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Research Article

Stability of Equilibrium Points of Fractional Difference Equations with Stochastic Perturbations

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It is supposed that the fractional difference equation $x_{n+1} = (\mu + \sum_{j=0}^{k} a_j x_{n-j})/(\lambda + \sum_{j=0}^{k} b_j x_{n-j})$, $n = 0, 1, \ldots$, has an equilibrium point $\hat{x}$ and is exposed to additive stochastic perturbations type of $\sigma(x_n - \hat{x})\xi_{n+1}$ that are directly proportional to the deviation of the system state x_n from the equilibrium point $\hat{x}$. It is shown that known results in the theory of stability of stochastic difference equations that were obtained via V. Kolmanovskii and L. Shaikhet general method of Lyapunov functionals construction can be successfully used for getting of sufficient conditions for stability in probability of equilibrium points of the considered stochastic fractional difference equation. Numerous graphical illustrations of stability regions and trajectories of solutions are plotted.

1. Introduction—Equilibrium points

Recently, there is a very large interest in studying the behavior of solutions of nonlinear difference equations, in particular, fractional difference equations [1–38]. This interest really is so large that a necessity appears to get some generalized results.

Here, the stability of equilibrium points of the fractional difference equation

$$x_{n+1} = \frac{\mu + \sum_{j=0}^{k} a_j x_{n-j}}{\lambda + \sum_{j=0}^{k} b_j x_{n-j}}, \quad n \in Z = \{0, 1, \ldots\}, \tag{1.1}$$

with the initial condition

$$x_j = \phi_j, \quad j \in Z_0 = \{-k, -k+1, \ldots, 0\}, \tag{1.2}$$

is investigated. Here μ, λ, a_j, b_j, $j = 0, \dots, k$ are known constants. Equation (1.1) generalizes a lot of different particular cases that are considered in [1–8, 16, 18–20, 22–24, 32, 35, 37].

Put

$$A_j = \sum_{l=j}^{k} a_j, \quad B_j = \sum_{l=j}^{k} b_j, \quad j = 0, 1, \dots, k, \ A = A_0, \ B = B_0, \tag{1.3}$$

and suppose that (1.1) has some point of equilibrium $\hat{x}$ (not necessary a positive one). Then by assumption

$$\lambda + B\hat{x} \neq 0 \tag{1.4}$$

the equilibrium point $\hat{x}$ is defined by the algebraic equation

$$\hat{x} = \frac{\mu + A\hat{x}}{\lambda + B\hat{x}}. \tag{1.5}$$

By condition (1.4), equation (1.5) can be transformed to the form

$$B\hat{x}^2 - (A - \lambda)\hat{x} - \mu = 0. \tag{1.6}$$

It is clear that if

$$(A - \lambda)^2 + 4B\mu > 0, \tag{1.7}$$

then (1.1) has two points of equilibrium

$$\hat{x}_1 = \frac{A - \lambda + \sqrt{(A - \lambda)^2 + 4B\mu}}{2B}, \tag{1.8}$$

$$\hat{x}_2 = \frac{A - \lambda - \sqrt{(A - \lambda)^2 + 4B\mu}}{2B}. \tag{1.9}$$

If

$$(A - \lambda)^2 + 4B\mu = 0, \tag{1.10}$$

then (1.1) has only one point of equilibrium

$$\hat{x} = \frac{A - \lambda}{2B}. \tag{1.11}$$

And at last if

$$(A - \lambda)^2 + 4B\mu < 0, \tag{1.12}$$

then (1.1) has not equilibrium points.

Remark 1.1. Consider the case $\mu = 0$, $B \neq 0$. From (1.5) we obtain the following. If $\lambda \neq 0$ and $A \neq \lambda$, then (1.1) has two points of equilibrium:

$$\hat{x}_1 = \frac{A - \lambda}{B}, \qquad \hat{x}_2 = 0. \tag{1.13}$$

If $\lambda \neq 0$ and $A = \lambda$, then (1.1) has only one point of equilibrium: $\hat{x} = 0$. If $\lambda = 0$, then (1.1) has only one point of equilibrium: $\hat{x} = A/B$.

Remark 1.2. Consider the case $\mu = B = 0$, $\lambda \neq 0$. If $A \neq \lambda$, then (1.1) has only one point of equilibrium: $\hat{x} = 0$. If $A = \lambda$, then each solution $\hat{x} = \text{const}$ is an equilibrium point of (1.1).

2. Stochastic perturbations, centering, and linearization—Definitions and auxiliary statements

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a probability space and let $\{\mathfrak{F}_n, n \in Z\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, that is, $\mathfrak{F}_{n_1} \subset \mathfrak{F}_{n_2}$ for $n_1 < n_2$, let $\mathbf{E}$ be the expectation, let ξ_n, $n \in Z$, be a sequence of $\mathfrak{F}_n$-adapted mutually independent random variables such that $\mathbf{E}\xi_n = 0$, $\mathbf{E}\xi_n^2 = 1$.

As it was proposed in [39, 40] and used later in [41–43] we will suppose that (1.1) is exposed to stochastic perturbations ξ_n which are directly proportional to the deviation of the state x_n of system (1.1) from the equilibrium point $\hat{x}$. So, (1.1) takes the form

$$x_{n+1} = \frac{\mu + \sum_{j=0}^{k} a_j x_{n-j}}{\lambda + \sum_{j=0}^{k} b_j x_{n-j}} + \sigma(x_n - \hat{x})\xi_{n+1}. \tag{2.1}$$

Note that the equilibrium point $\hat{x}$ of (1.1) is also the equilibrium point of (2.1).

Putting $y_n = x_n - \hat{x}$ we will center (2.1) in the neighborhood of the point of equilibrium $\hat{x}$. From (2.1) it follows that

$$y_{n+1} = \frac{\sum_{j=0}^{k} (a_j - b_j\hat{x}) y_{n-j}}{\lambda + B\hat{x} + \sum_{j=0}^{k} b_j y_{n-j}} + \sigma y_n \xi_{n+1}. \tag{2.2}$$

It is clear that the stability of the trivial solution of (2.2) is equivalent to the stability of the equilibrium point of (2.1).

Together with nonlinear equation (2.2) we will consider and its linear part

$$z_{n+1} = \sum_{j=0}^{k} \gamma_j z_{n-j} + \sigma z_n \xi_{n+1}, \qquad \gamma_j = \frac{a_j - b_j\hat{x}}{\lambda + B\hat{x}}. \tag{2.3}$$

Two following definitions for stability are used below.

Definition 2.1. The trivial solution of (2.2) is called stable in probability if for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists $\delta > 0$ such that the solution $y_n = y_n(\phi)$ satisfies the condition $\mathbf{P}\{\sup_{n\in Z}|y_n(\phi)| > \epsilon_1\} < \epsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\sup_{j\in Z_0}|\phi_j| \le \delta\} = 1$.

