 1/(a+t^2+(m+1)t+mu) = 0$. One deduce from these two relations the equality $u(1+u)/t(1+t) = 1+t-u$, so we have $t = u$ and we apply property (i).

In order to prove (i) for J_a, the relation $\partial J_a/\partial x_1 = 0$ is $t/(1+t)(a+(q-1)t) = 1/t(1+t)$, which gives $t = \ell$. Now suppose that a point $(t, u, t, u, \ldots, u, t)$ is critical. When we write that the two first partial derivatives are zero, we obtain $t/(1+t)(a+m(t+u)) = [u(1+u)]^m/[t(1+t)]^{m+1}$ and $(1+u)/u(a+1+(m+1)(t+u)) = [u(1+u)]^m/[t(1+t)]^{m+1}$. So we have $a+m(t+u) = tu$, and by putting this equality in the first previous relation, we get $(1+u)^m u^{m+1} = (1+t)^m t^{m+1}$, that is, $t = u$: we can apply property (i). □

Conjecture 1 would have many consequences.

Definition 3.7. We put, for $M > M(a)$,

$$S_a(M) := \{x \mid H_a(x) = M\}, \qquad \widetilde{S}_a(M) := \{x \mid H_a(x) \leq M\}. \tag{3.11}$$

Obviously, these sets are compact and invariant by the action of F_a. From [9, Proposition 2.1], Conjecture 1 would imply other properties, which are assertions of Proposition 2.3 for the first invariant G_a and are still conjectures for H_a.

Conjecture 2. The sets $S_a(M)$ are the boundaries of the open sets $\{x \mid H_a(x) < M\}$, which are connected.

Conjecture 1 would imply an important property of $S_a(M)$, but we have no proof of it.

Conjecture 3. The sets $S_a(M)$ are, for $M > M(a)$, smooth manifolds of $\mathbb{R}^q$.

In fine , we think that Theorem 2.5 is valid for S_a.

Conjecture 4. The invariant sets $S_a(M)$ are, for $M > M(a)$, homeomorphic to the $(q-1)$-dimensional sphere $\mathbb{S}^{q-1}$.

In fact, Conjecture 4 will be a consequence of Conjecture 1 and of a Conjecture 6 which will be introduced later. First, we prove a result similar to Proposition 2.4.

Proposition 3.8. *Suppose $q = 3$. If $M > M(a)$ is sufficiently near to $M(a)$, then the set $\widetilde{S}_a(M)$ is starlike with respect to the point L, and a ray from L cuts $S_a(K)$ at exactly one point. Moreover, $S_a(M)$ is the boundary of $\widetilde{S}_a(M)$ and of the set $\{H_a < M\}$. Hence $\widetilde{S}_a(M)$ is homeomorphic to an euclidean ball, and $S_a(M)$ is homeomorphic to a sphere, if M is sufficiently near to $M(a)$.*

Proof. We start the proof for $q \geq 3$. Put, for $\rho > 0$, $\rho < \ell$, and $\overrightarrow{u}$ any unitary vector of $\mathbb{R}^q$, $\delta_{\overrightarrow{u}}(\rho) := \ln H_a(L + \rho\overrightarrow{u})$. As in the proof of Proposition 2.4, it is sufficient to prove the relation $\lim_{\rho\to 0}(\delta'_{\overrightarrow{u}}(\rho)/\rho) \geq D > 0$, uniformly with respect to $\overrightarrow{u}$, in order to get the starlikeness of

$\widetilde{S}_a(M)$, and the fact that a ray from L cuts $S_a(M)$ at exactly a point. So it is easy to see that $S_a(M)$ is the boundary of the two sets given in the proposition, and then the existence of the announced homeomorphisms is easy to prove.

Easy calculations give for the value of $\delta'_{\overrightarrow{u}}(\rho)$: $(2\rho u_1 u_q + \ell(u_1+u_q) + \sum_{i=1}^{q} u_i)/(2\ell^2+\ell+\rho(\ell(u_1+u_q)+\sum_{i=1}^{q} u_i)+\rho^2 u_1 u_q) + \sum_{j=1}^{q-1}(u_j+u_{j+1})/(1+2\ell+\rho(u_j+u_{j+1})) - \sum_{i=1}^{q} u_i/(\ell+\rho u_i)$. The asymptotic expansion of $\delta'_{\overrightarrow{u}}(\rho)$ at 0 has the form $\rho A(\overrightarrow{u}) + o(\rho)$, where $o(\rho)$ is uniform with respect to $\overrightarrow{u}$, and where one has, with the relation $\sum_{i=1}^{q} u_i^2 = 1$,

$$\begin{aligned}
Q := \ell^2(2\ell+1)^2 A &= (2\ell+1)^2 + 2\ell(2\ell+1)u_1u_q - \left[\ell(u_1+u_q) + \sum_{i=1}^{q} u_i\right]^2 - \ell^2\sum_{j=1}^{q-1}(u_j+u_{j+1})^2 \\
&= 2\ell^2 + 4\ell - \left[2\sum_{2\le i<k\le q-1} u_iu_k + 2l(u_1+u_q)\sum_{i=2}^{q-1} u_i + 2\left\{\ell\left(u_1^2+u_q^2\right) - \left(\ell^2-\ell-1\right)u_1u_q\right\}\right. \\
&\qquad \left. +2\ell^2\sum_{j=1}^{q-1} u_j + u_{j+1}\right].
\end{aligned} \tag{3.12}$$

Now we suppose that $q=3$. We have to find an upper bound on the unit sphere of the quantity in the sqare brackets. It is a quadratic form B, and the eigenvalues of its matrix are form of $\lambda_3 < 0 < \lambda_2 < \lambda_1 = (\ell^2+3\ell-1)/2 = (1/4)[-\ell^2+5\ell+1+\sqrt{(\ell^2-5\ell-1)^2+32(\ell^2+\ell+1)^2}]$ So the quadratic form B is majorized by λ_3. Hence the proposition results from the following easy lemma. □

So it is clear from this proof that Proposition 3.8 would be true for every $q \ge 4$ if the following conjecture was true.

Lemma 3.9. *If $\ell \ge 2$, one has*

$$2\ell^2 + 4\ell - \frac{-\ell^2+5\ell+1+\sqrt{(\ell^2-5\ell-1)^2+32(\ell^2+\ell+1)^2}}{4} \ge 2 \tag{3.13}$$

Conjecture 5. For $\ell > q-1 \ge 3$ the quadratic form on $\mathbb{R}^q$

$$\begin{aligned}
&\left(2\ell^2+4\ell\right)\sum_{i=1}^{q} x_i^2 - 2\left[\sum_{2\le i<k\le q-1} u_iu_k + \ell(u_1+u_q)\sum_{i=2}^{q-1}u_i\right. \\
&\qquad\qquad \left. +\ell\left(u_1^2+u_q^2\right) - \left(\ell^2-\ell-1\right)u_1u_q + \ell^2\sum_{j=1}^{q-1}u_ju_{j+1}\right]
\end{aligned} \tag{3.14}$$

is positively defined.

Corollary 3.10. *Conjectures 1 and 5 imply Conjecture 4 (and then all the four conjectures).*

Proof. It will be an easy consequence of the classical following result in differential geometry (see [10, Theorem 50, pages 109-110]). □

Fact 1. Let V be a differential manifold, and $f : V \mapsto \mathbb{R}$ a smooth function, such that

(i) f has a unique critical point α, at its absolute minimum $m = f(\alpha)$;

(ii) for every λ, μ such that $m \leq \lambda < \mu$, the subset $V_\lambda^\mu := \{x \mid \lambda \leq f(x) \leq \mu\}$ is compact.

Then there is a diffeomorphism of V which maps V_m^λ onto V_m^μ.

We apply this fact with $V = \mathbb{R}_*^{+q}$, $f = H_a$, $\alpha = L$, and $m = M(a)$. If M is sufficiently near to $M(a)$, $\widetilde{S_a}(M)$ is homeomorphic to the ball, and for $M' > M$ the set $\{M \leq H_a \leq M'\}$ is compact. So for every $M' > M$ the set $\widetilde{S_a}(M')$ is homeomorphic to a ball, and the set $S_a(M)$ is homeomorphic to the unit sphere $\mathbb{S}^{q-1}$: this is Conjecture 4.

Remark 3.11. Of course the previous corollary is true for the invariant G_a, and then gives an other proof of Theorem 2.5 which does not use Reeb's theorem. The analogue of Conjecture 6 is in the proof of Proposition 2.4.

3.3. *The Truth of the Four Conjectures for* $q = 3$

Theorem 3.12. *If $q = 3$, Conjectures 1, 2, 3, and 4 are true.*

Proof. As we saw before, it is sufficient to prove Conjecture 1 (Conjecture 6 is true for $q = 3$). We have

$$H_a(x, y, z) = \frac{(1 + x + y)(1 + y + z)(a + xz + x + y + z)}{xyz}. \tag{3.15}$$

We write the two equalities $(H_a)'_x = 0$, $(H_a)'_z = 0$. Easy calculations give first the relations $x(z+1)(x+y+1) = (y+1)(a+xz+x+z+y)$ and $z(x+1)(z+y+1) = (y+1)(a+xz+x+z+y)$. So we get $x = z$. Now the third equation $(H_a)'_y = 0$ is written as

$$2y\left(a + x^2 + 2x + y\right) = (x + y + 1)\left(a + x^2 + 2x\right), \tag{3.16}$$

and the two first equations reduced to the only one

$$x(x + 1)(x + y + 1) = (y + 1)\left(a + x^2 + 2x + y\right). \tag{3.17}$$

We write (3.16) in the form $(y - x)(y + 2x + a + 1) - (x + 1)(x^2 - 2x - a) = 0$ and (3.17) in the form $(y - x)(x^2 + 4x + 2y + a) + (x^2 - 2x - a) = 0$. These two relations imply the equalities $x = y$ and $x^2 - 2x - a = 0$. So we obtain $x = y = z = \ell$. □

Corollary 3.13. *Let $q = 3$. If $G_a(M_0) = K$ and $H_a(M_0) = M$, then the orbit of M_0 remains on the invariant compact curve (part in $\mathbb{R}_*^{+3}$ of the intersection of two regular surfaces)*

$$C_a^+(K, M) := \Sigma_a(K) \cap S_a(M). \tag{3.18}$$

From Theorems 2.5 and 3.12, the two surfaces of previous corollary are homeomorphic to the 2-dimensional sphere $\mathbb{S}^2$. In [4], other and more complicated proofs of these facts are given.

3.4. The Truth of the First Three Conjectures for $q = 4$

Theorem 3.14. *If $q = 4$, then Conjectures 1, 2, and 3 are true.*

Proof. It is sufficient to prove Conjecture 1. We have now

$$H_a(x, y, z, t) = \frac{(1 + x + y)(1 + y + z)(1 + z + t)(a + xt + x + y + z + t)}{xyzt}. \tag{3.19}$$

We will use Proposition 3.4; so it suffices to see that a critical point (x, y, z, t) satisfies $y = z$ and $x = t$. We put $D := a + xt + x + y + z + t$. The logarithmic derivation gives for a critical point of H_a the four equations (they exchange if we exchange x and t, and y and z):

$$\frac{\partial H_a}{\partial x} = 0 \quad \text{or} \quad \frac{1}{D} = \frac{1 + y}{x(1 + x + y)(1 + t)}, \tag{3.20}$$

$$\frac{\partial H_a}{\partial t} = 0 \quad \text{or} \quad \frac{1}{D} = \frac{1 + z}{t(1 + t + z)(1 + x)}, \tag{3.21}$$

$$\frac{\partial H_a}{\partial y} = 0 \quad \text{or} \quad \frac{1}{D} = \frac{(1 + x)(1 + z) - y^2}{y(1 + x + y)(1 + y + z)}, \tag{3.22}$$

$$\frac{\partial H_a}{\partial z} = 0 \quad \text{or} \quad \frac{1}{D} = \frac{(1 + t)(1 + y) - z^2}{z(1 + t + z)(1 + y + z)}. \tag{3.23}$$

From these equations, we will prove two fondamental relations:

$$xz(z + 1) = ty(y + 1), \tag{3.24}$$

$$t^2(x + 1) = (z + 1)(a + x + y + z). \tag{3.25}$$

In order to prove (3.24) we make the ratio of relations (3.20) and (3.21), subtract (3.23) from (3.22), and make the ratio of the two results. The second one is a consequence of Lemma 3.5: the point $F_a(x, y, z, t) = ((a + x + y + z)/t, x, y, z)$ is a critical point, so its coordinates satisfy relation (3.24): $((a + x + y + z)/t)y(y + 1) = zx(x + 1)$. We use once more (3.24) for (x, y, z, t) and obtain (3.25). Now we put $A := a+y+z$. Relation (3.25) becomes $t^2(x+1) = (z+1)(A+x)$. By exchange of y and z and of x and t we obtain also $x^2(t + 1) = (y + 1)(A + t)$. We substract these two relations and obtain

$$(t - x)(x + 1)(t + 1) = A(z - y) + xz - ty. \tag{3.26}$$

But we have $t-x = t(1-x/t) = t(1-y(y+1)/z(z+1)) = t(z-y)(y+z+1)/z(z+1)$, from relation (3.24), and $xz = ty(y+1)/(z+1)$, and then $xz-ty = (ty/(z+1))(y-z)$. We put these relations in

(3.26) and obtain the factorization $((z-y)/z(z+1))[t(x+1)(t+1)(y+z+1)+tyz-Az(z+1)] = 0$. But from the value of A and relation (3.25) we have $A(z+1) = t^2(x+1) - x(z+1)$. We put these value of $A(z+1)$ in the previous formula and obtain

$$\frac{(z-y)}{z(z+1)}\left[t(x+1)(t+1)(y+z+1) + tyz + xz(z+1) - zt^2(x+1)\right] = 0. \tag{3.27}$$

The development of the first term in square brackets in formula (3.27) contains the last term $zt^2(x+1)$, and so the factor of $(z-y)$ is strictly positive. So we obtain $y = z$, but the relation which follows formula (3.26) gives $x = t$, and this proves the proposition. □

3.5. The Truth of the First Three Conjectures for $q = 5$

In this section we prove the following result.

