Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 104310, 13 pages
doi:10.1155/2009/104310

Research Article

Exponential Stability of Difference Equations with Several Delays: Recursive Approach

Leonid Berezansky[1] and Elena Braverman[2]

[1] *Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel*
[2] *Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary AB, Canada T2N 1N4*

Correspondence should be addressed to Elena Braverman, maelena@math.ucalgary.ca

Received 8 January 2009; Revised 16 April 2009; Accepted 23 April 2009

Recommended by Istvan Gyori

We obtain new explicit exponential stability results for difference equations with several variable delays and variable coefficients. Several known results, such as Clark's asymptotic stability criterion, are generalized and extended to a new class of equations.

1. Introduction and Preliminaries

In this paper we study stability of a scalar linear difference equation with several delays,

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l(n)x(h_l(n)), \quad h_l(n) \le n,\ n \ge n_0, \tag{1.1}$$

where $h_l(n)$ is an integer for any $l = 1, \ldots, m$, n is an integer, $n_0 \ge 0$. Stability of (1.1) and relevant nonlinear equations has been an intensively developed area during the last two decades.

Let us compare stability methods for delay differential equations and delay difference equations. Many of the methods previously used for differential equations have also been applied to difference equations. However, there are at least two methods which are specific for difference equations. The first approach is reducing a solution of a delay difference equation to the values of a solution of a delay differential equation with piecewise constant arguments at integer points. The second method is based on a recursive form of difference equations and is described in detail later. In this paper we obtain new stability results based on the recursive solution representation.

For (1.1) everywhere below we assume that $h_l(n) \geq n - T$, $n \geq n_0 \geq 0$, that is, the system has a finite memory, and the following initial conditions are defined

$$x(n) = \varphi(n), \qquad n_0 - T \leq n \leq n_0. \tag{1.2}$$

Definition 1.1. Equation (1.1) is *exponentially stable* if there exist constants $M > 0$, $\lambda \in (0,1)$ such that for every solution $\{x(n)\}$ of (1.1) and (1.2) the inequality

$$|x(n)| \leq M\lambda^{n-n_0}\left(\max_{n_0-T\leq k\leq n_0}\{|\varphi(k)|\}\right) \tag{1.3}$$

holds for all $n \geq n_0$, where M, λ do not depend on $n_0 \geq 0$.

Equation (1.1) is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\max_{n_0-T\leq k\leq n_0}\{|\varphi(k)|\} < \delta$ implies $|x(n)| < \varepsilon$, $n \geq n_0$; if δ does not depend on n_0, then (1.1) is *uniformly stable*.

Equation (1.1) is *attractive* if for any $\varphi(k)$ a solution tends to zero $\lim_{n\to\infty}x(n) = 0$. It is *asymptotically stable* if it is both stable and attractive.

One of the methods to establish stability of difference equations is based on a recursive form of these equations; see the monographs [1, 2]. The following result was also obtained by this method.

Lemma 1.2 ([3, 4]). *Consider the nonlinear delay difference equation*

$$x_{n+1} = f(n, x_n, \ldots, x_{n-T}), \qquad n \geq 0. \tag{1.4}$$

Assume that $f : \mathbb{N} \times \mathbb{R}^{T+1} \to \mathbb{R}$ *satisfies*

$$\left|f(n, u_0, \ldots, u_T)\right| \leq b\max\{|u_0|, \ldots, |u_T|\}, \tag{1.5}$$

for some constant $b < 1$, *and for all* $(n, u_0, \ldots, u_T) \in \mathbb{N} \times \mathbb{R}^{T+1}$. *Then*

$$|x_n| \leq b^{n/(T+1)}M_0, \qquad n \geq 0, \tag{1.6}$$

for every solution $\{x_n\}$ *of* (1.4), *where* $M_0 = \max_{-T\leq i\leq 0}\{|x_i|\}$. *In particular, the zero solution of* (1.4) *is globally exponentially stable.*

This result was applied to a more general than (1.1) nonlinear difference equation

$$x_{n+1} - x_n = -\sum_{k=0}^{N} a_k(n)x_{n-k} + f(n, x_n, \ldots, x_{n-T}). \tag{1.7}$$

Without loss of generality, we can suppose that $N \le T$. We assume that there exist constants $b_n \ge 0$ such that

$$\left| f(n, u_0, \ldots, u_T) \right| \le b_n \max\{|u_0|, \ldots, |u_T|\}, \tag{1.8}$$

for all $n \ge 0$ and $(u_0, \ldots, u_T) \in \mathbb{R}^{T+1}$.