Definition 2.2. The trivial solution of (2.3) is called mean square stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that the solution $z_n = z_n(\phi)$ satisfies the condition $\mathbf{E}|z_n(\phi)|^2 < \epsilon$ for any initial function ϕ such that $\sup_{j\in Z_0}\mathbf{E}|\phi_j|^2 < \delta$. If, besides, $\lim_{n\to\infty}\mathbf{E}|z_n(\phi)|^2 = 0$, for any initial function ϕ, then the trivial solution of (2.3) is called asymptotically mean square stable.

The following method for stability investigation is used below. Conditions for asymptotic mean square stability of the trivial solution of constructed linear equation (2.3) were obtained via V. Kolmanovskii and L. Shaikhet general method of Lyapunov functionals construction [44–46]. Since the order of nonlinearity of (2.2) is more than 1, then obtained stability conditions at the same time are [47–49] conditions for stability in the probability of the trivial solution of nonlinear equation (2.2) and therefore for stability in probability of the equilibrium point of (2.1).

Lemma 2.3. *(see [44]). If*

$$\sum_{j=0}^{k}|\gamma_j| < \sqrt{1-\sigma^2}, \tag{2.4}$$

then the trivial solution of (2.3) *is asymptotically mean square stable.*

Put

$$\beta = \sum_{j=0}^{k}\gamma_j, \qquad \alpha = \sum_{j=1}^{k}|G_j|, \quad G_j = \sum_{l=j}^{k}\gamma_l. \tag{2.5}$$

Lemma 2.4. *(see [44]). If*

$$\beta^2 + 2\alpha|1-\beta| + \sigma^2 < 1, \tag{2.6}$$

then the trivial solution of (2.3) *is asymptotically mean square stable.*

Consider also the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (2.3).

Let U and Γ be two square matrices of dimension $k+1$ such that $U = \|u_{ij}\|$ has all zero elements except for $u_{k+1,k+1} = 1$ and

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \gamma_k & \gamma_{k-1} & \gamma_{k-2} & \cdots & \gamma_1 & \gamma_0 \end{pmatrix}. \tag{2.7}$$

Lemma 2.5 ([46]). *Let the matrix equation*

$$\Gamma' D\Gamma - D = -U \tag{2.8}$$

has a positively semidefinite solution D with $d_{k+1,k+1} > 0$. Then the trivial solution of (2.3) *is asymptotically mean square stable if and only if*

$$\sigma^2 d_{k+1,k+1} < 1. \tag{2.9}$$

Corollary 2.6. *For $k = 1$ condition* (2.9) *takes the form*

$$|\gamma_1| < 1, \qquad |\gamma_0| < 1-\gamma_1, \tag{2.10}$$

$$\sigma^2 < d_{22}^{-1} = \frac{(1+\gamma_1)\left((1-\gamma_1)^2 - \gamma_0^2\right)}{1-\gamma_1}. \tag{2.11}$$

If, in particular, $\sigma = 0$, then condition (2.10) *is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of* (2.3) *for $k = 1$.*

Remark 2.7. Put $\sigma = 0$. If $\beta = 1$, then the trivial solution of (2.3) can be stable (e.g., $z_{n+1} = z_n$ or $z_{n+1} = 0.5(z_n + z_{n-1})$), unstable (e.g., $z_{n+1} = 2z_n - z_{n-1}$) but cannot be asymptotically stable.

Really, it is easy to see that if $\beta \geq 1$ (in particular, $\beta = 1$), then sufficient conditions (2.4) and (2.6) do not hold. Moreover, necessary and sufficient (for $k = 1$) condition (2.10) does not hold too since if (2.10) holds, then we obtain a contradiction

$$1 \leq \beta = \gamma_0 + \gamma_1 \leq |\gamma_0| + \gamma_1 < 1. \tag{2.12}$$

Remark 2.8. As it follows from results of [47–49] the conditions of Lemmas 2.3, 2.4, 2.5 at the same time are conditions for stability in probability of the equilibrium point of (2.1).

3. Stability of equilibrium points

From conditions (2.4), (2.6) it follows that $|\beta| < 1$. Let us check if this condition can be true for each equilibrium point.

Suppose at first that condition (1.7) holds. Then (2.1) has two points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ defined by (1.8) and (1.9) accordingly. Putting $S = \sqrt{(A-\lambda)^2 + 4B\mu}$ via (2.5), (2.3), (1.3), we obtain that corresponding β_1 and β_2 are

$$\begin{aligned} \beta_1 &= \frac{A - B\hat{x}_1}{\lambda + B\hat{x}_1} = \frac{A - (1/2)(A - \lambda + S)}{\lambda + (1/2)(A - \lambda + S)} = \frac{A + \lambda - S}{A + \lambda + S}, \\ \beta_2 &= \frac{A - B\hat{x}_2}{\lambda + B\hat{x}_2} = \frac{A - (1/2)(A - \lambda - S)}{\lambda + (1/2)(A - \lambda - S)} = \frac{A + \lambda + S}{A + \lambda - S}. \end{aligned} \tag{3.1}$$

So, $\beta_1\beta_2 = 1$. It means that the condition $|\beta| < 1$ holds only for one from the equilibrium points $\hat{x}_1$ and $\hat{x}_2$. Namely, if $A + \lambda > 0$, then $|\beta_1| < 1$; if $A + \lambda < 0$, then $|\beta_2| < 1$; if $A + \lambda = 0$, then $\beta_1 = \beta_2 = -1$. In particular, if $\mu = 0$, then via Remark 1.1 and (2.3) we have $\beta_1 = \lambda A^{-1}$, $\beta_2 = \lambda^{-1}A$. Therefore, $|\beta_1| < 1$ if $|\lambda| < |A|$, $|\beta_2| < 1$ if $|\lambda| > |A|$, $|\beta_1| = |\beta_2| = 1$ if $|\lambda| = |A|$.

So, via Remark 2.7, we obtain that equilibrium points $\hat{x}_1$ and $\hat{x}_2$ can be stable concurrently only if corresponding β_1 and β_2 are negative concurrently.

Suppose now that condition (1.10) holds. Then (2.1) has only one point of equilibrium (1.11). From (2.5), (2.3), (1.3), (1.11) it follows that corresponding β equals

$$\beta = \frac{A - B\hat{x}}{\lambda + B\hat{x}} = \frac{A - (1/2)(A - \lambda)}{\lambda + (1/2)(A - \lambda)} = \frac{A + \lambda}{\lambda + A} = 1. \tag{3.2}$$

As it follows from Remark 2.7 this point of equilibrium cannot be asymptotically stable.