Theorem 3.15. *If $q = 5$, then Conjectures 1, 2, and 3 are true.*

Proof. As usual, it suffices to prove Conjecture 1. The proof follows the same line as for $q = 4$. First we put $D := a+x_1x_5+x_1+x_2+x_3+x_4+x_5$, and we write the five equations $\partial H_a/\partial x_j = 0$:

$$\frac{1}{1+x_1+x_2} - \frac{1}{x_1} + \frac{1+x_5}{D} = 0, \tag{3.28}$$

$$\frac{1}{1+x_1+x_2} + \frac{1}{1+x_2+x_3} - \frac{1}{x_2} + \frac{1}{D} = 0, \tag{3.29}$$

$$\frac{1}{1+x_2+x_3} + \frac{1}{1+x_3+x_4} - \frac{1}{x_3} + \frac{1}{D} = 0, \tag{3.30}$$

$$\frac{1}{1+x_3+x_4} + \frac{1}{1+x_4+x_5} - \frac{1}{x_4} + \frac{1}{D} = 0, \tag{3.31}$$

$$\frac{1}{1+x_4+x_5} - \frac{1}{x_5} + \frac{1+x_1}{D} = 0. \tag{3.32}$$

The difference of (3.28) and (3.29) gives $1/x_1 - x_5/D - (1+x_3)/x_2(1+x_2+x_3) = 0$, that is, with $S := \sum_{j=1}^{5} x_j$, $(a+S)/D = x_1(1+x_3)/x_2(1+x_2+x_3)$. But the left-hand member is invariant by exchange of x_1 with x_5 and of x_2 with x_4, and so we have $x_1(1+x_3)/x_2(1+x_2+x_3) = x_5(1+x_3)/x_4(1+x_4+x_3)$. So we obtain the fundamental relation, analogous to (3.24) in case $q = 4$,

$$x_1x_4(1+x_3+x_4) = x_5x_2(1+x_3+x_2). \tag{3.33}$$

We have three useful consequences of (3.33). By writing from (3.33) the ratio x_2x_5/x_1x_4, we obtain

$$x_1x_4 - x_2x_5 = (x_2 - x_4)\frac{x_1x_4}{1+x_2+x_3}. \tag{3.34}$$

By writing $x_1 - x_5 = x_1(1 - x_5/x_1)$ and using (3.33) for x_5/x_1, we have also

$$x_1 - x_5 = (x_2 - x_4)\frac{x_1(1 + x_2 + x_3 + x_4)}{x_2(1 + x_2 + x_3)}. \tag{3.35}$$

Finally, we write $x_2x_1 - x_4x_5 = x_2x_1(1 - (x_4/x_2)(x_5/x_1))$ and put in this relation the ratio x_5/x_1 from relation (3.33); we obtain after easy calculations

$$x_2x_1 - x_4x_5 = (x_2 - x_4)\frac{x_1}{x_2}\,\frac{(1 + x_3)(x_2 + x_4) + x_2^2 + x_2x_4 + x_4^2}{1 + x_2 + x_3}. \tag{3.36}$$

The essential fact in formulas (3.34), (3.35), and (3.36) is that the three factors of $(x_2 - x_4)$ are positive. Now (3.28) gives $D = x_1(1 + x_5)(1 + x_1 + x_2)/(1 + x_2)$, and from invariance of the left-hand member by exchange of x_1 with x_5 and of x_2 with x_4 we get

$$x_1(1 + x_4)(1 + x_5)(1 + x_1 + x_2) - x_5(1 + x_2)(1 + x_1)(1 + x_5 + x_4) = 0. \tag{3.37}$$

The goal is to make appear the three first member of (3.34), (3.35) and (3.36) in (3.37). We obtain $(1+x_1)(1+x_5)([x_1-x_5]+[x_1x_4-x_2x_5])+x_1x_2(1+x_4)(1+x_5)-x_5x_4(1+x_1)(1+x_2) = 0$, or $(1+x_1)(1+x_5)([x_1-x_5]+[x_1x_4-x_2x_5])+[x_2x_1-x_4x_5]+x_2x_4[x_1-x_5]+x_1x_5(x_2-x_4) = 0$. From formulas (3.34), (3.35), and (3.36) we obtain *in fine* a relation $(x_2 - x_4)A = 0$, where $A > 0$. So if x is a critical point we have $x_2 = x_4$, and formula (3.35) gives $x_1 = x_5$.

Then it results from Proposition 3.4(b) that the only critical point is L. □

4. The Third Invariant of Order q Lyness' Equation When $q = 2m + 1 \geq 5$ is Odd

In [5], the authors give a simple third invariant of order q Lyness' equation for odd $q \geq 5$.

Proposition 4.1 ([11]). *When $q = 2m + 1 \geq 5$ is odd, the function*

$$J_a(x_1, x_2, \ldots, x_q) := \frac{\left(a + \sum_{i=1}^{q} x_i\right)\prod_{j=1}^{m} x_{2j}(1 + x_{2j}) + \prod_{j=0}^{m} x_{2j+1}(1 + x_{2j+1})}{\sum_{i=1}^{q} x_i} \tag{4.1}$$

is invariant under the action of F_a.

See Remark 7.2 of Section 7.1 for an easy proof of Proposition 4.1. It is to be noticed that J_a is invariant under a permutation of variables of odd ranks, and the same fact holds for even ranks. Then the analogue of Propositions 2.3 and 3.2 is the following.

Proposition 4.2. *(1) The quantity $J_a(x)$ tends to infinity when x tends to the point at infinity of $\mathbb{R}_*^{+q}$ $(q = 2m + 1 \geq 5)$.*

(2) J_a has a strict minimum at the point $L = (\ell, \ldots, \ell)$, whose value is

$$N(a) = 2(\ell + 1)^{m+1}\,\ell^{-m}. \tag{4.2}$$

(3) J_a has only one critical point, its minimum $L = (\ell, \ldots, \ell)$.

(4) For $N > N(a)$ the sets

$$\Delta_a(N) := \{x \mid J_a(x) = N\} \tag{4.3}$$

are compact $(q-1)$-dimensional smooth manifolds in $\mathbb{R}_*^{+q}$ which are invariant under the action of F_a. They are the boundaries of the open sets $\{x \mid J_a(x) < N\}$, which are connected and form a fundamental system of neighborhoods of point L.

Proof. (i) It is to see that the closed set $\widetilde{\Delta}_a(N) := \{x \mid J_a(x) \le N\}$ is compact. We denote by Π_{od} the product of the x_{2j+1}, by Π_{ev} the analogue for even ranks, and by Π the product of all the x_i .

From inequality $J_a(x) \le N$, we deduce $N \ge (x_{j_0}\prod_{\text{ev}}^2 + \prod_{\text{od}}^2)/\prod \ge 2(\sqrt{x_{j_0}}\sqrt{\prod_{\text{ev}}^2 \times \prod_{\text{od}}^2}/\prod) = 2\sqrt{x_{j_0}}$, and so all the x_{j_0} are bounded by some $\alpha > 0$ on $\widetilde{\Delta}_a(N)$. But we have also $N \ge (a\prod_{\text{ev}} + \prod_{\text{od}})/\prod \ge 2\sqrt{a}(\sqrt{\Pi}/\Pi) = 2\sqrt{a}/\sqrt{\Pi}$, and so $\Pi \ge \beta$ for some $\beta > 0$. We have then $\alpha^{q-1}x_{j_0} \ge \Pi \ge \beta$, and so $x_{j_0} \ge \gamma > 0$. These inequalities prove the compacity of $\widetilde{\Delta}_a(N)$.

(ii) Now it suffices to prove point (3) of the proposition: of course, we deduce of it that J_a has a strict minimum at L, whose value $N(a)$ is easy to calculate. Then point (3) and [17, Proposition 2.1] together imply point (4). *In fine*, the fact that $\widetilde{\Delta}_a(N)$ is a manifold comes from point (3).

(iii) *Proof of point* (3). We will prove that at a critical point $(x_1, x_2, \ldots x_{2m+1})$ we have $x_1 = x_{2m+1}$ and $x_2 = x_{2m}$. Then, Proposition 3.4 (b) and Lemma 3.6 will give the result.

We have, with evident denotations, $J_a(x) = (a + \Sigma)\prod_{\text{ev}}(1 + x_j)/\prod_{\text{od}}x_j + \prod_{\text{od}}(1 + x_j)/\prod_{\text{ev}}x_j$. We write the relation $\partial J_a/\partial x_1 = 0$, which by an easy calculation becomes

$$\frac{x_1}{(1 + x_1)(a + x_2 + x_3 + \cdots + x_{2m+1})} = \frac{\prod_{\text{ev}}x_j(1 + x_j)}{\prod_{\text{od}}x_j(1 + x_j)}. \tag{4.4}$$

But the right member of (4.4) is invariant under the exchange of x_1 and x_{2m+1}, and so the relation $\partial J_a/\partial x_{2m+1} = 0$ is

$$\frac{x_{2m+1}}{(1 + x_{2m+1})(a + x_1 + x_2 + \cdots + x_{2m})} = \frac{\prod_{\text{ev}}x_j(1 + x_j)}{\prod_{\text{od}}x_j(1 + x_j)}. \tag{4.5}$$

If we put $A = a + x_2 + x_3 + \cdots + x_{2m}$, (4.4) and (4.5) give $x_1(A + x_1)/(1 + x_1) = x_{2m+1}(A + x_{2m+1})/(1 + x_{2m+1})$. The strict monotonicity of the function $s \mapsto s(A + s)/(1 + s)$ on $\mathbb{R}_*^+$, if $A > 0$, gives the equality $x_1 = x_{2m+1}$. By similar calculations, the relation $\partial J_a/\partial x_2 = 0$ gives by using symmetry

$$\frac{\prod_{\text{ev}}x_j(1 + x_j)}{\prod_{\text{od}}x_j(1 + x_j)} = \frac{1 + x_2}{x_2}\,\frac{1}{1 + a + x_2 + \sum_{i=1}^{q} x_i} = \frac{1 + x_{2m}}{x_{2m}}\,\frac{1}{1 + a + x_{2m} + \sum_{i=1}^{q} x_i}. \tag{4.6}$$

If we put $B = 1 + a + \sum_{i \ne 2, i \ne 2m} x_i > 0$, we obtain $(1 + 1/x_2)/(1 + 1/x_{2m}) = (2x_2 + x_m + B)/(x_2 + 2x_{2m} + B) = 1 + (x_2 - x_{2m})/(x_2 + 2x_{2m} + B)$, and so $(1/x_2 - 1/x_{2m})/(1 + 1/x_{2m}) = (x_2 - x_{2m})/(x_2 + 2x_{2m} + B)$. Clearly this relation implies the equality $x_2 = x_{2m}$. □

As for invariant H_a we make the following conjecture.

Conjecture 6. For $N > N(a)$, the invariant compact $(q-1)$-dimensional manifold $\Delta_a(N) = \{x \mid J_a(x) = N\}$ is homeomorphic to the sphere $\mathbb{S}^{q-1}$.

We can reduce this conjecture to the following, which can be tested on a computer.

Conjecture 7. For integer $m \geq 2$, real numbers $a > 0$ and $N > 0$ the two curves in the positive plane $\mathbb{R}_*^{+2}$

$$u(1+t) + t(1+u) = Nu^{m+1}(1+t)^{-m}, \tag{4.7}$$

$$a + (m+1)t + mu = t^{m+2}(1+t)^m u^{-(m+1)}(1+u)^{1-m} \tag{4.8}$$

have at most two common points (t_1, u_1) and (t_2, u_2).

Tests with a graphic computer give a great evidence to Conjecture 7, but its proof seems not easy.

Proof of Conjecture 7⇒ Conjecture 6. As for invariant G_a we will use a smooth function which will have, by Conjecture 7, exactly two critical points on manifold $\Delta_a(N)$; so the theorem of Reeb will give us the result similarly to the proof of Theorem 2.5.

Let $f(x) = \sum_{i=1}^{q} x_i$. We search critical point x of f on $\Delta_a(N)$. The relation $df = \lambda dJ_a$ means that all the partial derivatives of J_a are equal. We write with evident denotations $J_a(x) = (a+\Sigma)(\prod_{ev}(1+x_j)/\prod_{od} x_j) + \prod_{od}(1+x_j)/\prod_{ev} x_j$ and put $A := a + \Sigma$, $B := \prod_{ev}(1+x_j)/\prod_{od} x_j$ and $C := \prod_{od}(1+x_j)/\prod_{ev} x_j$. The equality of $\partial J_a/\partial x_{2i+1}$ with $\partial J_a/\partial x_{2j}$ is $B - AB/x_{2i+1} + C/(1+x_{2i+1}) = B + AB/(1+x_{2j}) - C/x_{2j}$, that is

$$\frac{AB}{C} = \frac{x_{2i+1}(1+x_{2j})}{x_{2j}(1+x_{2i+1})}. \tag{4.9}$$

By writing (4.9) for $2i+1$ and $2j'$, we obtain $x_{2j} = x_{2j'}$, and in the same manner we have $x_{2i+1} = x_{2i'+1}$. So all the odd coordinates of a critical point have the same value t, and all the even coordinates of a critical point have the same value u.