Similar argument leads to the following result.

Lemma 1.3 ([4]). *Assume that for n large enough inequality* (1.8) *holds and there exists a constant* $\gamma \in (0,1)$ *such that*

$$c_n := \left| 1 - \sum_{k=0}^{N} a_k(n) \right| + \sum_{k=1}^{N} |a_k(n)| \sum_{m=n-k}^{n-1} \left(b_m + \sum_{k=0}^{N} |a_k(m)| \right) + b_n \le \gamma. \tag{1.9}$$

Then the zero solution of (1.7) *is globally exponentially stable. Moreover, if* (1.8) *and* (1.9) *hold for* $n \ge 0$, *then*

$$|x_n| \le \gamma^{n/(N+T+1)} \max\{|x_N|, \ldots, |x_{-T}|\}, \quad n \ge N, \tag{1.10}$$

for every solution $\{x_n\}$ *of* (1.7).

Recently several results on exponential stability of high-order difference equations appeared where the results are based on the recursive representations; see, for example, [5, 6]. In particular, in [6, Corollary 7] contains the following statement.

Lemma 1.4 ([6]). *If there exists* $\lambda \in (0,1)$ *such that*

$$\Lambda_n = \left| \prod_{j=0}^{N} a(n-j) - c(n) \right| + \sum_{s=1}^{N} \left| \prod_{j=0}^{s-1} a(n-j) \right| |c(n-s)| \le \lambda, \tag{1.11}$$

for large n, *then the zero solution of the equation*

$$x(n+1) = a(n)x(n) - c(n)x(n-N) \tag{1.12}$$

is globally exponentially stable.

In the present paper we obtain some new stability results for (1.1) with several variable delays. In contrast to many other stability tests, we consider the case when the sum of coefficients $\sum_{k=1}^{m} a_k(n)$ or some of its subsum is allowed to be in the interval $(0,2]$, not just $(0,1]$. We illustrate our results with several examples.

2. Main Results

Now we can proceed to the main results of this paper. Let us note that any sum where the lower index exceeds the upper index is assumed to vanish.

Theorem 2.1. *Suppose that there exist a set of indices $I \subset \{1, 2, \ldots, m\}$ and $\gamma \in (0,1)$ such that for n sufficiently large*

$$\sum_{k\in I}|a_k(n)|\sum_{j=h_k(n)}^{n-1}\sum_{l=1}^{m}\left|a_l(j)\right| + \sum_{l\notin I}|a_l(n)| + \left|1 - \sum_{k\in I}a_k(n)\right| \le \gamma. \tag{2.1}$$

Then (1.1) *is exponentially stable.*

Proof. Since

$$\begin{aligned} x(n+1) &= x(n) - \sum_{k\in I} a_k(n)x(h_k(n)) - \sum_{l\notin I} a_l(n)x(h_l(n)) \\ &= \sum_{k\in I} a_k(n)\sum_{j=h_k(n)}^{n-1}\left[x(j+1) - x(j)\right] + x(n) - \sum_{k\in I} a_k(n)x(n) - \sum_{l\notin I} a_l(n)x(h_l(n)) \\ &= -\sum_{k\in I} a_k(n)\sum_{j=h_k(n)}^{n-1}\sum_{l=1}^{m} a_l(j)x(h_l(j)) + x(n)\left[1 - \sum_{k\in I} a_k(n)\right] - \sum_{l\notin I} a_l(n)x(h_l(n)), \end{aligned} \tag{2.2}$$