Corollary 3.1. *Let $\hat{x}$ be an equilibrium point of* (2.1) *such that*

$$\sum_{j=0}^{k} |a_j - b_j\hat{x}| < |\lambda + B\hat{x}|\sqrt{1 - \sigma^2}, \quad \sigma^2 < 1. \tag{3.3}$$

Then the equilibrium point $\hat{x}$ is stable in probability.

The proof follows from (2.3), Lemma 2.3, and Remark 2.8.

Corollary 3.2. *Let $\widehat{x}$ be an equilibrium point of* (2.1) *such that*

$$\left|A - B\widehat{x}\right| < \left|\lambda + B\widehat{x}\right|, \tag{3.4}$$

$$2\sum_{j=1}^{k}\left|A_j - B_j\widehat{x}\right| < |\lambda + A| - \sigma^2\frac{(\lambda + B\widehat{x})^2}{\left|\lambda - A + 2B\widehat{x}\right|}. \tag{3.5}$$

Then the equilibrium point $\widehat{x}$ is stable in probability.

Proof. Via (1.3), (2.3), (2.5) we have

$$\alpha = \left|\lambda + B\widehat{x}\right|^{-1}\sum_{j=1}^{k}\left|A_j - B_j\widehat{x}\right|, \qquad \beta = \frac{A - B\widehat{x}}{\lambda + B\widehat{x}}. \tag{3.6}$$

Rewrite (2.6) in the form

$$2\alpha < 1 + \beta - \frac{\sigma^2}{1-\beta}, \quad |\beta| < 1, \tag{3.7}$$

and show that it holds. From (3.4) it follows that $|\beta| < 1$. Via $|\beta| < 1$ we have

$$\begin{aligned} 1 + \beta &= 1 + \frac{A - B\widehat{x}}{\lambda + B\widehat{x}} = \frac{\lambda + A}{\lambda + B\widehat{x}} > 0, \\ 1 - \beta &= 1 - \frac{A - B\widehat{x}}{\lambda + B\widehat{x}} = \frac{\lambda - A + 2B\widehat{x}}{\lambda + B\widehat{x}} > 0. \end{aligned} \tag{3.8}$$

So,

$$2\sum_{j=1}^{k}\left|A_j - B_j\widehat{x}\right| < \left|\lambda + B\widehat{x}\right|\left(\frac{\lambda + A}{\lambda + B\widehat{x}} - \sigma^2\frac{\lambda + B\widehat{x}}{\lambda - A + 2B\widehat{x}}\right) = |\lambda + A| - \sigma^2\frac{(\lambda + B\widehat{x})^2}{\left|\lambda - A + 2B\widehat{x}\right|}. \tag{3.9}$$

It means that the condition of Lemma 2.4 holds. Via Remark 2.8 the proof is completed. □

Corollary 3.3. *An equilibrium point $\widehat{x}$ of the equation*

$$x_{n+1} = \frac{\mu + a_0x_n + a_1x_{n-1}}{\lambda + b_0x_n + b_1x_{n-1}} + \sigma(x_n - \widehat{x})\xi_{n+1} \tag{3.10}$$

is stable in probability if and only if

$$\begin{gathered} \left|a_1 - b_1\widehat{x}\right| < \left|\lambda + B\widehat{x}\right|, \\ \left|a_0 - b_0\widehat{x}\right| < (\lambda - a_1 + (b_0 + 2b_1)\widehat{x})\,sign(\lambda + B\widehat{x}), \end{gathered} \tag{3.11}$$

$$\sigma^2 < \frac{(\lambda + a_1 + b_0\widehat{x})(\lambda + a_0 - a_1 + 2b_1\widehat{x})(\lambda - A + 2B\widehat{x})}{(\lambda - a_1 + (b_0 + 2b_1)\widehat{x})(\lambda + B\widehat{x})^2}. \tag{3.12}$$

The proof follows from (2.3), (2.10), (2.11).

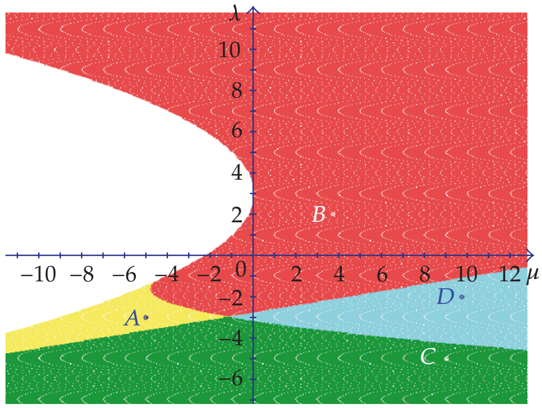

Figure 1: Stability regions, $\sigma^2 = 0$.

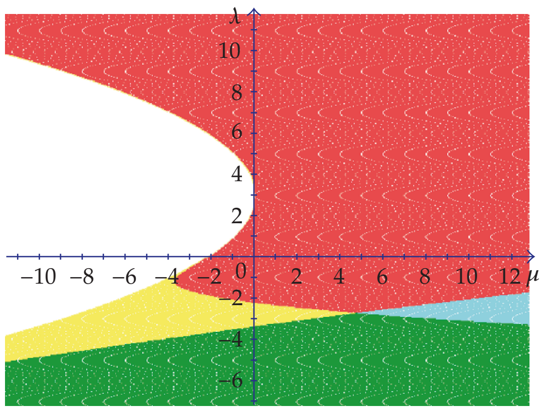

Figure 2: Stability regions, $\sigma^2 = 0.3$.

4. Examples

Example 4.1. Consider (3.10) with $a_0 = 2.9$, $a_1 = 0.1$, $b_0 = b_1 = 0.5$. From (1.3) and (1.7)–(1.9) it follows that $A = 3$, $B = 1$ and for any fixed μ and λ such that $\mu > -(1/4)(3-\lambda)^2$ equation (3.10) has two points of equilibrium

$$\hat{x}_1 = \frac{1}{2}\left(3 - \lambda + \sqrt{(3-\lambda)^2 + 4\mu}\right), \qquad \hat{x}_2 = \frac{1}{2}\left(3 - \lambda - \sqrt{(3-\lambda)^2 + 4\mu}\right). \tag{4.1}$$

In Figure 1, the region where the points of equilibrium are absent (white region), the region where both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ are there but unstable (yellow region), the region where the point of equilibrium $\hat{x}_1$ is stable only (red region), the region where the point of equilibrium $\hat{x}_2$ is stable only (green region), and the region where both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ are stable (cyan region) are shown in the space of (μ, λ). All regions are obtained via condition (3.11) for $\sigma^2 = 0$. In Figures 2, 3 one can see similar regions for $\sigma^2 = 0.3$ and $\sigma^2 = 0.8$, accordingly, that were obtained via conditions (3.11), (3.12). In Figure 4 it is shown that sufficient conditions (3.3) and (3.4), (3.5) are enough close to necessary and sufficient conditions (3.11), (3.12): inside of the region where the point of equilibrium $\hat{x}_1$ is stable (red

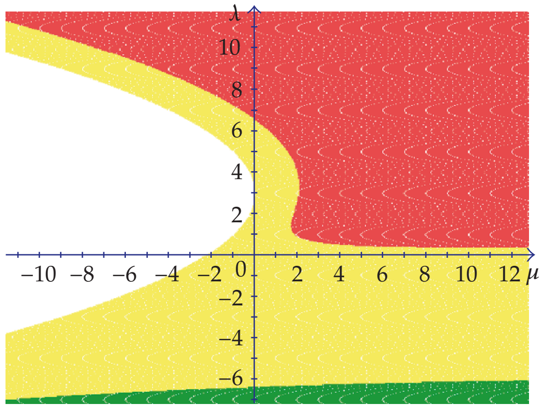

Figure 3: Stability regions, $\sigma^2 = 0.8$.