Then relation (4.9) becomes $(a + (m+1)t + mu)((1+u)^m/t^{m+1})(u^m/(1+t)^{m+1}) = t(1+u)/u(1+t)$, that is, relation (4.8) of Conjecture 7. The relation $(t, u, t, \ldots, u, t) \in \Delta_a(N)$ gives

$$N = (a + (m+1)t + mu)\frac{(1+u)^m}{t^{m+1}} + \frac{(1+t)^{m+1}}{u^m}. \tag{4.10}$$

We put the value of $a + (m+1)t + mu$ from (4.8) in this relation and obtain relation (4.7) of Conjecture 7. This conjecture implies that there is at most two critical points $(t_1, u_1, \ldots, t_1)$ and $(t_2, u_2, \ldots, t_2)$ of function f on $\Delta_a(N)$. But there is at least two such critical points: the maximum and minimum of f on the manifold; these two numbers are different if $N > N(a)$, because a compact $(q-1)$-manifold in $\mathbb{R}^q$ cannot be included in an hyperplane. So we can conclude by Reeb's theorem. □

5. Nature of the Differential $dF_a(L)$ at the Equilibrium

Now we look at the eigenvectors of $dF_a(L)$. The matrix of the differential $dF_a(L)$ is

$$A = \begin{pmatrix} \frac{1}{\ell} & \frac{1}{\ell} & \frac{1}{\ell} & \cdots & \frac{1}{\ell} & -1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{5.1}$$

So we search a vector $x = (x_1, \ldots, x_q) \neq 0$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$. We get easily the relations

$$\left[\lambda^q + 1 - \frac{1}{\ell}\left(\lambda + \lambda^2 + \cdots + \lambda^{q-1}\right)\right] x_q = 0, \quad x_i = \lambda^{q-i} x_q \quad \text{for } i = 1, 2, \ldots, q-1. \tag{5.2}$$

So $x \neq 0 \Leftrightarrow x_q \neq 0$, and we have the equation of the eigenvalues

$$\lambda^q + 1 - \frac{1}{\ell}\left(\lambda + \lambda^2 + \cdots + \lambda^{q-1}\right) = 0, \tag{5.3}$$

and, for each such eigenvalue λ, a 1-dimensional eigenspace generated by the eigenvector $V_\lambda = (\lambda^{q-1}, \lambda^{q-2}, \ldots, \lambda, 1)$.

So we have to determine the nature of solutions of (5.3). The results are not the same if q is odd or if it is even. Recall that from Lemma 2.2 we have $1/\ell < 1/(q-1)$.

5.1. The Eigenvalues When q Is Even

First we study the case where $q = 2m$ is even. We use the following result.

Lemma 5.1. *If $0 < z \leq 1/(2m-1)$, the polynomial*

$$P_{2m}(\lambda) := \lambda^{2m} + 1 - z\left(\lambda + \lambda^2 + \cdots + \lambda^{2m-1}\right) \tag{5.4}$$

has $2m$ distinct roots, with modulus 1, different from 1 and -1, hence of the form $e^{\pm i\,\theta_j}$, $j = 1, 2, \ldots, m$, with $\theta_j \neq \theta_k$ if $j \neq k$, and $0 < \theta_j < \pi$ for all j.

Proof. First it is obvious that 0, 1, -1 are not roots of P_{2m}. Then the proof has three steps.

First step. P_{2m} has no real roots.

Because P_{2m} is reciprocal, it suffices to show that there is no real root λ with $0 < |\lambda| < 1$. If such a root exists, then one has $1 < \lambda^{2m} + 1 = \lambda z(1 + \lambda + \lambda^2 + \cdots + \lambda^{2m-2}) < z(2m-1)$, and so $z > 1/(2m-1)$, which is false.

Second step. The roots of P_{2m} have modulus 1.

The number $z = (\lambda^{2m} + 1)(\lambda - 1)/\lambda(\lambda^{2m-1} - 1)$ is real, and $z + 1$ also. Hence we have $(\lambda^{2m+1} - 1)((\overline{\lambda})^{2m} - \overline{\lambda}) = ((\overline{\lambda})^{2m+1} - 1)(\lambda^{2m} - \lambda)$, or

$$\left(\lambda - \overline{\lambda}\right)\left(\lambda\overline{\lambda} - 1\right)\left\{\left(\lambda\overline{\lambda}\right)^{2m-1} + \left(\lambda\overline{\lambda}\right)^{2m-2} + \cdots + 1 - \left(\lambda^{2m-1} + \lambda^{2m-2}\overline{\lambda} + \cdots + \left(\overline{\lambda}\right)^{2m-1}\right)\right\} = 0. \tag{5.5}$$

But $\lambda \neq \overline{\lambda}$; if $\lambda\overline{\lambda} = 1$, we have $|\lambda| = 1$. If not, put $r := \lambda\overline{\lambda} > 0$. Because P_{2m} is reciprocal, we can suppose that $|\lambda| < 1$. From the equality

$$\left(\lambda\overline{\lambda}\right)^{2m-1} + \left(\lambda\overline{\lambda}\right)^{2m-2} + \cdots + 1 - \left(\lambda^{2m-1} + \lambda^{2m-2}\overline{\lambda} + \cdots + \left(\overline{\lambda}\right)^{2m-1}\right) = 0, \tag{5.6}$$

we get $r^{2m-1} + r^{2m-2} + \cdots + 1 \leq 2mr^{m-1/2}$, or $r^{m-1/2} + 1/r^{m-1/2} + \cdots + r^{1/2} + 1/r^{1/2} \leq 2m$. But if $r \neq 1$, the first member is greater than $2m$, which gives a contradiction.

Third step. Roots of P_{2m} are all distinct.

The proof of this step will be given in common with the one of the third step in the proof of Lemma 5.3. □

5.2. The Eigenvalues When q Is Odd

Now we study the case where $q = 2m + 1$ is odd. First we have the following result.

Lemma 5.2. *If $q = 2m + 1$ is odd, then -1 is an eigenvalue of $dF_a(L)$, for the eigen vector $V_{-1} = (1, -1, 1, \ldots, -1, 1)$, which is a vector tangent at the point L to the hyperbola $\mathscr{H}_a$ defined in Theorem 6.1 in Section 6.1.*

Proof. A first proof is to see that the vector V_{-1} satisfies the relation $(A + I)V_{-1} = 0$, which is obvious. One can also see that -1 is a root of (5.3), which is also obvious. A deeper proof is in relation with the 2-periodic points of the hyperbola $\mathscr{H}_a$ (see Theorem 6.1 in Section 6.1). We have, by Theorem 6.1, the relations $(F_a(h(t)) - F_a(h(\ell)))/(t - \ell) = (h(\phi_a^m(t)) - h(\phi_a^m(\ell)))/(t - \ell)$ (see the definition of h and ϕ_a^m in Theorem 6.1 and formula (6.4)). But the first term tends to the vector $dF_a(L) \circ h'(\ell)$ when $t \to \ell$, and the second tends to the vector $(\phi_a^m)'(\ell) \circ h'(\ell) = -h'(\ell)$. So $h'(\ell)$ is an eigenvector of $dF_a(L)$ for the eigenvalue -1. And it is obvious to see that $h'(\ell) = (1, -1, \ldots, -1, 1) = V_{-1}$. □

So (5.3) can be written $(\lambda+1)(\lambda^{2m} - \lambda^{2m-1} + \cdots - \lambda + 1) = (\lambda/\ell)(\lambda+1)(1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2m-2})$. Then we cancel the factor $\lambda + 1$ and add to the two members the quantity $\lambda + \lambda^3 + \cdots + \lambda^{2m-1}$. *In fine*, we see that the eigenvalues of $dF_a(L)$ distinct of -1 are the roots of the polynomial $P_{2m+1}(\lambda) := \lambda^{2m} + \lambda^{2m-2} + \cdots + \lambda^2 + 1 - (1 + z)(\lambda + \lambda^2 + \lambda^3 + \cdots + \lambda^{2m-1})$, for $z = 1/\ell$. We have now the following result, analogous to Lemma 5.1.

Lemma 5.3. *If $0 < z \leq 1/2m$, the polynomial*

$$P_{2m+1}(\lambda) = \lambda^{2m} + \lambda^{2m-2} + \cdots + \lambda^2 + 1 - (1 + z)\left(\lambda + \lambda^3 + \cdots + \lambda^{2m-1}\right) \tag{5.7}$$

has $2m$ distinct roots, with modulus 1, different from 1 and -1, hence of the form $e^{\pm i\,\theta_j}$, $j = 1, 2, \ldots, m$, with $\theta_j \neq \theta_k$ if $j \neq k$, and $0 < \theta_j < \pi$ for all j.

Proof. First, it is obvious that $0, 1$, and -1 are not roots of P_{2m+1}. Then, the proof has three steps, as for Lemma 5.1.

First step. P_{2m+1} has no real roots.

A real root λ is necessarily positive; because P_{2m+1} is reciprocal, we can suppose $\lambda < 1$. So we have $1 + 1/2m \geq 1 + z = (1 + \lambda^2 + \cdots + \lambda^{2m})/\lambda(1 + \lambda^2 + \cdots + \lambda^{2m-2}) = 1/\lambda + \lambda^{2m-1}/(1 + \lambda^2 + \cdots + \lambda^{2m-2}) > 1/\lambda + \lambda^{2m-1}/m$. But the function of λ in the right-hand member has for minimum on $]0, +\infty[$ the positive quantity $(2m/(2m-1))((2m-1)/m)^{1/2m} > 1 + 1/2m$, and this is a contradiction.

Second step. Roots of P_{2m+1} have modulus 1.

Remark that $1 + z = (\lambda^{2m+2} - 1)/\lambda(\lambda^{2m} - 1)$ is real, hence equals to its conjugated. As in proof of Lemma 5.1, this gives the relation

$$\left(\lambda - \overline{\lambda}\right)\left(\lambda\overline{\lambda} - 1\right)\left\{\left(\lambda\overline{\lambda}\right)^{2m} + \left(\lambda\overline{\lambda}\right)^{2m-1} + \cdots + \lambda\overline{\lambda} + 1 - \left(\lambda^{2m} + \lambda^{2m-1}\overline{\lambda} + \cdots + \left(\overline{\lambda}\right)^{2m}\right)\right\} = 0. \tag{5.8}$$

The same method as in Lemma 5.1, with $r := \lambda\overline{\lambda}$, supposed to satisfy $r < 1$, gives the inequality $r^m + 1/r^m + r^{m-1} + 1/r^{m-1} + \cdots + r + 1/r + 1 \leq 2m + 1$. But if $r \neq 1$, the first member is greater than $2m + 1$, and this is a contradiction.

Third step. roots of P_{2m+1} are all distinct.

The proof of this step is the same of that one of third step of the proof of Lemma 5.1 and is given below. □

Common proof of third steps of Lemmas 5.1 and 5.3. Roots of equation $\lambda^q - z(\lambda^{q-1} + \cdots + \lambda) + 1 = 0$ are distinct if and only if it is the case for equation $\lambda^{q+1} - u\,\lambda^q + u\,\lambda - 1 = 0$, where $u = z + 1$, obtained by multiplication by $(\lambda - 1)$. But a multiple root is a common root to this equation and to the derivative one $(q+1)\lambda^q - q\,u\,\lambda^{q-1} + u = 0$. We eliminate u between these two equations, and obtain $q = (1 - \lambda^{2q})/\lambda^{q-1}(1 - \lambda^2)$. So we have $\lambda^{q-1} = (1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2q-2})/q$. By steps 1 and 2 of the proofs of Lemmas 5.1 and 5.3 we know that λ^{q-1} is an extremal point of the unit disk, and so the previous barycentric relation implies that $\lambda^2 = 1$, and $\lambda = \pm 1$, which is impossible.

5.3. The Global Nature of $dF_a(L)$

From Lemmas 5.1, 5.2, and 5.3, we get the description of $dF_a(L)$.

Theorem 5.4. *(1) The q eigenvalues of $dF_a(L)$ are all distinct and have modulus 1.*

(2) When $q = 2m$ is even, $dF_a(L)$ is linearly conjugated to a product of m rotations in mutually orthogonal planes, with angles $\theta_1, \theta_2, \ldots, \theta_m \in]0, \pi[$ all distinct.

(3) When $q = 2m + 1$ is odd, $dF_a(L)$ is linearly conjugated to a product of m rotations in mutually orthogonal planes in the hyperplane $x_q = 0$, of angles $\theta_1, \theta_2, \ldots, \theta_m \in]0, \pi[$ all distinct, and of an orthogonal symmetry with respect to the hyperplane $x_q = 0$.