then

$$\begin{aligned} |x(n+1)| &\le \left[\sum_{k\in I}|a_k(n)|\sum_{j=h_k(n)}^{n-1}\sum_{l=1}^{m}\left|a_l(j)\right| + \sum_{l\notin I}|a_l(n)| + \left|1 - \sum_{k\in I} a_k(n)\right|\right]\max_{j\le n}\left|x(j)\right| \\ &\le \gamma \max_{j\le n}\left|x(j)\right|. \end{aligned} \tag{2.3}$$

By Lemma 1.2, (1.1) is exponentially stable. □

We can reformulate Theorem 2.1 in the following equivalent form.

Theorem 2.2. *Suppose that there exist a set of indices $I \subset \{1, 2, \ldots, m\}$ and $\varepsilon \in (0,1)$ such that for n sufficiently large*

$$\sum_{k\in I}|a_k(n)|\sum_{j=h_k(n)}^{n-1}\sum_{l=1}^{m}\left|a_l(j)\right| + \sum_{l\notin I}|a_l(n)| \le \min\left\{\sum_{k\in I} a_k(n),\, 2 - \sum_{k\in I}a_k(n)\right\} - \varepsilon. \tag{2.4}$$

Then (1.1) *is exponentially stable.*

Assuming first $I = \{1\}$ and then $I = \{1, 2, \ldots, m\}$, we obtain the following two corollaries for an equation with a nondelay term.

Corollary 2.3. *Let $0 < \gamma < 1$ and $\sum_{l=2}^{m}|a_l(n)| + |1 - a_1(n)| \le \gamma$ for n large enough. Then the equation*

$$x(n+1) - x(n) = -a_1(n)x(n) - \sum_{l=2}^{m} a_l(n)x\big(g_l(n)\big) \tag{2.5}$$

is exponentially stable.

Example 2.4. Consider the equation

$$x(n+1) - x(n) = -1.5x(n) - 0.3\sin(n)x(h(n)), \tag{2.6}$$

with an arbitrary bounded delay $h(n)$: $n - T \le h(n) \le n$ for some integer $T > 0$ and any n. Since $0.3|\sin n| + |1 - 1.5| \le 0.8 < 1$, then by Corollary 2.3 this equation is exponentially stable. Lemma 1.4 is formulated for a constant delay, some other tests do not apply since $|-1.5| > 1$.

Corollary 2.5. *Suppose that for some $\gamma \in (0,1)$ the following inequality is satisfied for n large enough:*

$$\sum_{k=2}^{m}|a_k(n)| \sum_{j=g_k(n)}^{n-1} \sum_{l=1}^{m}\big|a_l(j)\big| + \left|1 - \sum_{k=1}^{m} a_k(n)\right| \le \gamma. \tag{2.7}$$

Then (2.5) *is exponentially stable.*

Example 2.6. By Corollary 2.5 the equation

$$x(n+1) - x(n) = -(0.5 + 0.1\sin n)x(n) - (0.6 - 0.1\sin n)x(n-1) \tag{2.8}$$

is exponentially stable, since $|0.5+0.1\sin n|+|0.6-0.1\sin n| = 1.1$ and $0.7\cdot 1.1+|1-1.1| = 0.87 < 1$.

Now let us assume that all coefficients are proportional. Such equations arise as linear approximations of nonlinear difference equations in mathematical biology. Then a straightforward computation leads to the following result.

Corollary 2.7. *Suppose that all coefficients are proportional $a_l(n) = A_l r(n)$, $l = 1, \ldots, m$, there exist $r_0 > 0$, $\varepsilon > 0$ and a set of indices $I \subset \{1, 2, \ldots, m\}$, such that $r(n) \ge r_0 > 0$ and*

$$\sum_{l\in I}|A_l| \sum_{k=h_l(n)}^{n-1} r(k) \le \frac{\min\{r(n)\sum_{l\in I} A_l,\ 2 - r(n)\sum_{l\in I} A_l\} - r(n)\sum_{l\notin I}|A_l| - \varepsilon}{r(n)\sum_{l=1}^{m}|A_l|}. \tag{2.9}$$

Then (1.1) *is exponentially stable.*

Assuming constant coefficients and $I = \{1, 2, \ldots, m\}$ we obtain the following corollary.