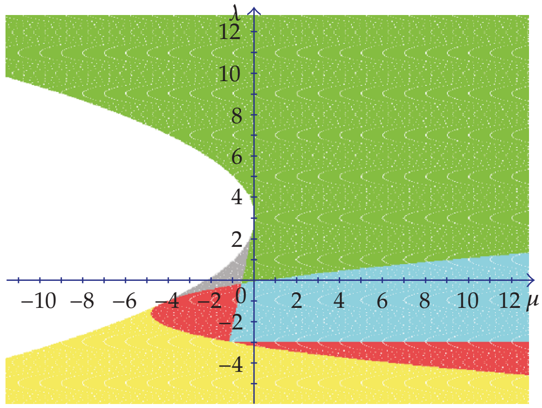

Figure 4: Stability regions, $\sigma^2 = 0$.

region) one can see the regions of stability of the point of equilibrium $\hat{x}_1$ that were obtained by condition (3.3) (grey and green regions) and by conditions (3.4), (3.5) (cyan and green regions). Stability regions obtained via both sufficient conditions of stability (3.3) and (3.4), (3.5) give together almost whole stability region obtained via necessary and sufficient stability conditions (3.11), (3.12).

Consider now the behavior of solutions of (3.10) with $\sigma = 0$ in the points A, B, C, D of the space of (μ, λ) (Figure 1). In the point A with $\mu = -5$, $\lambda = -3$ both equilibrium points $\hat{x}_1 = 5$ and $\hat{x}_2 = 1$ are unstable. In Figure 5 two trajectories of solutions of (3.10) are shown with the initial conditions $x_{-1} = 5$, $x_0 = 4.95$, and $x_{-1} = 0.999$, $x_0 = 1.0001$. In Figure 6 two trajectories of solutions of (3.10) with the initial conditions $x_{-1} = -3$, $x_0 = 13$, and $x_{-1} = -1.5$, $x_0 = -1.500001$ are shown in the point B with $\mu = 3.75$, $\lambda = 2$. One can see that the equilibrium point $\hat{x}_1 = 2.5$ is stable and the equilibrium point $\hat{x}_2 = -1.5$ is unstable. In the point C with $\mu = 9$, $\lambda = -5$ the equilibrium point $\hat{x}_1 = 9$ is unstable and the equilibrium point $\hat{x}_2 = -1$ is stable. Two corresponding trajectories of solutions are shown in Figure 7 with the initial conditions $x_{-1} = 7$, $x_0 = 10$, and $x_{-1} = -8$, $x_0 = 8$. In the point D with $\mu = 9.75$, $\lambda = -2$ both equilibrium points $\hat{x}_1 = 6.5$ and $\hat{x}_2 = -1.5$ are stable. Two corresponding trajectories of solutions are shown in Figure 8 with the initial conditions $x_{-1} = 2$, $x_0 = 12$, and $x_{-1} = -8$, $x_0 = 8$. As it was noted above in this case, corresponding β_1 and β_2 are negative: $\beta_1 = -7/9$ and $\beta_2 = -9/7$.

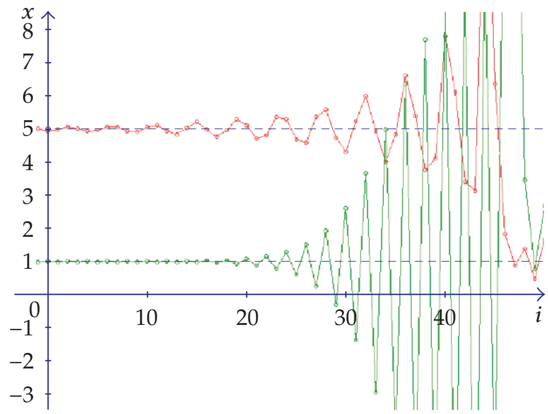

Figure 5: Unstable equilibrium points $\hat{x}_1 = 5$ and $\hat{x}_2 = 1$ for $\mu = -5$, $\lambda = -3$.

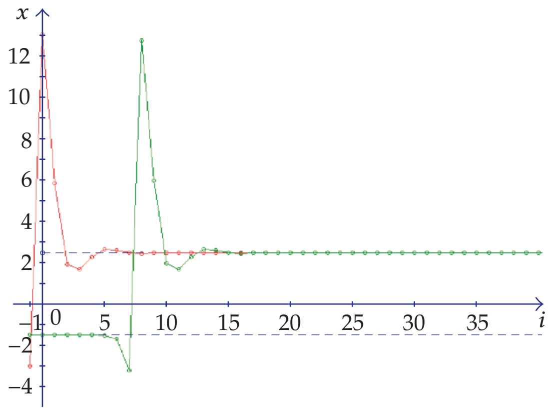

Figure 6: Stable equilibrium point $\hat{x}_1 = 2.5$ and unstable $\hat{x}_2 = -1.5$ for $\mu = 3.75$, $\lambda = 2$.

Consider the difference equation

$$x_{n+1} = p + q\frac{x_{n-m}}{x_{n-r}} + \sigma\left(x_n - \hat{x}\right)\xi_{i+1}. \tag{4.2}$$

Different particular cases of this equation were considered in [2–5, 16, 22, 23, 37].