Proof. Eigenvalues of $dF_a(L)$ are distinct in the two cases, and so the matrix is diagonalizable in a basis of eigenvectors in $\mathbb{C}^q$. But it is classical that, with conjugated eigenvalues of modulus 1 and a real matrix for $dF_a(L)$, one can obtain the forms given in the theorem. □

6. Results about Possible Periods of Solutions of (1.1)

6.1. Role of 2-Periodic Points When $q = 2m + 1$ Is Odd

In [12], the authors found the existence of 2-periodic points when $q = 2m + 1$ is odd (see also [4, 13]). There is a theoretical reason to the existence of these points, from Theorem 2.5 and the following result.

Fact 2. Every continuous map $f : \mathbb{S}^{2m} \to \mathbb{S}^{2m}$ has a 2-periodic point.

Proof. It is sufficient to prove the result for smooth maps. Put $g = f \circ f$ and $id(x) = x$ on $\mathbb{S}^{2m}$. If g has no fixed point, then for all $x \in \mathbb{S}^{2m}$, $g(x) \neq -(-id(x))$; so g and $-id$ are homotopic, and then have the same degree. But $\deg(g) = [\deg(f)]^2$ and $\deg(-id) = (-1)^{2m+1}$, and this is impossible (for elementary properties of the degree, see [10, pages 237–239]). □

These 2-periodic points have particular properties. The relations (6.1) of the following theorem are already in [12]. See also [9] for applications to *lacunary* Lyness' difference equations.

Theorem 6.1. *Let $q = 2m + 1$, $m \geq 1$, and $a > 0$. The locus of 2-periodic points for map F_a is the branch of hyperbola $\mathscr{H}_a \subset \mathbb{R}_*^{+q}$ whose equations are*

$$x_1 = x_3 = \cdots = x_q, \quad x_2 = x_4 = \cdots = x_{q-1} = \frac{mx_1 + a}{x_1 - m}, \quad x_1 > m. \tag{6.1}$$

This branch of hyperbola $\mathscr{H}_a$ passes through the equilibrium L and is invariant by the map F_a. The map

$$h : t \longmapsto \left(t, \frac{mt + a}{t - m}, t, \frac{mt + a}{t - m}, \ldots, \frac{mt + a}{t - m}, t\right) :]m, +\infty[\longrightarrow]m, +\infty[^q \subset \mathbb{R}_*^{+q} \tag{6.2}$$

is a parametrization of $\mathscr{H}_a$, which has the property that

$$F_a(h(t)) = h\left(\frac{mt + a}{t - m}\right), \qquad F_a\left(h\left(\frac{mt + a}{t - m}\right)\right) = h(t). \tag{6.3}$$

Moreover, if $K > K(a)$, $\mathscr{H}_a$ cuts the manifold $\Sigma_a(K)$ at exactly two points $P_a^1(K)$ and $P_a^2(K)$ which are exchanged by F_a, their parameters are t_1 and $t_2 = (mt_1 + a)/(t_1 - m)$ with $m < t_1 < \ell < t_2$.

Proof. The q equations satisfied by a point $M_0 = (u_{q-1}, u_{q-2}, \ldots, u_0)$ for being 2-periodic are $u_0 = u_2 = \cdots = u_{q-1}$, $u_1 = u_3 = \cdots = u_q$ and $a + u_q + \cdots + u_2 = u_1 u_{q-1}$. If we put $u_0 = x_q = t$ and $u_1 = x_{q-1} = s$, we obtain $t = (a + m(t + s))/s$, or $s = (mt + a)/(t - m)$, that is, (6.1) and the parametrization (6.2). Conversely the point $h(t)$ is obviously 2-periodic for $t > m$.

Now we put, for $t > m$,

$$\phi_a^m(t) = \frac{mt + a}{t - m}, \tag{6.4}$$

which is an involution with fixed point ℓ. We can write $h(t) = (t, \phi_a^m(t), t, \phi_a^m(t), \ldots, t)$, and hence we have $h(\phi_a^m(t)) = (\phi_a^m(t), t, \phi_a^m(t), \ldots, \phi_a^m(t)) = F_a(h(t))$, for $h(t)$ is 2-periodic. We have also $F_a(h(\phi_a^m(t))) = F_a \circ F_a(h(t)) = h(t)$.

At last it is to find the points where $\mathscr{H}_a$ cuts $\Sigma_a(K)$. First, we change the parameter for $\mathscr{H}_a$; we have $\ell = m + \sqrt{m^2 + a}$ and we put $b := \sqrt{m^2 + a} = \ell - m$; so we take the new parameter $v := (t - m)/b$. Hence, $v = 1 \Leftrightarrow t = \ell$, and if $\psi := (\phi_a^m - m)/b$, then $\psi(v) = 1/v$. So we put the values $t = bv + m$ for $x_1, x_3, \ldots, x_q$ and $\phi_a^m(t) = m + b/v$ for $x_2, x_4, \ldots, x_{q-1}$ in the equation $G_a(x) = K$ of $\Sigma_a(K)$. Some easy calculations (using the relation $a + m^2 = b^2$ for simplifying) give the equation $(1/v)([b(m+1)v^2 + ((m+1)^2 + b^2)v + b(m+1)]^{m+1}/[bmv^2 + (m^2 + b^2)v + bm]^m) = K$. But the two quadratic polynomials in v are reciprocal, and so we put $X = v + 1/v \geq 2$ (and $X = 2 \Leftrightarrow v = 1 \Leftrightarrow t = \ell$). Then we define $f : [2, +\infty[\to \mathbb{R}$ by $f(X) := (m+1)\ln[b(m+1)X + (m+1)^2 + b^2] - m\ln[bmX + m^2 + b^2]$. We have $f'(X) = b(m+1)^2/(b(m+1)X + (m+1)^2 + b^2) - bm^2/(bmX + m^2 + b^2) = \gamma(m+1) - \gamma(m)$, where the function $m \mapsto \gamma(m) = bm^2/(bmX + m^2 + b^2)$ is obviously increasing. So we have $f'(X) > 0$, and f is increasing, with $f(+\infty) = +\infty$. But $f(2) = \ln\,(b+m+1)^{2m+2}/(b+m)^{2m} = \ln\,((\ell+1)^{q+1}/\ell^{q-1}) = \ln\,K(a)$, and so there is a unique solution $X > 2$ to the equation $f(X) = \ln K$ if $K > K(a)$. This solution gives two solutions v to the equation $v + 1/v = X$, $0 < v_1 < 1 < v_2$. And this gives two solutions $m < t_1 < \ell < t_2$, and from the invariance of $\mathscr{H}_a$ and $\Sigma_a(K)$ by F_a we have $t_2 = \phi_a^m(t_1)$. ☐

6.2. The General Case $q \geq 2$

We start with a formula easy to prove.

Lemma 6.2. *If $(u_n)_n$ is a solution of* (1.1) *and $K = G_a(u_{q-1}, \ldots, u_1, u_0)$, one has the relation*

$$\forall n \geq 0 \quad \sum_{i=0}^{q} u_{n+i} + a + 1 = K \prod_{i=1}^{q-1} \frac{u_{n+i}}{1 + u_{n+i}}. \tag{6.5}$$

We deduce from (6.5) *the possibility for $q+1$ to be a period of some solutions of* (1.1).

Proposition 6.3. *The only $(q+1)$-periodic solutions of* (1.1) *are the following:*

(i) *if q is even, the equilibrium L;*

(ii) *if q is odd, the 2-periodic solutions of Theorem 5.4.*

Proof. If $(u_n)_n$ is $(q+1)$-periodic, the left-hand member of (6.5) is constant when n varies, and so $v_n := \prod_{i=1}^{q-1}(u_{n+i}/(1 + u_{n+i}))$ is also constant; so we have $v_{n+1}/v_n = 1 = (u_{n+q}/(1 + u_{n+q}))((1 + u_{n+1})/u_{n+1})$, and then for all n $u_{n+1} = u_{n+q}$, that is, $(u_n)_n$ is $(q-1)$-periodic. If p is the minimal period of the sequence u_n, p divides $q+1$ and $q-1$, and then $p = 1$ or $p = 2$. If p is odd, then $p = 1$ and the solution is the equilibrium. If p is even, then $p = 2$, and $q - 1$ is a multiple of

2, and then q is odd; in this case, u_n is a 2-periodic solution of Theorem 5.4 (which may be constant). □

As a consequence, we see that order 2 Lyness' equation has no solution with minimal period 3, which was already known. For order 3 equation, consequences are in the following.

6.3. The Case $q = 3$

Proposition 6.4. *(1) The only 4-periodic solutions of order 3 Lyness' equation are 2-periodic, and given by Theorem 5.4.*

(2) Order 3 Lyness' equation has no solution with minimal period 3.

(3) Order 3 Lyness' equation has no solution with minimal period 5.

Remark that in [13] authors find an infinity of starting points which are 5-periodic for order 3 Lyness' equation, but in [13] this equation is studied in $\mathbb{R}$ and not, as we make here, in $\mathbb{R}_*^+$: their 5-periodic solutions take negative values. In Section 7, we will find again points (2) and (3) of Proposition 6.4.

Proof. (1) This point is an immediate consequence of Proposition 6.3.

(2) We put $u_0 = x,\ u_1 = y,\ u_2 = z$, and for finding 3-periodic solutions we must write the relations $u_3 = x,\ u_4 = y,\ u_5 = z$, that is, $x^2 = a+y+z,\ y^2 = a+z+x$, and $z^2 = a+x+y$. By substracting we get $(y-x)(x+y+1) = 0$, and then $x = y$. In the same way we obtain $y = z$, and so the solution is the equilibrium, whose minimal period is 1.

(3) We put also $u_3 = s,\ u_4 = t$. We have to write three conditions: $u_5 = u_0,\ u_6 = u_1,\ u_7 = u_2$, and the definition of s and t. We obtain (i) $xz = a+s+t$, (ii) $ys = a+t+x$, (iii) $zt = a+x+y$, (iv) $sx = a+y+z$, and (v) $ty = a+s+z$. From (i), (ii), and (iv) we obtain (vi) $z\ (x^2-y-1) = (y+1)(a+y)-x^2$. Relation (ii) where s and t are replaced by their value obtained from (iv) and (v) gives (vii) $z\ (y^2-x-1) = xy\ (a+x)+ax+a+y-y^2(a+y)$. In the same way, relation (i) gives (viii) $z = (y+1)(ax+y+a)/(x+1)(xy-y-1)$, for the denominator cannot be zero because the numerator is positive. The difference of equations (vi) and (vii) gives, after factorization, $(x-y)[z(x+y+1)+(y+1)(a+x+y)] = 0$. So we have $x-y = 0$.

In this case (viii) becomes (ix) $\ z = (ax+x+a)/(x^2-x-1)$. The difference of (i) and (v) gives $x(z-t) = t-z$, that is, $z = t$. So (iii) becomes $z^2 = a+2x$, or $x = (z^2-a)/2$. We put this value of x in (ix) and obtain, after factorization, the relation (x) $\ (z^2-2z-a)(z^3+2z^2-(a-2)z-2(a-1)) = 0$. But $z^2 > a$, so the second factor is greater than $a(z+2)-(a-2)z-2(a-1) = 2z+2 > 2$, and we have $z^2-2z-a = 0$, that is, $z = \ell$. But the two relations $z^2 = 2x+a$ and $z^2 = 2z+a$ imply the relation $x = z$, and so $x = y = z = \ell$: the only 5-periodic solution is the point L. □

6.4. Eigenvalues of $dF_a(L)$ and the Possible Common Periods of Solutions of (1.1)

The question is for which values of $a > 0$ and $q \geq 2$ have all solutions of (1.1) a common period p? What are these common periods? In the case where such a common period exists, one say that (1.1) is "globally periodic" or "globally p-periodic".

The following result is a special case of a deep theorem of Csörnyei and Laczkovich (see [12, 14] for an extension), which gives the answer to the question.

Proposition 6.5. *There is no period p common to all solutions of order q Lyness' equation* (1.1) *with parameter $a > 0$, except the cases $q = 2$, $a = 1$ (with $p = 5$), and $q = 3$, $a = 1$ (with $p = 8$).*

The proof of [14] is difficult. For the case $a = 1$, there is in [15] a nice short proof, which uses the characteristic polynomial of $dF_1(L^*)$, with $L^* = (\ell^*, \ldots, \ell^*)$, ℓ^* being the negative root of $X^2 - (q-1)X - 1 = 0$. Unfortunately this proof does not work for $a \neq 1$.

For every value of $a > 0$, the existence of a common period p to all positive solutions of order q Lyness' equation would imply that the characteristic polynomial $B_q(\lambda) := \lambda^q + 1 - (1/\ell)(\lambda^{q-1} + \cdots + \lambda^2 + \lambda)$ would be a *factor* of the polynomial $A_p := \lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda + 1$. Indeed, the relation $F_a^p = Id$ on $\mathbb{R}_*^{+q}$; gives obviously the equality $dF_a(L)^p = Id$ on $\mathbb{R}^q$, so eigenvalues of $dF_a(L)$ are pth-roots of unity or equal to 1, and by Lemmas 5.1, 5.2, and 5.3, roots of B_q are all distinct. So Proposition 6.5 is related to the following question (where in fact $z = 1/\ell$).