Corollary 2.8. *Suppose that all coefficients are constants $a_l(n) \equiv a_l$ and*

$$\limsup_{n\to\infty} \sum_{l=1}^{m} |a_l| \sum_{l=1}^{m} |a_l| (n - h_l(n)) < \min\left\{ \sum_{l=1}^{m} a_l,\ 2 - \sum_{l=1}^{m} a_l \right\}. \tag{2.10}$$

Then (1.1) *is exponentially stable.*

Remark 2.9. Corollary 2.8 for the case $0 < \sum_{l=1}^{m} a_l < 1$ was obtained in Proposition 4.1 of [7].

Now let us consider the equation with one nondelay and one delay terms

$$x(n+1) - x(n) = -a(n)x(n) - b(n)x(h(n)). \tag{2.11}$$

Choosing $I = \{1\}$, $I = \{1,2\}$, we obtain Parts (1) and (2) of Corollary 2.10, respectively.

Corollary 2.10. *Suppose that there exists $\gamma \in (0,1)$ such that at least one of the following conditions holds for n sufficiently large:*

(1) $|b(n)| + |1 - a(n)| \le \gamma$;

(2) $|b(n)| \sum_{k=h(n)}^{n-1} [|a(k)| + |b(k)|] + |1 - a(n) - b(n)| \le \gamma$.

Then (2.11) *is exponentially stable.*

Let us now proceed to equations with three terms in the right-hand side

$$x(n+1) - x(n) = -a(n)x(n) - b(n)x(h(n)) - c(n)x(g(n)). \tag{2.12}$$

For $I = \{1\}$, $\{1,2,3\}$, $\{1,2\}$ and $\{1,3\}$ we obtain Parts (1), (2), (3), and (4), respectively.

Corollary 2.11. *Suppose that there exists $\gamma \in (0,1)$ such that at least one of the following conditions holds for n sufficiently large:*

(1) $|b(n)| + |c(n)| + |1 - a(n)| \le \gamma$;

(2) $|b(n)| \sum_{k=h(n)}^{n-1} [|a(k)| + |b(k)| + |c(k)|] + |c(n)| \sum_{k=g(n)}^{n-1} [|a(k)| + |b(k)| + |c(k)|] + |1 - a(n) - b(n) - c(n)| \le \gamma$;

(3) $|b(n)| \sum_{k=h(n)}^{n-1} [|a(k)| + |b(k)| + |c(k)|] + |c(n)| + |1 - a(n) - b(n)| \le \gamma$;

(4) $|c(n)| \sum_{m=g(n)}^{n-1} [|a(k)| + |b(k)| + |c(k)|] + |b(n)| + |1 - a(n) - c(n)| \le \gamma$.

Then (2.12) *is exponentially stable.*

Theorem 2.1 and its corollaries imply new explicit conditions of exponential stability for autonomous difference equations with several delays, as well as a new justification for known ones.

Consider the autonomous equation

$$x(n+1) - x(n) = -a_1 x(n) - \sum_{l=2}^{m} a_l x(n - h_l), \tag{2.13}$$

where $h_l > 0$. Choosing $I = \{1\}$ we immediately obtain the following stability test.

Corollary 2.12. *Let* $\sum_{l=2}^{m}|a_l| < \min\{a_1, 2-a_1\}$*. Then* (2.13) *is exponentially stable.*

Remark 2.13. This result is well known; see, for example, [8] for $m = 2$ as well as some results for autonomous equations below. We presented it just to illustrate our method.

Further, Theorem 2.1 and Corollaries 2.8 and 2.11 can be reformulated for (2.13) as follows.