Equation (4.2) is a particular case of (2.1) with

$$\begin{aligned} &a_r = p, \qquad a_m = q, \qquad a_j = 0 \quad \text{if } j \neq r,\ j \neq m, \\ &\mu = \lambda = 0, \qquad b_r = 1, \qquad b_j = 0 \quad \text{if } j \neq r,\ \hat{x} = p + q. \end{aligned} \tag{4.3}$$

Suppose firstly that $p + q \neq 0$ and consider two cases: (1) $m > r \geq 0$, (2) $r > m \geq 0$. In the first case,

$$\begin{aligned} &A_j = p + q \quad \text{if } j = 0, \dots, r, \qquad A_j = q \quad \text{if } j = r + 1, \dots, m, \\ &B_j = 1 \quad \text{if } j = 0, \dots, r, \qquad B_j = 0 \quad \text{if } j = r + 1, \dots, m. \end{aligned} \tag{4.4}$$

In the second case,

$$A_j = p + q \quad \text{if } j = 0, \dots, m, \qquad A_j = p \quad \text{if } j = m + 1, \dots, r, \qquad B_j = 1 \quad \text{if } j = 0, \dots, r. \tag{4.5}$$

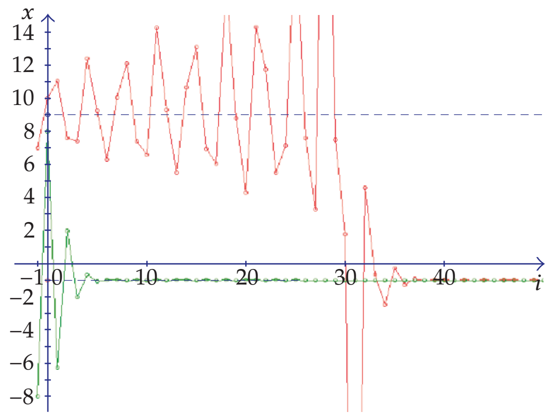

Figure 7: Unstable equilibrium point $\hat{x}_1 = 9$ and stable $\hat{x}_2 = -1$ for $\mu = 9$, $\lambda = -5$.

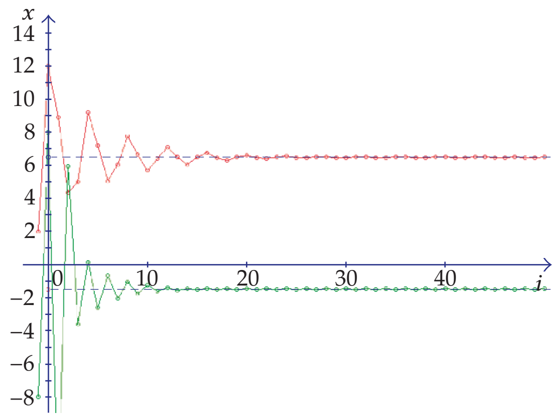

Figure 8: Stable equilibrium points $\hat{x}_1 = 6.5$ and $\hat{x}_2 = -1.5$ for $\mu = 9.75$, $\lambda = -2$.

In both cases, Corollary 3.1 gives stability condition in the form $2|q| < \sqrt{1-\sigma^2}|p+q|$ or

$$p \in (-\infty, -q - \theta|q|) \cup (-q + \theta|q|, \infty) \tag{4.6}$$

with

$$\theta = \theta_1 = \frac{2}{\sqrt{1-\sigma^2}}. \tag{4.7}$$

Corollary 3.2 in both cases gives stability condition in the form $2|q||m-r| < (1-\sigma^2)|p+q|$ or (4.6) with

$$\theta = \theta_2 = \frac{2|m-r|}{1-\sigma^2}. \tag{4.8}$$

Since $\theta_2 > \theta_1$ then condition (4.6), (4.7) is better than (4.6), (4.8).

In the case $m = 1$, $r = 0$ Corollary 3.3 gives stability condition in the form

$$|q| < |p+q|, \ |q| < p \operatorname{sign}(p+q), \quad \sigma^2 < \frac{(p+2q)(p-q)}{p(p+q)} \tag{4.9}$$

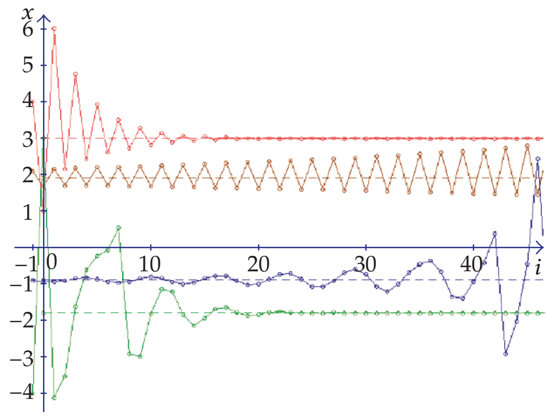

Figure 9: Stable equilibrium points $\hat{x} = 3$ and $\hat{x} = -1.8$, unstable $\hat{x} = 1.93$ and $\hat{x} = -0.9$.

or

$$p \in \left(-\infty, \frac{1}{2}(-q - \theta|q|)\right) \cup \left(\frac{1}{2}(-q + \theta|q|), \infty\right), \quad \theta = \sqrt{\frac{9-\sigma^2}{1-\sigma^2}}. \tag{4.10}$$

In particular, from (4.10) it follows that for $q = 1$, $\sigma = 0$ (this case was considered in [3, 23]) the equilibrium point $\hat{x} = p + 1$ is stable if and only if $p \in (-\infty, -2) \cup (1, \infty)$. Note that in [3] for this case the condition $p > 1$ only is obtained.

In Figure 9 four trajectories of solutions of (4.2) in the case $m = 1$, $r = 0$, $\sigma = 0$, $q = 1$ are shown: (1) $p = 2$, $\hat{x} = 3$, $x_{-1} = 4$, $x_0 = 1$ (red line, stable solution); (2) $p = 0.93$, $\hat{x} = 1.93$, $x_{-1} = 2.1$, $x_0 = 1.7$ (brown line, unstable solution); (3) $p = -1.9$, $\hat{x} = -0.9$, $x_{-1} = -0.89$, $x_0 = -0.94$ (blue line, unstable solution); (4) $p = -2.8$, $\hat{x} = -1.8$, $x_{-1} = -4$, $x_0 = 3$ (green line, stable solution).

In the case $r = 1$, $m = 0$, Corollary 3.3 gives stability condition in the form

$$|q| < |p+q|, \quad |q| < (p+2q)\operatorname{sign}(p+q), \quad \sigma^2 < \frac{p(p+3q)}{(p+q)(p+2q)} \tag{4.11}$$

or

$$p \in \left(-\infty, \frac{1}{2}(-3q - \theta|q|)\right) \cup \left(\frac{1}{2}(-3q + \theta|q|), \infty\right), \quad \theta = \sqrt{\frac{9-\sigma^2}{1-\sigma^2}}. \tag{4.12}$$

Example 4.2. For example, from (4.12) it follows that for $q = -1$, $\sigma = 0$ (this case was considered in [22, 37]), the equilibrium point $\hat{x} = p - 1$ is stable if and only if $p \in (-\infty, 0) \cup (3, \infty)$. In Figure 10 four trajectories of solutions of (4.2) in the case $r = 1$, $m = 0$, $\sigma = 0$, $q = -1$ are shown: (1) $p = 3.5$, $\hat{x} = 2.5$, $x_{-1} = 3.5$, $x_0 = 1.5$ (red line, stable solution); (2) $p = 2.2$, $\hat{x} = 1.2$, $x_{-1} = 1.2$, $x_0 = 1.2001$ (brown line, unstable solution); (3) $p = 0.3$, $\hat{x} = -0.7$, $x_{-1} = -0.7$, $x_0 = -0.705$ (blue line, unstable solution); (4) $p = -0.2$, $\hat{x} = -1.2$, $x_{-1} = -2$, $x_0 = -0.4$ (green line, stable solution).