Question 1. Let integers $q \geq 2$, $p \geq 2$, and a number $z \in]0, 1/(q-1)[$. Can it happen that

$$B_q(\lambda) = \lambda^q - z\left(\lambda^{q-1} + \cdots + \lambda\right) + 1 \tag{6.6}$$

were a factor of the polynomial

$$A_p(\lambda) = \lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda + 1? \tag{6.7}$$

Of course a negative answer to Question 1 when $q \geq 4$, or $q = 3$ and $a \neq 1$, or $q = 2$ and $a \neq 1$, would prove Proposition 6.5, because it is known that for $a = 1$ and $q = 2$ or $q = 3$ (1.1) is globally p-periodic, with $p = 5$ or 8.

But we can have a positive answer to Question 1: some a, p, and q for which B_q divides A_p, that is, all eigenvalues of $dF_a(L)$ are pth root of unity, but without (1.1) being globally p-periodic, such cases are given by the following proposition.

Proposition 6.6. (1) *If $q = 2$, for every $p \geq 13$ there is a number (perhaps not unique) $z(p)$, $0 < z(p) < 1$ (and then an $a(p)$), such that the polynomial $B_2 = \lambda^2 - z(p)\lambda + 1$ is a factor of A_p; that is, eigenvalues of $dF_{a(p)}(L)$ are pth root of unity.*

(2) *If $q = 3$, for every even $p \geq 20$ there is a number (perhaps not unique) $z(p)$, $0 < z(p) < 1/2$ (and so an $a(p)$), such that the polynomial $B_3 = \lambda^3 - z(p)(\lambda^2 + \lambda) + 1 = (\lambda + 1)(\lambda^2 - (z(p) + 1)\lambda + 1)$ is a factor of A_p; that is, eigenvalues of $dF_{a(p)}(L)$ are pth root of unity.*

Proof of Proposition 6.6. (1) If polynomial $\lambda^2 - z\lambda + 1$ divides A_p, then it is $\lambda^2 - 2\cos(2k\pi/p)\lambda + 1$ for some integer k satisfying $1 \leq k < p/4$. So we have $z = 2\cos(2k\pi/p)$ and $a = (1-z)/z^2$, with only the constraint that $0 < z < 1$ and $a > 0$, that is, $1 \leq k < p/4$ and $k > p/6$. This is possible if $p = 5, 9, 10, 11$, and if $p \geq 13$. This last case gives the first point of Proposition 6.5.

(2) If polynomial $(\lambda + 1)(\lambda^2 - (z+1)\lambda + 1)$ divides A_p, then $p = 2m$ is even, and the roots of $\lambda^2 - (z+1)\lambda + 1$ are pth roots of unity, and then this polynomial has the form $\lambda^2 - 2\cos(k\pi/m)\lambda + 1$, for some integer k satisfying $1 \leq k \leq m-1$. So we have $z = 2\cos(k\pi/m) - 1$ and $a = (1-2z)/z^2$. The only constraints are $a > 0$ and $1 + 1/2 > z + 1 > 1$, that is, $1 \leq k < m/3$ and $k > m\theta_0$ where $\theta_0 = (1/\pi)\cos^{-1}(3/4) = 0.230$. But if $m > 1/(1/3 - \theta_0) = 9.68$, there is an integer $k \geq 1$ satisfying the constraints. So if $p = 2m \geq 20$, we have the point (2) of Proposition 6.6. But one can see that some k works also for $p = 8, 14$, and 16 ($k = 2$), and only in these cases. □

Example 6.7. (1) If $q = 2$ and $p = 24$, we find that for the equation $u_{n+2} = (2 + \sqrt{3} - \sqrt{2+\sqrt{3}} + u_{n+1})/u_n$, the eigenvalues of $dF_a(L)$ are 24th roots of unity, but 24 is not a common period to all solutions.

(2) For $p = 30$, we find the same fact for (1.1) with $q = 3$ and $a(30) = (3 - 4\cos(4\pi/15))/(2\cos(4\pi/15) - 1)^2$.

We make the following conjecture.

Conjecture 8. If $q \geq 4$, the polynomial B_q does not divide the polynomial A_p for $0 < z < 1/(q-1)$ (and then for all $a > 0$ (1.1) is not globally periodic if $q \geq 4$).

7. Order 3 Lyness' Difference Equation

Before to study this case, we give general results for odd q.

7.1. Invariants for F_a^2 When $q = 2m + 1$ Is Odd

If q is odd, we have nice invariants for F_a^2.

Proposition 7.1. *When $q = 2m + 1$ is odd, the quantities*

$$R^0(x_1, \ldots, x_q) := \frac{(1+x_1)(1+x_3)\cdots(1+x_{2m+1})}{x_2 x_4 \cdots x_{2m}}, \tag{7.1}$$

$$R_a^1(x_1, \ldots, x_q) := \frac{(1+x_2)(1+x_4)\cdots(1+x_{2m})(a + x_1 + x_2 + \cdots + x_{2m} + x_{2m+1})}{x_1 x_3 \cdots x_{2m+1}} \tag{7.2}$$

are exchanged by F_a:

$$R^0 \circ F_a = R_a^1, \qquad R_a^1 \circ F_a = R^0 \tag{7.3}$$

and then are invariant under the action of F_a^2.

Moreover one has the relations

$$R^0 R_a^1 = G_a, \quad \text{if } q \geq 5 \quad R^0 + R_a^1 = J_a. \tag{7.4}$$

Proof. Relations (7.3) come from and easy calculation, and relations (7.4) are obvious from the definitions of G_a and J_a. □

Remark 7.2. Relations (7.4) give simple proof of the invariance of G_a and J_a under the action of F_a and also of every symmetric function of R^0 and R_a^1.

Proposition 7.1 has significant consequences in the case $q = 3$.

7.2. The Case $q = 3$

This case was intensively studied in the nice paper [4], where the authors proved, for the first time, that there are two curves in $\mathbb{R}_*^{+^3}$ invariant by the map F_a^2 and homeomorphic to two circles, such that the restriction of F_a^2 to these curves is conjugated to rotations on the circles, and that these curves are exchanged by the map F_a. The authors gave also some interessant results about possible periods for F_a^2 of some starting points for some values of parameter a and about initial points whose orbits are dense in the invariant curves.

Authors of [16] have found a remarkable property of even terms u_{2n} and odd terms u_{2n+1} of the order 3 Lyness' sequence: they are solutions of two-order 2 difference equations related to two elliptic quartics in the plane. In [4], the authors make the conjecture, in a note added in proof, that their (difficult) paper would be simpler with the aid of [16]. We will see that in fact this is right, at least partially, with the aid of invariants R^0 and R_a^1 of F_a^2 (these invariants are implicit in [16] for the case $q = 3$). In the case $q = 3$ these invariants become

$$R^0(x,y,z) = \frac{(1+x)(1+z)}{y}, \qquad R_a^1(x,y,z) = \frac{(1+y)(a+x+y+z)}{xz}. \tag{7.5}$$

If we denote, as in [16],

$$c_0 := R^0(M_0), \qquad c_1 := R_a^1(M_0), \tag{7.6}$$

the orbit of the starting point M_0 under the action of F_a^2 is on the curve $\mathfrak{D}(M_0)$ defined by the two equations

$$(1+x)(1+z) - c_0 y = 0, \qquad (1+y)(a+x+y+z) - c_1 xz = 0. \tag{7.7}$$

These two surfaces are just quadratic surfaces ("quadrics") Q_0 and Q_1, and then $\mathfrak{D}(M_0)$ is merely a quartic curve. In fact, this curve is included in the curve $\mathcal{C}(M_0)$ given in (3.18).

Theorem 7.3. *(1) In addition to the first relation* (7.5) *one has an expression of H_a in terms of R^0 and R_a^1, and so one has:*

$$G_a = R^0 R_a^1, \qquad H_a = R^0 R_a^1 + R^0 + R_a^1 + 2 - a. \tag{7.8}$$

(2) The polynomial equations $g(x,y,z) = 0$ and $h(x,y,z) = 0$ of $\Sigma_a(K)$ and of $S_a(M)$ given by formulas (2.5) *and* (3.11) *(where $K = G_a(M_0)$ and $M = H_a(M_0)$) are polynomial combinations of the two equations of the curve $\mathfrak{D}(M_0)$ given in* (7.7)*:*

$$2g(x,y,z) = A(x,y,z)\left[(1+x)(1+z) - c_0 y\right] + B(x,y,z)\left[(1+y)(a+x+y+z) - c_1 xz\right], \tag{7.9}$$

where $A(x,y,z) = c_1 xz + (1+y)(a+x+y+z)$ and $B(x,y,z) = (1+x)(1+z) + c_0 y$, and

$$h(x,y,z) = C(x,y,z)\left[(1+x)(1+z) - c_0 y\right] + D(x,y,z)\left[(1+y)(a+x+y+z) - c_1 xz\right], \tag{7.10}$$

where $C(x,y,z) = y^2 + xy + yz + zx + x + (a+1)y + z + a$ and $D(x,y,z) = (c_0+1)y$.

(3) One has the inclusion

$$\mathfrak{D}(M_0) \cup F_a(\mathfrak{D}(M_0)) \subset \mathcal{C}, \tag{7.11}$$

and the part $\mathfrak{D}^+(M_0)$ of $\mathfrak{D}(M_0)$ which is in $\mathbb{R}_*^{+3}$ is compact.

Proof. (1) The first relation (7.8) is in (7.4), and an easy but tedious calculation proves the second relation.

(2) The proofs of the two announced expressions for $2g$ and h come from simple but also tedious calculations using the two relations $K = c_0c_1$ and $M = c_0c_1 + c_0 + c_1 + 2 - a$, which come from relations (7.8) applied to the point M_0.

(3) The inclusion (7.11) and then the announced compacity are obvious consequences of point (2) and of the compacity of $\mathcal{C}^+(M_0) = \mathbb{R}_*^{+3} \cap \mathcal{C}(M_0)$. $\square$

Remark 7.4. (1) The second formula (7.8) gives, for $q = 3$, an easy proof of the invariance of H_a under the action of F_a.

(2) It is not the case for $q \geq 5$, because the second relation (7.8) has no corresponding one: we know from [5] that G_a, H_a, and J_a are independent and so cannot be all three regular functions of R^0 and R_a^1.

We will see that significant properties of maps F_a^2 and F_a can be deduce only with the use of invariants R^0 and R_a^1. We start with the identification of $F_a(\mathfrak{D}(M_0))$.

Lemma 7.5. *(1) If $\mathfrak{D}(M_0)$ has equations $R^0(x,y,z) = c_0$ and $R_a^1(x,y,z) = c_1$, then the curve $F_a(\mathfrak{D}(M_0))$ is a quartic curve whose equations are*

$$R^0(x,y,z) = c_1, \qquad R_a^1(x,y,z) = c_0. \tag{7.12}$$

These two curves are disjoint if and only if M_0 does not belong to the surface $\mathcal{W}_a$ of $\mathbb{R}_*^{+3}$ with equation

$$\omega(x,y,z) := R^0(x,y,z) - R_a^1(x,y,z) = 0, \tag{7.13}$$

that is,

$$\widetilde{\omega}(x,y,z) := xz(1+x)(1+z) - y^3 - y^2(1+a+x+z) - y(a+x+z) = 0, \tag{7.14}$$

where $\widetilde{\omega}(x,y,z) = xyz\,\omega(x,y,z)$.

(2) The two parts of $\mathbb{R}_*^{+3} \setminus \mathcal{W}_a$, denoted by

$$\mathcal{W}_a^+ = \left\{R^0 > R_a^1\right\}, \qquad \mathcal{W}_a^- = \left\{R^0 < R_a^1\right\}, \tag{7.15}$$

are two open sets whose union is dense in $\mathbb{R}_*^{+3}$, which are exchanged by F_a, and so both globally invariant under the action of F_a^2. If $M_0 \notin \mathcal{W}_a$, one has

$$\begin{gathered}\mathfrak{D}^+(M_0) \subset \mathcal{W}_a^+ \Longleftrightarrow c_0 > c_1 \Longleftrightarrow F_a(\mathfrak{D}^+(M_0)) \subset \mathcal{W}_a^-,\\ \mathfrak{D}^+(M_0) \subset \mathcal{W}_a^- \Longleftrightarrow c_0 < c_1 \Longleftrightarrow F_a(\mathfrak{D}^+(M_0)) \subset \mathcal{W}_a^+.\end{gathered} \tag{7.16}$$

(3) The point L lies on $\mathcal{W}_a$, and $\mathcal{W}_a$ cuts the diagonal in $\mathbb{R}_*^{+3}$ only at L.

(4) One has $\mathcal{W}_a \cap \mathcal{H}_a \cap \mathbb{R}_*^{+3} = \{L\}$, where $\mathcal{H}_a$ is the hyperbola of 2-periodic points.

(5) In fine , $\mathcal{W}_a$ is the graph of a continuous function $\phi : (x,z) \mapsto \phi(x,z)$ on $\mathbb{R}_*^{+2}$, whose epigraph is $\mathcal{W}_a^-$ and hypograph is $\mathcal{W}_a^+$, which so are connected sets.

Proof. One has $M \in F_a(\mathfrak{D}(M_0)) \iff R^0(F_a^{-1}(M)) = c_0$ and $R_a^1(F_a^{-1}(M)) = c_1$. With the expression $F_a^{-1}(x,y,z) = (y,z,(a+y+z)/x)$ this last condition is easily translated as $R_a^1(x,y,z) = c_0$ and $R^0(x,y,z) = c_1$. Now conditions (7.13) or (7.14) are obvious, and the conditions $\mathfrak{D}^+(M_0) \subset \mathcal{W}_a^+$ and $\mathfrak{D}^+(M_0) \subset \mathcal{W}_a^-$ also.