Corollary 2.14. *Suppose that there exists a set of indices* $I \subset \{1, 2, \dots, m\}$*, with* $1 \in I$*, such that*

$$\sum_{l=1}^{m}|a_l|\sum_{k\in I}|a_k|h_k + \sum_{l\notin I}|a_l| + \left|1 - \sum_{k\in I}a_k\right| < 1. \tag{2.14}$$

Then (2.13) *is exponentially stable.*

Corollary 2.15. *If* $\sum_{l=1}^{m}|a_l|\sum_{k=2}^{m}|a_k|h_k < \min\{\sum_{k=1}^{m}a_k,\ 2-\sum_{k=1}^{m}a_k\}$*, then* (2.13) *is exponentially stable.*

Consider now an autonomous equation with two delays:

$$x(n+1) - x(n) = -a_0x(n) - a_1x(n-h_1) - a_2x(n-h_2), \tag{2.15}$$

where $h_1 > 0$, $h_2 > 0$.

Corollary 2.16. *Suppose that at least one of the following conditions holds*

(1) $|a_1| + |a_2| + |1-a_0| < 1$;

(2) $(|a_0| + |a_1| + |a_2|)(|a_1|h_1 + |a_2|h_2) + |1-a_0-a_1-a_2| < 1$;

(3) $|a_1|h_1(|a_0| + |a_1| + |a_2|) + |1-a_0-a_1| < 1$;

(4) $|a_2|h_2(|a_0| + |a_1| + |a_2|) + |1-a_0-a_2| < 1$.

Then (2.15) *is exponentially stable.*

Let us present two more results which can be easily deduced from the recursive representation of solutions. To this end we consider the equation

$$x(n+1) = \sum_{l=1}^{m}a_l(n)x(h_l(n)), \tag{2.16}$$

which is a different form of (1.1).

We recall that we assume $n - h_l(n) \le T$ for all delays $h_l(n)$ in this paper.

Theorem 2.17. *Suppose that there exists* $\lambda \in (0,1)$ *such that*

$$\limsup_{n\to\infty}\sum_{l=1}^{m}|a_l(n)|\sum_{j=1}^{m}\left|a_j(h_l(n)-1)\right| \le \lambda. \tag{2.17}$$

Then (2.16) *is exponentially stable.*

Proof. Without loss of generality we can assume that the expression under lim sup in (2.17) does not exceed some $\lambda < 1$ for $n \geq n_0$. Since

$$x(h_l(n)) = \sum_{j=1}^{m} a_j(h_l(n)-1)x\big(h_j(h_l(n)-1)\big), \tag{2.18}$$

then

$$\begin{aligned} x(n+1) &= \sum_{l=1}^{m} a_l(n)x(h_l(n)) \\ &= \sum_{l=1}^{m} a_l(n)\sum_{j=1}^{m} a_j(h_l(n)-1)x\big(h_j(h_l(n)-1)\big). \end{aligned} \tag{2.19}$$

Hence for $n \geq n_0 + 2T + 1$ we have

$$\begin{aligned} |x(n+1)| &\leq \sum_{l=1}^{m} |a_l(n)|\sum_{j=1}^{m} \big|a_j(h_l(n)-1)\big| \max_{n-2T-1\leq k\leq n} |x(k)| \\ &\leq \lambda \max_{n-2T-1\leq k\leq n} |x(k)|. \end{aligned} \tag{2.20}$$

Thus by Lemma 1.2 $|x(n)| \leq M_0\mu^{n-n_0}$, where $\mu = \lambda^{1/(2T+2)}$, $M_0 = \lambda^{-2T-1}\max_{-T\leq k\leq n_0}|x(k)|$, for $n \geq n_0$, so (2.16) is exponentially stable. □

Theorem 2.18. *Suppose that there exists $\lambda \in (0,1)$, $N \in \mathbb{N}$ such that*

$$\limsup_{n\to\infty} \prod_{j=0}^{N} \sum_{l=1}^{m} \big|a_l(n-j)\big| \leq \lambda. \tag{2.21}$$