Via simulation of a sequence of mutually independent random variables ξ_n consider the behavior of the equilibrium point by stochastic perturbations. In Figure 11 one thousand trajectories are shown for $p = 4$, $q = -1$, $\sigma = 0.5$, $x_{-1} = 3.5$, $x_0 = 2.5$. In this case, stability

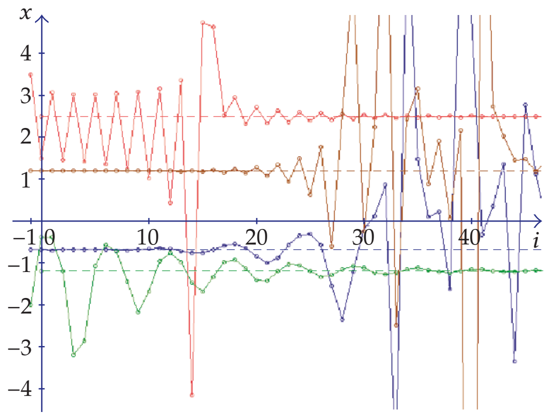

Figure 10: Stable equilibrium points $\hat{x} = 2.5$ and $\hat{x} = -1.2$, unstable $\hat{x} = 1.2$ and $\hat{x} = -0.7$.

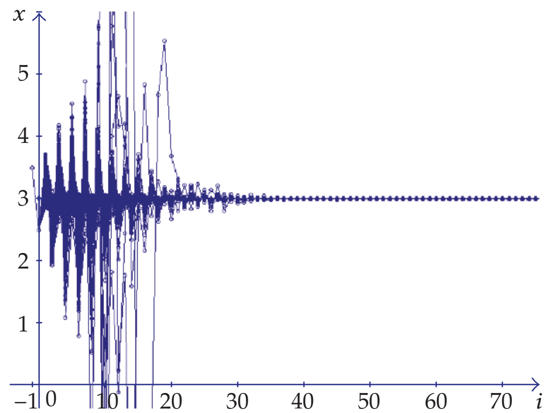

Figure 11: Stable equilibrium point $\hat{x} = 3$ for $p = 4$, $q = -1$, $\sigma = 0.5$.

condition (4.12) holds ($4 \in (-\infty, -0.2) \cup (3.2, \infty)$) and therefore the equilibrium point $\hat{x} = 3$ is stable: all trajectories go to $\hat{x}$. Putting $\sigma = 0.9$, we obtain that stability condition (4.12) does not hold ($4 \notin (-\infty, -1.78) \cup (4.78, \infty)$). Therefore, the equilibrium point $\hat{x} = 3$ is unstable: in Figure 12 one can see that 1000 trajectories fill the whole space.

Note also that if $p+q$ goes to zero all obtained stability conditions are violated. Therefore, by conditions $p + q = 0$ the equilibrium point is unstable.

Example 4.3. Consider the equation

$$x_{n+1} = \frac{\mu + a x_{n-1}}{\lambda + x_n} + \sigma\left(x_n - \hat{x}\right)\xi_{n+1} \tag{4.13}$$

(its particular cases were considered in [18, 19, 35]). Equation (4.13) is a particular case of (2.1) with $k = 1$, $a_0 = b_1 = 0$, $a_1 = a$, $b_0 = 1$. From (1.7)–(1.9) it follows that by condition $\mu > -(1/4)(a - \lambda)^2$ it has two equilibrium points

$$\hat{x}_1 = \frac{a - \lambda + S}{2}, \quad \hat{x}_2 = \frac{a - \lambda - S}{2}, \quad S = \sqrt{(a - \lambda)^2 + 4\mu}. \tag{4.14}$$

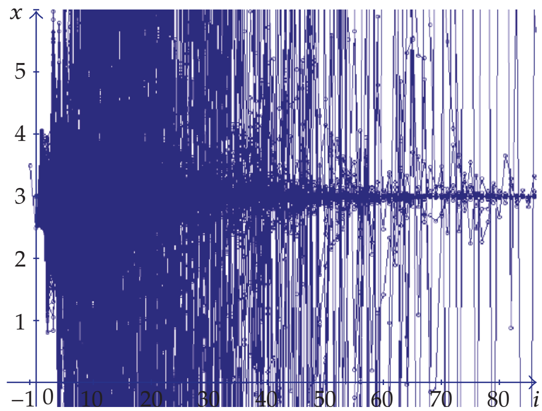

Figure 12: Unstable equilibrium point $\hat{x} = 3$ for $p = 4$, $q = -1$, $\sigma = 0.9$.

For equilibrium point $\hat{x}$ sufficient conditions (3.3) and (3.4), (3.5) give

$$\begin{gathered} |\hat{x}| + |a| < |\lambda + \hat{x}|\sqrt{1-\sigma^2}, \\ 2|a| < |\lambda + a| - \sigma^2 \frac{(\lambda + \hat{x})^2}{|\lambda + 2\hat{x} - a|}, \quad |a - \hat{x}| < |\lambda + \hat{x}|. \end{gathered} \tag{4.15}$$

From (3.11), (3.12) it follows that an equilibrium point $\hat{x}$ of (4.13) is stable in probability if and only if

$$\begin{gathered} |\lambda + \hat{x}| > |a|, \quad |\hat{x}| < (\lambda + \hat{x} - a)\operatorname{sign}(\lambda + \hat{x}), \\ \sigma^2 < \frac{(\lambda + \hat{x} + a)(\lambda - a)(\lambda + 2\hat{x} - a)}{(\lambda + \hat{x} - a)(\lambda + \hat{x})^2}. \end{gathered} \tag{4.16}$$

For example, for $\hat{x} = \hat{x}_1$ from (4.15) we obtain

$$\begin{gathered} |a - \lambda + S| + 2|a| < |a + \lambda + S|\sqrt{1-\sigma^2}, \\ 2|a| < \lambda + a - \sigma^2 \frac{(\lambda + a + S)^2}{4S}, \quad \lambda + a > 0. \end{gathered} \tag{4.17}$$

From (4.16) it follows

$$\begin{gathered} |a + \lambda + S| > 2|a|, \\ |a - \lambda + S| < (\lambda - a + S)\operatorname{sign}(a + \lambda + S), \\ \sigma^2 < \frac{4S(\lambda - a)(\lambda + 3a + S)}{(\lambda - a + S)(\lambda + a + S)^2}. \end{gathered} \tag{4.18}$$