For the point (3), the equation $\widetilde{\omega}(t,t,t) = 0$ is $t^2 - 2t - a = 0$, whose only positive solution is ℓ.

For $t > 1$ the equation $\widetilde{\omega}(t,(t+a)/(t-1),t) = 0$ is $(t^2 - 2t - a)(t^4 + t^3 + at^2 + (3a-2)t + (a-1)^2) = 0$. The polynome of degree 4 in t is a degree 2 polynomial in a, which is positive for $t > 1$ because its discriminant is $-t(t-1)^2(3t+4)$. So the only solution is $t = \ell$.

In fine the last assertion is obvious by the study of the function $y \mapsto \widetilde{\omega}(x,y,z)$. □

Remark 7.6. (1) In [4] the surface $\mathcal{W}_a$ with (7.14) is obtained as part of the locus of points where the gradients of G_a and H_a are parallel (the other points are these ones of the hyperbola $\mathcal{H}_a$).

(2) Points (1), (2), and (3) of Lemma 7.5 are obviously true for every odd q, from Proposition 7.10. Some easy calculations show that the point (4) is also true for an odd dimension q.

(3) The curves $\mathfrak{D}(M_0)$ and $F_a(\mathfrak{D}(M_0))$ are not disjoint in $\mathbb{P}^3$; they have in common four points at infinity: $(0,0,1,0)$, $(0,1,-1,0)$, $(1,0,0,0)$, and $(1,-1,0,0)$.

From the fact that if $M_0 \notin \mathcal{W}_a$, $\mathfrak{D}(M_0) \cap F_a(\mathfrak{D}(M_0)) = \emptyset$, the following result is obvious (and has an obvious generalization to order q Lyness' equation when q is odd, which is in [4]) and gives again points (2) and (3) of Proposition 6.4.

Corollary 7.7. *No point of $\mathbb{R}_*^{+3} \setminus \mathcal{W}_a$ is periodic under the action of F_a with an odd period.*

Now we can assert the principal qualitative result on the map F_a^2. Via Proposition 7.13 below we find again a result of [4], but it comes in this last paper from new and interessant, but sophisticated, arguments, whereas we use here only previous results of [17].

Theorem 7.8. *Let $M_0 \notin \mathcal{W}_a \cup \mathcal{H}_a$. Then the two quartics $\mathfrak{D}(M_0)$ and $F_a(\mathfrak{D}(M_0))$ are disjoint, exchanged by F_a, both globally invariant under the action of F_a^2. The positive and compact parts $\mathfrak{D}^+(M_0)$ and $F_a(\mathfrak{D}^+(M_0))$ are both homeomorphic to the circle, and the restrictions of F_a^2 to each of these two curves are conjugated to rotations on the circle, with the same angle.*

Proof. We can suppose that $M_0 \in \mathcal{W}_a^+$, that is, $c_0 > c_1$, the proof is the same in the other case.

(1) Let π be the projection on the (x,z)-plane, and put

$$E_0(M_0) := \pi(\mathfrak{D}(M_0)). \tag{7.17}$$

First we remark that π is injective on the surface $R^0(x,y,z) = c_0$, because it is the graph of the function $(x,z) \mapsto y = (1+x)(1+z)/c_0$. So the projection π is a *homeomorphism* of the compact curve $\mathfrak{D}^+(M_0)$ onto its projection $E_0^+(M_0) = E_0(M_0) \cap \mathbb{R}_*^{+2}$.

(2) We search the conjugated of the restriction of F_a^2 to $\mathfrak{D}^+(M_0)$ by the projection π. We have easily $F_a^2(x,y,z) = ((a+x+y+az+xz)/yz, (a+x+y)/z, x)$, and if $(x,y,z) \in \mathfrak{D}(M_0)$,

we have $y = (1+x)(1+z)/c_0$, and so we obtain (after simplification by the factor $1+z$) that if $(x,z) \in E_0^+(M_0)$, then

$$\pi\left[F_a^2\left(\pi^{-1}(x,z)\right)\right] = \left(\frac{(1+c_0a)+(1+c_0)x}{z(1+x)}, x\right). \tag{7.18}$$

So we conclude that the restriction of F_a^2 to $\mathfrak{D}^+(M_0)$ is conjugated by π to the restriction to $E_0^+(M_0)$ of the map

$$T_a(c_0) : \mathbb{R}_*^{+2} \longrightarrow \mathbb{R}_*^{+2} : (x,z) \longmapsto \left(\frac{(1+c_0a)+(1+c_0)x}{z(1+x)}, x\right). \tag{7.19}$$

(3) But this map $T_a(c_0)$ in $\mathbb{R}_*^{+2}$ is studied in [17]. We know from this paper (see its introduction and part 4: "The homographic case") that there is an invariant of this map and of the associated difference equation $x_{n+2}x_n = ((1+c_0a)+(1+c_0)x_{n+1})/(1+x_{n+1})$. More precisely, put

$$p_0 = 1 + c_0 a, \qquad q_0 = 1 + c_0; \tag{7.20}$$

then one has

$$\frac{p_0 + q_0 x}{1+x} = \frac{(p_0+q_0x)(q_0+x)}{(1+x)(q_0+x)} = \frac{a' + b'x + c'x^2}{c' + d'x + x^2}, \tag{7.21}$$

$$a' = p_0 q_0, \qquad b' = p_0 + q_0^2, \qquad c' = q_0, \qquad d' = q_0 + 1. \tag{7.22}$$

Then we see in [17] that the quantity

$$V_0(x,z) := \frac{1}{xz}\left(x^2z^2 + d'xz(x+z) + c'\left(x^2+z^2\right) + b'(x+z) + a'\right) \tag{7.23}$$

is invariant under the action of $T_a(c_0)$. So the curves Γ_k with equations $V_0(x,z) = k$ are invariant by $T_a(c_0)$, and it is proved in [17] that these curves are, if $k > k_m$, elliptic quartics, which are homeomorphic to circles, and that the restriction of $T_a(c_0)$ to Γ_k is conjugated to a rotation. So the theorem will be proved if we show that the curve $E_0(M_0)$ is a curve Γ_k, for a k satisfying $k > k_m$, that is, if Γ_k does not reduce to the fixed point of $T_a(c_0)$.

So we will determine the equation of $E_0(M_0)$. We take y from the relation $R^0(x,y,z) = c_0$ and put it in the relation $R_a^1(x,y,z) = c_1$. We obtain for equation of $E_0(M_0)$

$$\left(1 + \frac{(1+x)(1+z)}{c_0}\right)\left(a + x + z + \frac{(1+x)(1+z)}{c_0}\right) = c_1 xz. \tag{7.24}$$

An easy calculation gives the form of the equation

$$x^2z^2 + (2+c_0)xz(x+z) + (c_0+1)\left(x^2+z^2\right) \\ + \left(1 + c_0a + (c_0+1)^2\right)(x+z) - xz\left(c_0^2c_1 - (3+a)c_0 - 4\right) = 0 \tag{7.25}$$

or, from (7.20) and (7.22), $V_0(x,z) = k_0$; that is, $E_0(M_0)$ is the curve Γ_{k_0} where

$$k_0 = c_0^2c_1 - (3+a)c_0 - 4. \tag{7.26}$$

It remains to show that this curve Γ_{k_0}, which passes through the point $m_0 = (x_0, z_0) = \pi(M_0)$, does not reduce to the fixed point of $T_a(c_0)$, which has the form $x = z = t$ where

$$t^3 + t^2 - (1+c_0)t - (1+c_0a) = 0. \tag{7.27}$$

If it were the case, we would have $x_0 = z_0 = t$, and so $c_0 = (1+t)^2/y_0$. If we put this value in the equation giving t, we obtain the relation $((1+t)^2/y_0)(a+t) = t^3 + t^2 - t - 1$, and this gives $y_0 = (t+a)/(t-1)$. So the point $M_0 = (t, (t+a)/(t-1)\ , t)$ would be in the hyperbola $\mathscr{H}_a$, because it is easy to see that the solution t of (7.28) is greater than 1. But we have supposed that $M_0 \notin \mathscr{H}_a$, and so the curve $E_0(M_0)$ does not reduce to the fixed point of $T_a(c_0)$ (besides, it was clear that if $M_0 \in \mathscr{H}_a$, it would be a fixed point for F_a^2). Remark that we do not use the hypothesis $M_0 \notin \mathcal{W}_a$; so the precedent results are true if only $M_0 \notin \mathscr{H}_a$, but we will see in Proposition 7.11 what happens when $M_0 \in \mathcal{W}_a \setminus \{L\}$.

(4) Then we know from [17] that there is a homeomorphism ψ of $E_0^+(M_0)$ onto the unit circle $\mathbb{T}$ (built with the Weierstrass' function) such that the map $T_a(c_0)_{|E_0^+(M_0)}$ is conjugated by ψ to a rotation on $\mathbb{T}$ by an angle $2\pi\theta_a(M_0)$ well defined, where $\theta_a(M_0) \in]0, 1/2[$. So by the map $\psi \circ \pi$ the restriction of F_a^2 to the curve $\mathfrak{D}^+(M_0)$ is conjugated to the same rotation on $\mathbb{T}$. Moreover the map $\psi \circ \pi \circ F_a^{-1}$ is an homeomorphism of $F_a(\mathfrak{D}^+(M_0))$ onto $\mathbb{T}$, which conjugates $F^2_{a|F_a(\mathfrak{D}^+(M_0))}$ to the same rotation on $\mathbb{T}$. □

One can see in Figure 2 the arrangement of the curves and surfaces of Lemma 7.5 and Theorem 7.8.

From the parts (1) and (2) of the proof, which do not use the hypothesis $M_0 \notin \mathcal{W}_a \cup \mathscr{H}_a$, we find again the result of [16], which is proved in this paper by the "QRT method" (remark that the value of k_0 is not given in [16]).

Corollary 7.9 ([16]). *If $(u_n)_n$ is a Lyness' sequence of order 3 (a solution of* (1.1)*), then the sequences $x_n = u_{2n}$ and $y_n = u_{2n+1}$ are solutions of two second order "homographic" difference equations*

$$x_{n+2}x_n = \frac{(1+ac_0) + (1+c_0)x_{n+1}}{1+x_{n+1}}, \qquad y_{n+2}y_n = \frac{(1+ac_1) + (1+c_1)x_{n+1}}{1+x_{n+1}}, \tag{7.28}$$

where $c_0 = (1+u_0)(1+u_2)/u_1$ *and* $c_1 = (1+u_1)(1+u_3)/u_2 = (1+u_1)(a+u_0+u_1+u_2)/u_2u_0$. *The orbit of* (x_{n+1}, x_n) *is on* $E_0^+(M_0)$, *and the orbit of* (y_{n+1}, y_n) *is on the quartic*

$$E_1^+(M_0) := E_0^+(M_1) = \pi(F_a(\mathfrak{D}^+(M_0))), \tag{7.29}$$

and the equation of $E_1(M_0)$ *is the same as this one of* $E_0(M_0)$ *with* c_1 *and* c_0 *exchanged.*

Proof. The proof is obvious for $x_n = u_{2n}$, because $\pi(M_{2n}) = (u_{2n+2}, u_{2n})$ and that $\pi(M_{2n+2}) = \pi(F_a^2(M_{2n})) = T_a(\pi(M_{2n}))$. For $y_n = u_{2n+1}$ the proof is the same, starting from $M_1 = F_a(M_o) = (u_3, u_2, u_1)$, with the quartic curve $E_1(M_0) := E_0(M_1) = \pi(F_a(\mathfrak{D}(M_0)))$, whose equation is the same as this one of $E_0(M_0)$ with $(1+u_1)(1+u_3)/u_2$ in place of c_0, and this number is $c_1 = (1+u_1)(a+u_0+u_1+u_2)/u_0u_2$. The value k_1 in the equation $V_1(x,z) = k_1$ of $E_1(M_0)$ is, by (7.26), $c_1^2c_2 - (3+a)c_1 - 4$, where $c_2 = c_1(M_1) = R_a^1(M_1) = R_a^1(F_a(M_0)) = R_0(M_0) = c_0$; so k_1 is nothing that k_0 where c_0 and c_1 are exchanged. □

Now there are *some questions*. First, what does happen if $M_0 \in \mathscr{H}_a \setminus \mathcal{W}_a = \mathscr{H}_a \setminus \{L\}$? Secondly, what does happen if $M_0 \in \mathcal{W}_a \setminus \{L\}$? Third, which information can one have about the numbers $\theta_a(M_0)$ *with our method* (such informations are of course in [4])? Fourth, what are exactly the relationships between the curves $\mathfrak{D}(M_0)$ and $\mathcal{C}(M_0)$?

For the first question, we know the action of F_a and F_a^2 on M_0, by Theorem 5.4: if $M_0 = (t, (t+a)/(t-1)\ , t)$ for some $t > 1$, then $F_a(M_0) = ((t+a)/(t-1), t, (t+a)/(t-1))$ and $F_a^2(M_0) = M_0$. But it is not clear *a priori* that $\mathfrak{D}^+(M_0)$ reduces to the point M_0. It is actually the case.

Proposition 7.10. *If* $M_0 \in \mathscr{H}_a$, *then* $\mathfrak{D}^+(M_0) = \{M_0\}$.