Then (2.16) *is exponentially stable.*

Proof. Without loss of generality we can assume that the expression under lim sup in (2.21) does not exceed some $\lambda < 1$ for $n \geq n_0$. Since

$$\begin{aligned} |x(n+1)| &\leq \sum_{l=1}^{m} |a_l(n)| \max_{n-T\leq k\leq n} |x(k)| \\ &\leq \prod_{j=0}^{N} \sum_{l=1}^{m} \big|a_l(n-j)\big| \max_{n-NT\leq k\leq n} |x(k)| \\ &\leq \lambda \max_{n-NT\leq k\leq n} |x(k)|, \end{aligned} \tag{2.22}$$

then the reference to Lemma 1.2 completes the proof. □

3. Discussion and Examples

Let us comment that Theorem 2.18 (see also Corollary 2.12) generalizes the result of Clark [8] that $|p| + |q| < 1$ is a sufficient condition for the asymptotic stability of the difference equation

$$x(n+1) + px(n) + qx(n-N) = 0. \tag{3.1}$$

We note that there are not really many publications on difference equations with variable delays, and the present paper partially fills up this gap. In particular, Theorem 2.18 gives the same stability condition for the equation with variable delays

$$x(n+1) + px(h(n)) + qx\big(g(n)\big) = 0 \tag{3.2}$$

once the delays are bounded: $n - h(n) \leq T_1$, $n - g(n) \leq T_2$.

The following example outlines the sharpness of the condition that the delays are bounded in Theorems 2.1, 2.2, 2.17, and 2.18.

Example 3.1. The equation with constant coefficients

$$x(n+1) = 0.4x(n) + 0.1x(0), \quad n = 0, 1, \ldots \tag{3.3}$$

satisfies all assumptions of Theorems 2.1, 2.2, 2.17, and 2.18 but the boundedness of the delay. Since the solution $x(n)$ with $x(0) = 6$ tends to 1 as $n \to \infty$, then the zero solution of (3.3) is neither asymptotically nor exponentially stable. Here even the condition $\lim_{n\to\infty} h_l(n) = \infty$ is not satisfied.

Example 3.2. Let us demonstrate that in the case when the arguments tend to infinity but the delays are not bounded and all other conditions of Theorem 2.18 are satisfied, this does not imply exponential stability. The equation

$$x(n) = 0.5x\left(\left[\frac{n}{2}\right]\right), \quad n = 1, 2, \ldots, \tag{3.4}$$

where $[x]$ is the integer part of x, is asymptotically, but not exponentially stable. Really, its solution

$$x(0), \frac{1}{2}x(0), \frac{1}{4}x(0), \frac{1}{4}x(0), \frac{1}{8}x(0), \frac{1}{8}x(0), \frac{1}{8}x(0), \frac{1}{8}x(0), \frac{1}{16}x(0), \ldots \tag{3.5}$$

is nonincreasing by the absolute value and

$$x(n) = \frac{x(1)}{n}, \tag{3.6}$$

for any $n = 2^k$, $k = 0, 1, 2, \ldots$, so $\lim_{n\to 0} x(n) = 0$ for any $x(1)$; the equation is asymptotically stable. Since the solution decay is not faster than $1/n$, then the equation is not exponentially stable.

Next, let us compare Corollary 2.10 and Theorem 4 of [6] which is also Lemma 1.4 of the present paper.

Example 3.3. Consider (1.12), where

$$a(n) = \begin{cases} 0.8, & \text{if } n \text{ is even,} \\ 0.9, & \text{if } n \text{ is odd,} \end{cases}$$
$$c(n) = \begin{cases} 0.2, & \text{if } n \text{ is even,} \\ 0.8, & \text{if } n \text{ is odd.} \end{cases} \tag{3.7}$$

Then, (1.12) is exponentially stable for any N by Corollary 2.10, Part (1); here $\lambda = 0.9$, $\gamma = 8/9$.