Similar for $\hat{x} = \hat{x}_2$ from (4.15) we obtain

$$\begin{gathered} |a - \lambda - S| + 2|a| < |a + \lambda - S|\sqrt{1-\sigma^2}, \\ 2|a| < |\lambda + a| - \sigma^2 \frac{(\lambda + a - S)^2}{4S}, \quad \lambda + a < 0. \end{gathered} \tag{4.19}$$

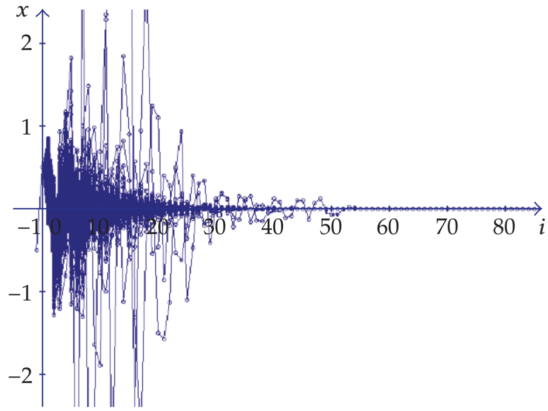

Figure 13: Stable equilibrium point $\hat{x}_2 = 0$ for $\mu = 0$, $\lambda = -2$, $a = 1$, $\sigma = 0.6$.

From (4.16) it follows

$$\begin{gathered} |a + \lambda - S| > 2|a|, \\ |a - \lambda - S| < (\lambda - a - S)\operatorname{sign}(a + \lambda - S), \\ \sigma^2 < \frac{4S(a - \lambda)(\lambda + 3a - S)}{(\lambda - a - S)(\lambda + a - S)^2}. \end{gathered} \tag{4.20}$$

Put, for example, $\mu = 0$. Then (4.13) has two equilibrium points: $\hat{x}_1 = a - \lambda$, $\hat{x}_2 = 0$. From (4.15)-(4.16) it follows that the equilibrium point $\hat{x}_1$ is unstable and the equilibrium point $\hat{x}_2$ is stable in probability if and only if

$$|\lambda| > \frac{|a|}{\sqrt{1 - \sigma^2}}. \tag{4.21}$$

Note that for particular case $\mu = 0$, $a = 1$, $\lambda > 0$, $\sigma = 0$ in [35] it is shown that the equilibrium point $\hat{x}_2$ is locally asymptotically stable if $\lambda > 1$; and for particular case $\mu = 0$, $a = -\alpha < 0$, $\lambda > 0$, $\sigma = 0$ in [18] it is shown that the equilibrium point $\hat{x}_2$ is locally asymptotically stable if $\lambda > \alpha$. It is easy to see that both these conditions follow from (4.21).

Similar results can be obtained for the equation $x_{n+1} = (\mu - ax_n)/(\lambda + x_{n-1})$ that was considered in [1].

In Figure 13 one thousand trajectories of (4.13) are shown for $\mu = 0$, $\lambda = -2$, $a = 1$, $\sigma = 0.6$, $x_{-1} = -0.5$, $x_0 = 0.5$. In this case stability condition (4.21) holds ($2 > 1.25$) and therefore the equilibrium point $\hat{x} = 0$ is stable: all trajectories go to zero. Putting $\sigma = 0.9$, we obtain that stability condition (4.21) does not hold ($2 < 2.29$). Therefore, the equilibrium point $\hat{x} = 0$ is unstable: in Figure 14 one can see that 1000 trajectories by the initial condition $x_{-1} = -0.1$, $x_0 = 0.1$ fill the whole space.

Example 4.4. Consider the equation

$$x_{n+1} = \frac{p + x_{n-1}}{qx_n + x_{n-1}} + \sigma(x_n - \hat{x})\xi_{n+1} \tag{4.22}$$

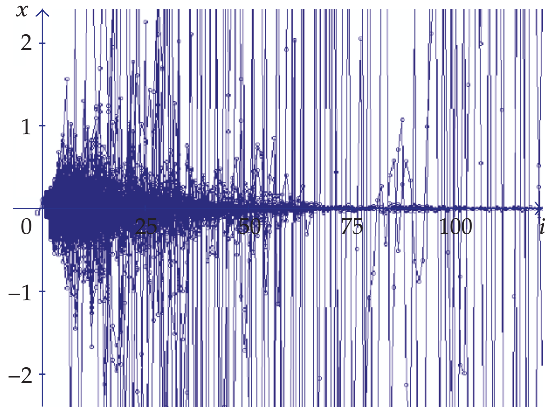

Figure 14: Unstable equilibrium point $\widehat{x}_2 = 0$ for $\mu = 0$, $\lambda = -2$, $a = 1$, $\sigma = 0.9$.

that is a particular case of (3.10) with $\mu = p$, $\lambda = 0$, $a_0 = 0$, $a_1 = 1$, $b_0 = q$, $b_1 = 1$. As it follows from (1.4), (1.7)–(1.9) by conditions $p(q+1) > -1/4$, $q \neq -1$, (4.22) has two equilibrium points

$$\widehat{x}_1 = \frac{1+S}{2(q+1)}, \quad \widehat{x}_2 = \frac{1-S}{2(q+1)}, \quad S = \sqrt{1+4p(q+1)}. \tag{4.23}$$

From (3.11), (3.12) it follows that an equilibrium point $\widehat{x}$ of (4.22) is stable in probability if and only if

$$\begin{gathered} |1-\widehat{x}| < |(q+1)\widehat{x}|, \\ |q\widehat{x}| < ((2+q)\widehat{x}-1)\operatorname{sign}((q+1)\widehat{x}), \\ \sigma^2 < \frac{(1+q\widehat{x})(2\widehat{x}-1)(2(q+1)\widehat{x}-1)}{((2+q)\widehat{x}-1)(q+1)^2\widehat{x}^2}. \end{gathered} \tag{4.24}$$

Substituting (4.23) into (4.24), we obtain stability conditions immediately in the terms of the parameters of considered equation (4.22): the equilibrium point $\widehat{x}_1$ is stable in probability if and only if

$$p \in \begin{cases} \left(\dfrac{q-1}{4}, \infty\right), & q \geq 0, \\ \left(-\dfrac{1}{4(q+1)}, \dfrac{2}{q} + \dfrac{1}{q^2}\right), & q \in \left(-\dfrac{2}{3}, 0\right), \end{cases} \quad \Bigg| \quad \sigma^2 < \frac{4S(S-q)((S+3)q+2)}{(S+1)^2(q+1)((q+2)S-q)}, \tag{4.25}$$

the equilibrium point $\widehat{x}_2$ is stable in probability if and only if

$$p \in \begin{cases} \left(\dfrac{2}{q} + \dfrac{1}{q^2}, \infty\right), & q > 0, \\ \left(\dfrac{q-1}{4}, \dfrac{2}{q} + \dfrac{1}{q^2}\right), & q < -2, \end{cases} \quad \Bigg| \quad \sigma^2 < \frac{4S(S+q)((S-3)q-2)}{(S-1)^2(q+1)((q+2)S+q)}. \tag{4.26}$$