Proof. We search the fixed point of $T_a(c_0)$ if $M_0 = (t, (t+a)/(t-1), t)$ for some $t > 1$. Its coordinates are $x = z = \mathrm{X}$, where X is the positive solution of the equation

$$\mathrm{X}^3 + \mathrm{X}^2 - (1+c_0)\mathrm{X} - (1+c_0a) = 0. \tag{7.30}$$

But this equation is satisfied by $\mathrm{X} = t$; so the fixed point is $(t,t) = \pi(M_0)$, and by [17] the quartic $E_0^+(M_0) = \pi(\mathfrak{D}^+(M_0))$ reduces to $\{(t,t)\}$, and so $\mathfrak{D}^+(M_0)$ reduces to $\{M_0\}$.

For the second question, we have a partially answer. □

Proposition 7.11. *If* $M_0 \in \mathcal{W}_a \setminus \{L\}$, *one has the following properties:*

(1) *the curves* $\mathfrak{D}(M_0)$ *and* $F_a(\mathfrak{D}(M_0))$ *coincide;*

(2) *the solution* (u_n) *of the order 3 Lyness' difference equation is such that the sequence* $s_n := (1+u_n)/c_0$ *is a solution of an order 2 Lyness' equation with negative parameter* $-1/c_0$: $s_{n+2}s_n = s_{n+1} - 1/c_0$;

(3) *the curve* $\mathfrak{D}^+(M_0)$ *is homeomorphic to the circle* $\mathbb{T}$, *globally stable by* F_a, *and the restriction of* F_a^2 *to this curve is, as in Theorem 7.8, conjugated by this homeomorphism to a rotation by angle* $2\pi\theta_a(M_0)$.

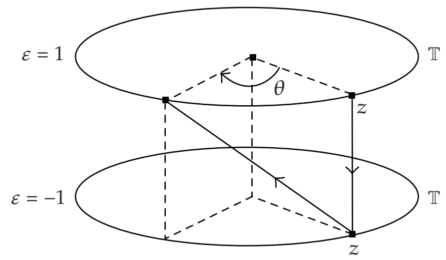

Figure 1

Proof. First we note that if $M_0 \in \mathcal{W}_a \setminus \{L\}$, then $R^0(M_0) = c_0 = c_1 = R_a^1(M_0)$. So the equations of $\mathfrak{D}(M_0)$ and $F_a(\mathfrak{D}(M_0))$ are the same, by (7.12). Moreover we have

$$\frac{(1+u_{2n+2})(1+u_{2n})}{u_{2n+1}} = \frac{(1+u_0)(1+u_2)}{u_1} = c_0 = c_1 = \frac{(1+u_1)(1+u_3)}{u_2} \\ = \frac{(1+u_{2n+1})(1+u_{2n+3})}{u_{2n+2}} \tag{7.31}$$

by the invariance under the action of F_a^2. So we have for all n $(1+u_n)(1+u_{n+2})/u_{n+1} = c_0$. If we put $s_n = (1+u_n)/c_0$, we obtain the announced order 2 Lyness' difference equation $s_{n+2}s_n = s_{n+1} - 1/c_0$. For the final assertion on the restriction of F_a^2 to $\mathfrak{D}^+(M_0)$, we note that the proof of Theorem 7.8 is valid in this case, and it uses only the hypothesis $M_0 \notin \mathcal{H}_a$. □

We have given in Theorem 7.8 and Proposition 7.11 a description of $F^2_{a|\mathfrak{D}^+(M_0)}$. We have also a good description of the restriction of F_a to $\mathfrak{D}^+(M_0) \cup F_a(\mathfrak{D}^+(M_0))$.

Theorem 7.12. *Suppose $a \neq 1$ and $M_0 \notin \mathcal{W}_a \cup \mathcal{H}_a$. Then the restriction of F_a to the set $\mathfrak{D}^+(M_0) \cup F_a(\mathfrak{D}^+(M_0))$ is conjugated to the following map in the set $\{-1,1\} \times \mathbb{T}$:*

$$(\varepsilon, z) \longmapsto \left(-\varepsilon, \phi_\varepsilon(z)\right), \quad \textit{where } \varepsilon = \pm 1,\ \phi_1 = Id,\ \phi_{-1}(z) = ze^{2i\pi\theta_a(M_0)}. \tag{7.32}$$

See Figure 1 for this map.

Proof. It is easy from the proof of Theorem 7.8, where now we distinguish the two circles by a product by $\{1\}$ or $\{-1\}$. □

Now we will give an answer to the question about relationship between curves $\mathfrak{D}(M_0)$ and $\mathcal{C}(M_0)$, which will give again, with Theorem 7.8, the principal result of [4].

Proposition 7.13. *When $M_0 \notin \mathcal{W}_a \cup \mathcal{H}_a$, the curve $\mathcal{C}(M_0)$ is the union of 8 straight lines $d_1, \ldots, d_8$ not included in $\mathfrak{D}(M_0)$ nor in $F_a(\mathfrak{D}(M_0))$, and of the two disjoint quartics $\mathfrak{D}(M_0)$ and $F_a(\mathfrak{D}(M_0))$:*

$$\mathcal{C}(M_0) = d_1 \cup d_2 \cup \cdots \cup d_8 \cup \mathfrak{D}(M_0) \cup F_a(\mathfrak{D}(M_0)), \tag{7.33}$$

and the straight lines d_i do not intersect $\mathbb{R}_*^{+3}$. So one has

$$\mathcal{C}^+(M_0) = \mathfrak{D}^+(M_0) \cup F_a(\mathfrak{D}^+(M_0)). \tag{7.34}$$

Proof. We write the equations of $\mathcal{C}(M_0)$ in homogeneous coordinates, for taking in account the points at infinity:

$$\begin{gathered}(t+x)(t+y)(t+z)(at+x+y+z) - Kxyzt = 0,\\ (t+x+y)(t+y+z)\left(at^2+xz+tx+ty+tz\right) - Mxyzt = 0.\end{gathered} \tag{7.35}$$

So it is obvious that the following straight lines (the two first are at infinity)

$$\begin{gathered}\{t=0,\ x=0\};\quad \{t=0,\ z=0\};\quad \{x=0,\ y=-1\};\quad \{x=0,\ a+y+z=0\};\\ \{z=0,\ y=-1\};\quad \{z=0,\ a+x+y=0\};\quad \{y=0,\ x=-1\};\quad \{y=0,\ z=-1\}\end{gathered} \tag{7.36}$$

satisfy the two equations of the surfaces whose intersection is $\mathcal{C}(M_0)$. But the curve $\mathcal{C}(M_0)$ has degree 16, and so we can write it as the union of the 8 straight lines and of a degree 8 algebraic curve $\widetilde{\mathcal{C}}(M_0)$. But we have the following result. □

Lemma 7.14. *The quartic curves $\mathfrak{D}(M_0)$ and $F_a(\mathfrak{D}(M_0))$ do not contain any straight lines.*

So, because the union of the two quartic curves has degree 8, we conclude that $\widetilde{\mathcal{C}}(M_0) = \mathfrak{D}(M_0) \cup F_a(\mathfrak{D}(M_0))$, and this proves the proposition.

Proof of Lemma 7.14. The proof is the same for the two curves. If $\mathfrak{D}(M_0)$ contains a straight line, then this line is a generatrix of the quadric Q_0 with equation $(1+x)(1+z) - c_0y = 0$. It is easy to see that these generatrices are the two families defined by

$$D_\lambda : 1+x=\lambda;\quad c_0y=\lambda(1+z);\quad D'_\mu : 1+z=\mu,\quad c_0y=\mu(1+x)\quad \left(\lambda,\mu\in\mathbb{P}^1(\mathbb{R})\right). \tag{7.37}$$

But it is obvious to see that none of these straight lines can be contained in the second quadric Q_1 with equation $(1+y)(a+x+y+z) - c_1xz = 0$ (the conditions of inclusion are $a=1$ and $c_1=-1$, or $M=0$, which are impossible). □

Now, we return to the curve $E_0(M_0)$ when $M_0 \notin \mathcal{H}_a$, by using [17]. First, we put $x = \mu X$ and $z = \mu Z$, where $\mu = \sqrt[4]{a\prime} = \sqrt{p_0} = \sqrt{1+c_0a}$. So the (X,Z) equation of the curve is

$$X^2Z^2 + dXZ(X+Z) + c\left(X^2+Z^2\right) + b(X+Z) + 1 = K_0XZ, \tag{7.38}$$

where one has

$$b = \frac{p_0+q_0^2}{p_0^{3/2}},\qquad c=\frac{q_0}{p_0},\qquad d=\frac{p_0+1}{\sqrt{p_0}},\qquad K_0=\frac{k_0}{p_0}. \tag{7.39}$$

It is proved in [17, Table 8.1, rows 2 and 7] that $T_a(c_0)$ is globally 4-periodic if and only if $b = d$ and $c = 1$, that is, $p_0 = q_0$, or $a = 1$. In this case, the restrictions of F_a^2 to $\mathfrak{D}^+(M_0)$ and to $F_a(\mathfrak{D}^+(M_0))$ are 4-periodic for every $M_0 \notin \mathcal{H}_a$, and so F_a is 8-periodic on $\mathbb{R}_*^{+3}$, as it is well known. In fact it is easy to prove, by using $T_a(c_0)$, that if some $M_0 = (x_0, y_0, z_0)$ with $x_0 \neq z_0$ is 8-periodic, then $a = 1$, but this result is already in [3].

Now suppose that $a \neq 1$. Then $p_0 \neq q_0$, and so $d \neq 1$ and $b \neq d$ from (7.39). Under this hypothesis it is proved in [17] that $\lim_{k\to+\infty}\theta(k) = 1/4$, where k is the value of the invariant function V_0 of (7.26) for the the dynamical system $(\mathbb{R}_*^{+2}, T_a(c_0))$.

Then we can look at the question about numbers $\theta_a(M_0)$. The first result comes from calculations which are made in [17], and that one can read again in the present context.

Lemma 7.15. *The function* $\Theta : (a, M_0) \mapsto \theta_a(M_0) :]0, +\infty[\times(\mathbb{R}_*^{+3} \setminus \mathcal{H}_a) \to]0, 1/2[$ *is analytical (as function of* a, x_0, y_0, z_0*) on each of the sets* $]0, 1[\times(\mathbb{R}_*^{+3} \setminus \mathcal{H}_a)$ *and* $]1, +\infty[\times(\mathbb{R}_*^{+3} \setminus \mathcal{H}_a)$.

Proof. The calculations of [17] show that Θ is an analytical function of K_0, and in fact, if we look at the formulas of [17], of the coefficients b, c, d of (7.38), if $a \neq 1$. So it is an analytical function of p_0, q_0, k_0, and then of a and c_0, c_1 (see (7.21)), that is, of a and x_0, y_0, z_0.

The following result will be useful. □

Lemma 7.16. *If* $a \neq 1$*, there is an* $\alpha > 1$ *and a map* $t \mapsto M_0(t) :]\alpha, +\infty[\to \mathcal{W}_a^-$ *such that* $\lim_{t\to+\infty}\theta_a(M_0(t)) = 1/4$.

Proof. The dynamical system in the (x, z)-plane $T_a(c_0)(x, z) = ((p + qx)/z(1 + x), x)$ depends only on c_0 and so is the same for all the points of the quadric Q_0. Let $M_0 \in \mathcal{W}_a^-$, with $R^0(M_0) = c_0 < c_1 = R_a^1(M_0)$. Put $M_0(t) := (t, (1 + t)^2/c_0, t)$, for $t > 0$. The point $M_0(t)$ lies on a parabola $\mathcal{P}_0 \subset Q_0$ (see Figure 2), and so for every t such that $c_1(t) := R_a^1(M_0(t)) > c_0$, the dynamical system in the (x, z)-plane associated to $M_0(t)$ is the same. Moreover the number $k(t) := c_0^2 c_1(t) - (3 + a)c_0 - 4$ of (7.27) has for value $(1 + (1 + t)^2/c_0)(a + 2t + (1 + t)^2/c_0)(1/t^2)$, which tends to $+\infty$ when $t \to +\infty$. So for $t > \alpha$ one has $c_1(t) > c_0$. Now we have seen that $\theta(k) \to 1/4$ when $k \to +\infty$, so $\theta_a(M_0(t)) = \theta(k(t)) \to 1/4$ when $t \to +\infty$. □

Now we have a dichotomy result.

Theorem 7.17. *One has only two mutually exclusive possibilities.*

(1) The function θ_a *is constant on the open dense set* $\mathbb{R}_*^{+3} \setminus \mathcal{H}_a$*, and then* $a = 1$ *and* F_a *is globally 8-periodic.*

(2) one has $a \neq 1$*, and the interior of the image of the map* θ_a *is a nonempty interval* $]\alpha, \beta[$*; in this case, one has the four properties:*

(a) the initial periodic points M_0 *for* F_a^2 *are dense in* $\mathbb{R}_*^{+3}$*;*

(b) the initial points M_0 *whose orbit for* F_a^2 *is dense in* $\mathfrak{D}^+(M_0)$ *are dense in* $\mathbb{R}_*^{+3} \setminus \mathcal{H}_a$*;*

(c) for every $a \neq 1$ *there exists an integer* $N(a)$ *such that for every integer* $n \geq N(a)$ *there is a starting point* M_0 *which is* n*-periodic for* F_a^2*;*

(d) each point $M_0 \in \mathbb{R}_*^{+3} \setminus \mathcal{H}_a$ *has sensitivity to initial conditions: it exists a number* $\delta_a(M_0) > 0$ *such that in every neighborhood of* M_0 *there exists* M_0'*, such that* $d(F_a^{2n}(M_0), F_a^{2n}(M_0')) \geq \delta_a(M_0)$ *for infinitely many* n*.*

These four properties are denoted in [17]: invariant pointwise chaotic behavior (IPCB).