If, for example, we assume $N = 1$, then (1.11) has the form

$$|a(n)a(n-1) - c(n)| + |a(n)|\ |c(n-1)| \leq \lambda < 1, \tag{3.8}$$

which is not satisfied for even n, so Lemma 1.4 cannot be applied to deduce exponential stability.

Example 3.4. We will modify Example 8 in [6] to compare Theorem 2.18 and Lemma 1.4. Consider

$$a(n) = \begin{cases} \dfrac{1}{25(N+1)}, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd,} \end{cases}$$
$$c(n) = \begin{cases} \dfrac{1}{25(N+1)}, & \text{if } n \text{ is even,} \\ d, & \text{if } n \text{ is odd.} \end{cases} \tag{3.9}$$

Let us compare constants d such that (1.12) is exponentially stable for $N = 1$,

$$x(n+1) = a(n)x(n) - c(n)x(n-1). \tag{3.10}$$

By Theorem 2.18 we obtain stability whenever

$$\left(\frac{1}{50} + \frac{1}{50}\right)(2 + |d|) < 1, \quad \text{or} \quad |d| < 23. \tag{3.11}$$

Condition (3.8) of Lemma 1.4 becomes

$$\left|\frac{1}{25} - \frac{1}{50}\right| + \frac{1}{50}|d| < 1 \tag{3.12}$$

for even n and

$$\left|\frac{1}{25} - d\right| + 2\,\frac{1}{50} < 1 \tag{3.13}$$

for odd n, which gives the intervals $|d| < 49$ and $-0.92 < d < 1$, respectively. Finally, Lemma 1.4 implies exponential stability for $d \in (-0.92, 1) \subset (-23, 23)$.

In addition to Theorems 2.1, 2.2, 2.17, and 2.18, let us review some other known stability conditions for equations with several delays. For comparison, we will cite the following two results.

Theorem A ([9–14]). *Suppose that*

$$\begin{gathered} a_j(n) \geq 0, \quad 0 \leq j \leq m, \quad \sum_{j=0}^{m} a_j(n) > 0, \quad \sum_{n=0}^{\infty}\sum_{j=0}^{m} a_j(n) = \infty, \\ \sup_{n\geq m} \sum_{k=n-m}^{n}\sum_{j=0}^{m} a_j(k) < \frac{3}{2} + \frac{1}{2(m+1)}. \end{gathered} \tag{3.14}$$

Then the equation with several delays

$$x(n+1) - x(n) = -\sum_{j=0}^{m} a_j(n)x(n-j), \quad n = 0, 1, 2, \ldots \tag{3.15}$$

is globally asymptotically stable.

Theorem B ([15]). *Suppose that* $a_l(n) \equiv a_l > 0$ *and*

$$\sum_{l=1}^{m} a_l \limsup_{n\to\infty}(n - h_l(n)) < 1 + \frac{1}{e} - \sum_{l=1}^{m} a_l. \tag{3.16}$$

Then (1.1) *is asymptotically stable.*

Let us note that unlike Theorems A and B we do not assume that coefficients are either nonnegative (as in Theorem A) or constant (as in Theorem B). Further, let us compare our stability tests with known results, including Theorems A and B.

Example 3.5. Consider the equation

$$x(n+1) = a(n)x(n-4) + c(n)x(n-6), \tag{3.17}$$

where

$$a(n) = \begin{cases} 2, & \text{if } n \text{ is even}, \\ 0.1, & \text{if } n \text{ is odd}, \end{cases}$$
$$c(n) = \begin{cases} 3, & \text{if } n \text{ is even}, \\ 0.05, & \text{if } n \text{ is odd}. \end{cases} \tag{3.18}$$

Repeating the proof of Theorem 2.17, we have

$$\begin{aligned} x(n+1) = a(n)[&a(n-5)x(n-9) + c(n-5)x(n-11)] \\ &+ c(n)[a(n-7)x(n-11) + c(n-7)x(n-13)]. \end{aligned} \tag{3.19}$$