Proof. Its principle is the same as in papers [1, 17]. From the analyticity, the two exclusive possibilities are obvious (recall that the values of θ are the same in $\mathcal{W}_a^+$ and in $\mathcal{W}_a^-$). What does

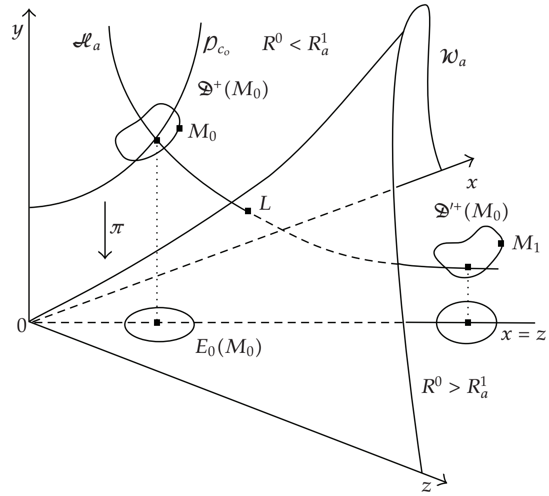

Figure 2

happen in each of them? In the first case, Lemma 7.16 shows that the constant value of θ_a is $1/4$. So all points of $\mathbb{R}_*^{+3} \setminus \mathscr{H}_a$ have the same period 4 for F_a^2, and by continuity all points have period 8 for F_a; so we have $a = 1$.

In the second case, the density of points M_0 of cases (a) and (b) comes from the nonconstancy of θ_a on each little ball. The result of point (c) is similar to this one of [1] or [17]: fractions r/n with prime integer r not factor of n exist in $]\alpha, \beta[$ if $n \geq N(a)$ (recall that the image of θ_a depends only on (a), by the classical Chebychef's inequalities and a classical majorization of the number of distinct prime factors of an integer n by $C(\ln n / \ln(\ln n))$. For the sensitivity to initial conditions, we know that the dynamical system $(\mathbb{R}_*^{+2}, T_a(c_0))$ has such a pointwise sensitivity. So if $M_0 \in \mathbb{R}_*^{+3} \setminus \mathscr{H}_a$, for every ball $B(M_0, \varepsilon)$ the set $\pi(B(M_0, \varepsilon) \cap Q_0)$ is a neighborhood of $m_0 = \pi(M_0)$ (Q_0 is the quadric which pass through M_0); so there is a point m'_0 in this neighborhood such that $d(T_a(c_0)^n(m_0), T_a(c_0)^n(m'_0)) \geq \delta(m_0)$. If we put $M'_0 = \pi^{-1}(m'_0)$ (remember that Q_0 is a graph), we have $d(F_a^{2n}(M_0), F_a^{2n}(M'_0)) \geq \delta(m_0)$ for infinitely many n. $\square$

It is also possible to have more information about the behavior of the function θ_a, by using Proposition 2.4, Theorem 7.8, Lemma 6.2, formula (6.3), and a natural 3-dimensional generalization of Zeeman's differential method easy to prove (see [11, 19]).

Proposition 7.18. *One has the following limit result:*

$$\lim_{M_0 \to L,\, M_0 \notin \mathscr{H}_a} \theta_a(M_0) = \frac{1}{\pi} \cos^{-1} \frac{\sqrt{1+a} + a - 1}{2a}. \tag{7.40}$$

So the image of the function θ_a contains the interval $I_a^+ :=]1/4, (1/\pi)\cos^{-1}((\sqrt{1+a} + a - 1)/2a)[$ if $a > 1$ or the interval $I_a^- :=](1/\pi)\cos^{-1}((\sqrt{1+a} + a - 1)/2a), 1/4[$ if $a < 1$.

Then one has

$$\bigcup_{a>0} Im(\theta_a) \supset \left] \frac{1}{\pi}\cos^{-1}\frac{3}{4}, \frac{1}{3} \right[, \tag{7.41}$$

and $(1/\pi)\cos^{-1}(3/4) = 0,230053456\ldots$. *Moreover,* $1/3 \notin \bigcup_{a>0} Im(\theta_a)$ *and* $1/5 \notin \bigcup_{a>0} Im(\theta_a)$.

Proof. The formula (7.40) comes from the extension of Zeeman's differential method and from the eigenvalues of $dF_a(L)$, the intervals I_a^+ and I_a^- comes from Lemma 7.16 and the connexity of $\mathbb{R}_*^{+3} \setminus \mathscr{H}_a$, and the formula (7.41) comes from the variations of the function of a in formula (7.40). □

Of course, formula (7.41) enables us to determine a set of even periods of solutions $(u_n)_n$ of order 3 Lyness' difference equation, for some initial points (u_2, u_1, u_0) and some $a > 0$.

Corollary 7.19. *Every even integer* $p \geq 42$ *is the minimal period of some solution of an order 3 Lyness' difference equation* $u_{n+3}u_n = a + u_{n+2} + u_{n+1}$, *for some* $a > 0$.

Proof. It sufficies to use exactly the same method as in [1, Theorem 4]. We do not make here the calculations, and we give only their principle: first, the use of the prime numbers theorem and of a computer gives $N = 1541$; then the method of successives blocks gives $N = 54$; at last, tests on the integers less than 54 give $N = 21$. □

Problem 1. What is exactly the interval $\bigcup_{a>0} \mathrm{Im}(\theta_a)$?

So we stop here the study of order 3 Lyness' difference equation by other methods than in [4] and refer the lector to [4] for another study and more informations.

8. General Conjectures about Order q Lyness' Difference Equation

We give here conjectures about the behavior of the map F_a in $\mathbb{R}_*^{+q}$, which extend known results of [1, 19] in case $q = 2$, and of [4], or Section 7 of the present paper, in case $q = 3$. These conjectures seem to be common with these ones of [6]. The first one extends a conjecture in [5].

Conjecture 9. Let $q = 2m - 1$ or $q = 2m$ ($m \geq 2$, $q \geq 4$). Then there are m differential invariants $G_1, G_2, \ldots, G_m$ which tend to $+\infty$ at the infinite point of the locally compact space $\mathbb{R}_*^{+q}$ and satisfy the Conjectures 1, 2, 3, and 4 of Section 3.2.

The following conjecture describes the behavior of the map F_a when restricted to the invariant set defined by the previous invariant.

Conjecture 10. Let $q = 2m$ or $q = 2m + 1$, $m \geq 1$. There is an open dense subset U_a^q of $\mathbb{R}_*^{+q}$ such that for every $M_0 \in U_a^q$ the following properties are satisfied by the invariant set $\mathcal{C}^+(M_0) = \bigcap_{i=1}^{[(q+1)/2]} \{M \in \mathbb{R}_*^{+3} \mid G_i(M) = G_i(M_0)\}$ (which contains M_0).

(1) If $q = 2m$ (there are m invariants), $\mathcal{C}^+(M_0)$ is a m-manifold which is homeomorphic to the m-dimensional torus $\mathbb{T}^m$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$), and the restriction of F_a to $\mathcal{C}^+(M_0)$ is conjugated to a well defined m-rotation $\mathcal{R}_{(\theta_1,\ldots,\theta_m)}$ on $\mathbb{T}^m$: $(z_1, z_2, \ldots, z_m) \mapsto (z_1e^{2i\pi\theta_1}, z_2e^{2i\pi\theta_2}, \ldots, z_me^{2i\pi\theta_m})$.

(2) If $q = 2m + 1$ (then there are $m + 1$ invariants), $\mathcal{C}^+(M_0)$ is a m-manifold which is homeomorphic to the union of two m-dimensional torus, that is, $\{-1, 1\} \times \mathbb{T}^m$, and the restriction of F_a to $\mathcal{C}^+(M_0)$ is conjugated to the following map:

$$(\varepsilon, z_1, z_2, \ldots, z_m) \longmapsto \left(-\varepsilon, \phi_\varepsilon^1(z_1), \phi_\varepsilon^2(z_2), \ldots, \phi_\varepsilon^m(z_m)\right), \tag{8.1}$$

where $\varepsilon = \pm 1$, $\phi_1^j = Id$, $\phi_{-1}^j(z) = ze^{2i\pi\theta_j}$.

(3) In the two cases, the map $M_0 \mapsto (\theta_1, \ldots, \theta_m)$ is continuous, and it is constant on no open set in U_a^q.

Moreover, the limit of $(\theta_1, \ldots, \theta_m)$ when $M_0 \mapsto L$ is, apart from a permutation, the angles of Theorem 5.4 on $dF_a(L)$ if q is even, and twice these angles if q is odd.

We give some comments. Put $p = [(q+1)/2]$. If we define $\mathcal{G} := (G_1, \ldots, G_p) : \mathbb{R}_*^{+q} \to \mathbb{R}_*^{+p}$, the set V of points where the rank of $d\mathcal{G}$ is maximum is open, and the set $\mathbb{R}_*^{+q} \setminus V$, is defined by $\binom{q}{p}$ algebraic equations (nullity of determinants), and so it is a closed set with empty interior: V is dense. The set V is invariant under the action of F_a, and so if $M_0 \in V$ its orbit is included in V. It is possible that one can take V for the set U_a^q of Conjecture 9. Perhaps another approach when q is odd could be to find sufficiently invariants for the map F_a^2.

References

[1] G. Bastien and M. Rogalski, "Global behavior of the solutions of Lyness' difference equation $u_{n+2}u_n = u_{n+1} + a$," *Journal of Difference Equations and Applications*, vol. 10, no. 11, pp. 977–1003, 2004.

[2] J. Milnor, *Morse Theory*, Annals of Mathematics Studies, no. 51, Princeton University Press, Princeton, NJ, USA, 1963.

[3] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[4] A. Cima, A. Gasull, and V. Mañosa, "Dynamics of the third order Lyness' difference equation," *Journal of Difference Equations and Applications*, vol. 13, no. 10, pp. 855–884, 2007.

[5] M. Gao, Y. Kato, and M. Ito, "Some invariants for kth-order Lyness equation," *Applied Mathematics Letters*, vol. 17, no. 10, pp. 1183–1189, 2004.

[6] A. Cima, A. Gasull, and V. Mañosa, "Some properties of the k-dimensional Lyness' map," *Journal of Physics A*, vol. 41, no. 28, Article ID 285205, 18 pages, 2008.

[7] G. Bastien and M. Rogalski, "New method for the study of solutions of the difference equation $u_{n+2} = (u_{n+1} + a)/(u_n + b)$," *Radovi Matematički*, vol. 12, no. 2, pp. 135–152, 2004.

[8] M. R. S. Kulenović, "Invariants and related Liapunov functions for difference equations," *Applied Mathematics Letters*, vol. 13, no. 7, pp. 1–8, 2000.

[9] G. Bastien and M. Rogalski, "Global behavior of the solutions of the k-lacunary order $2k$ Lyness' difference equation $u_n = (u_{n-k} + a)/(u_{n-2k})$ in $\mathbb{R}_*^+$, and of other more general equations," *Journal of Difference Equations and Applications*, vol. 13, no. 1, pp. 79–88, 2007.

[10] J. Lafontaine, *Introduction aux Variétés Différentielles*, Presses Universitaires de Grenoble, Grenoble, France, 1996.

[11] G. Bastien and M. Rogalski, "On the algebraic difference equations $u_{n+2} + u_n = \psi(u_{n+1})$ in $\mathbb{R}$, related to a family of elliptic quartics in the plane," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 822–844, 2007.

[12] A. Cima, A. Gasull, and F. Mañosas, "On periodic rational difference equations of order k," *Journal of Difference Equations and Applications*, vol. 10, no. 6, pp. 549–559, 2004.

[13] A. Cima, A. Gasull, and V. Mañosa, "Dynamics of some rational discrete dynamical systems via invariants," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 16, no. 3, pp. 631–645, 2006.

[14] M. Csörnyei and M. Laczkovich, "Some periodic and non-periodic recursions," *Monatshefte für Mathematik*, vol. 132, no. 3, pp. 215–236, 2001.

[15] L. Berg and S. Stević, “Periodicity of some classes of holomorphic difference equations,” *Journal of Difference Equations and Applications*, vol. 12, no. 8, pp. 827–835, 2006.
[16] J. Matsukidaira and D. Takahashi, “Third-order integrable difference equations generated by a pair of second-order equations,” *Journal of Physics A*, vol. 39, no. 5, pp. 1151–1161, 2006.
[17] G. Bastien and M. Rogalski, “On the algebraic difference equations $u_{n+2}u_n = \psi(u_{n+1})$ in $\mathbb{R}_*^+$, related to a family of elliptic quartics in the plane,” *Advances in Difference Equations*, no. 3, pp. 227–261, 2005.
[18] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjecture*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2002.
[19] E. C. Zeeman, *Geometric Unfolding of a Difference Equation*, Hertford College, Oxford, UK, 1996.