By Theorem 2.17, (3.17) is exponentially stable since

$$\begin{aligned} &|a(n)|\ [|a(n-5)| + |c(n-5)|] + |c(n)|[|a(n-7)| + |c(n-7)|] \\ &= [|a(n)| + |c(n)|]\ [|a(n-5)| + |c(n-5)|] \\ &= 0.15 \cdot 5 = 0.75 < 1. \end{aligned} \tag{3.20}$$

Lemma 1.4 cannot be applied since for even n

$$|a(n)a(n-1) - c(n)| + |a(n)||c(n-1)| = |0.2 - 3| + 2 \cdot 0.05 > 2.8 > 1. \tag{3.21}$$

Theorem A fails since

$$\sum_{j=n-6}^{n} [a(j) + b(j)] > a(n) + c(n) + a(n-1) + c(n-1) = 5.15 > \frac{3}{2} + \frac{1}{4}. \tag{3.22}$$

Theorem B is applicable to equations with constant coefficients only.

Acknowledgments

The authors are very grateful to the referee for valuable comments and remarks. This paper is partially supported by Israeli Ministry of Absorption and by the NSERC Research Grant.

References

[1] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[2] S. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 3rd edition, 2005.

[3] S. Stević, "Behavior of the positive solutions of the generalized Beddington-Holt equation," *Panamerican Mathematical Journal*, vol. 10, no. 4, pp. 77–85, 2000.
[4] L. Berezansky, E. Braverman, and E. Liz, "Sufficient conditions for the global stability of nonautonomous higher order difference equations," *Journal of Difference Equations and Applications*, vol. 11, no. 9, pp. 785–798, 2005.
[5] E. Liz, "On explicit conditions for the asymptotic stability of linear higher order difference equations," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 2, pp. 492–498, 2005.
[6] H. A. El-Morshedy, "New explicit global asymptotic stability criteria for higher order difference equations," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 262–276, 2007.
[7] I. Győri, F. Hartung, and J. Turi, "On the effects of delay perturbations on the stability of delay difference equations," in *Proceedings of the 1st International Conference on Difference Equations (San Antonio, Tex, 1994)*, pp. 237–253, Gordon and Breach, Luxembourg, UK, 1995.
[8] C. W. Clark, "A delayed-recruitment model of population dynamics, with an application to baleen whale populations," *Journal of Mathematical Biology*, vol. 3, no. 3-4, pp. 381–391, 1976.
[9] L. H. Erbe, H. Xia, and J. S. Yu, "Global stability of a linear nonautonomous delay difference equation," *Journal of Difference Equations and Applications*, vol. 1, no. 2, pp. 151–161, 1995.
[10] J. S. Yu, "Asymptotic stability for a linear difference equation with variable delay," *Computers & Mathematics with Applications*, vol. 36, no. 10–12, pp. 203–210, 1998.
[11] B. G. Zhang, C. J. Tian, and P. J. Y. Wong, "Global attractivity of difference equations with variable delay," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 6, no. 3, pp. 307–317, 1999.
[12] V. V. Malygina and A. Y. Kulikov, "On precision of constants in some theorems on stability of difference equations," *Functional Differential Equations*, vol. 15, no. 3-4, pp. 239–248, 2008.
[13] H. Matsunaga, T. Hara, and S. Sakata, "Global attractivity for a nonlinear difference equation with variable delay," *Computers & Mathematics with Applications*, vol. 41, no. 5-6, pp. 543–551, 2001.
[14] Y. Muroya, E. Ishiwata, and N. Guglielmi, "Global stability for nonlinear difference equations with variable coefficients," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 1, pp. 232–247, 2007.
[15] I. Győri and F. Hartung, "Stability in delay perturbed differential and difference equations," in *Topics in Functional Differential and Difference Equations (Lisbon, 1999)*, vol. 29 of *Fields Institute Communications*, pp. 181–194, American Mathematical Society, Providence, RI, USA, 2001.