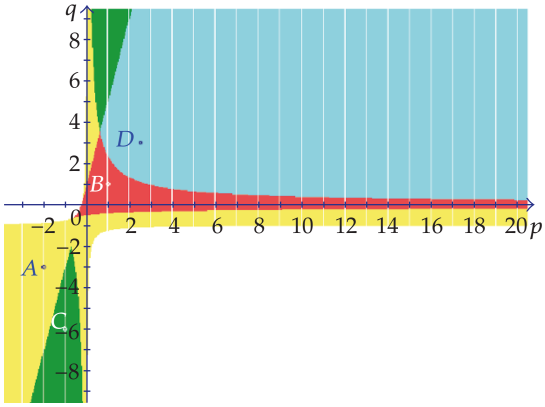

Figure 15: Stability regions, $\sigma = 0$.

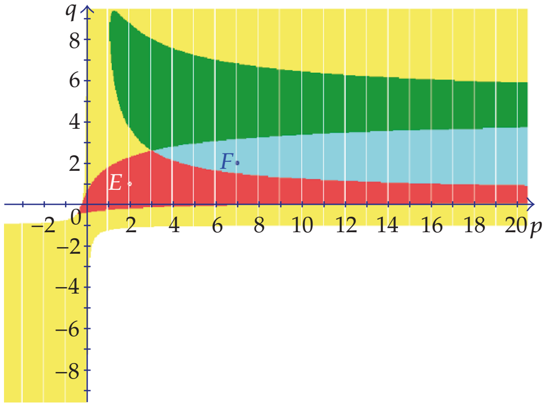

Figure 16: Stability regions, $\sigma = 0.7$.

Note that in [24] equation (4.18) was considered with $\sigma = 0$ and positive p, q. There it was shown that equilibrium point $\hat{x}_1$ is locally asymptotically stable if and only if $4p > q - 1$ that is a part of conditions (4.25).

In Figure 15 the region where the points of equilibrium are absent (white region), the region where the both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ are there but unstable (yellow region), the region where the point of equilibrium $\hat{x}_1$ is stable only (red region), the region where the point of equilibrium $\hat{x}_2$ is stable only (green region) and the region where the both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ are stable (cyan region) are shown in the space of (p,q). All regions are obtained via conditions (4.25), (4.26) for $\sigma = 0$. In Figures 16 similar regions are shown for $\sigma = 0.7$.

Consider the point A (Figure 15) with $p = -2$, $q = -3$. In this point both equilibrium points $\hat{x}_1 = -1.281$ and $\hat{x}_2 = 0.781$ are unstable. In Figure 17 two trajectories of solutions of (4.22) are shown with the initial conditions $x_{-1} = -1.28$, $x_0 = -1.281$ and $x_{-1} = 0.771$, $x_0 = 0.77$. In Figure 18 two trajectories of solutions of (4.22) with the initial conditions $x_{-1} = 4$, $x_0 = -3$ and $x_{-1} = -0.51$, $x_0 = -0.5$ are shown in the point B (Figure 15) with $p = q = 1$. One can see that the equilibrium point $\hat{x}_1 = 1$ is stable and the equilibrium point $\hat{x}_2 = -0.5$ is unstable. In the point C (Figure 15) with $p = -1$, $q = -6$ the equilibrium point $\hat{x}_1 = -0.558$ is unstable and the equilibrium point $\hat{x}_2 = 0.358$ is stable. Two corresponding trajectories of solutions are shown

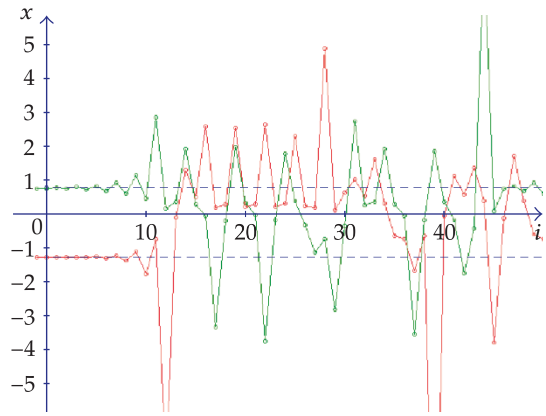

Figure 17: Unstable equilibrium points $\widehat{x}_1 = -1.281$ and $\widehat{x}_2 = 0.781$ for $p = -2$, $q = -3$.

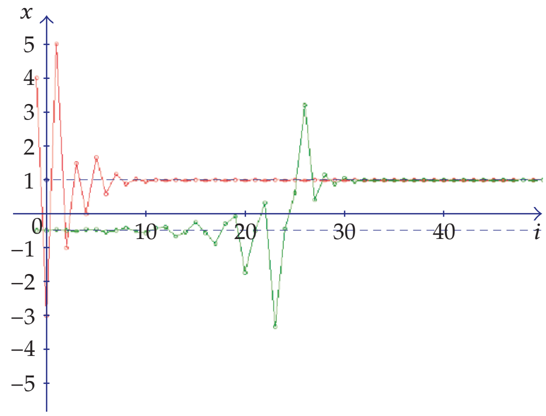

Figure 18: Stable equilibrium point $\widehat{x}_1 = 1$ and unstable $\widehat{x}_2 = -0.5$ for $p = 1$, $q = 1$.

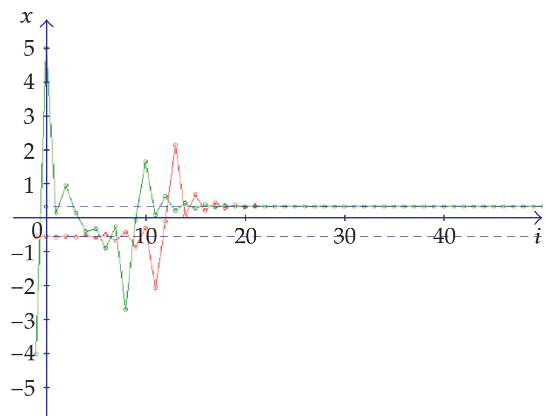

Figure 19: Unstable equilibrium point $\widehat{x}_1 = -0.558$ and stable $\widehat{x}_2 = 0.358$ for $p = -1$, $q = -6$.

in Figure 19 with the initial conditions $x_{-1} = x_0 = -0.55$ and $x_{-1} = -4$, $x_0 = 5$. In the point D (Figure 15) with $p = 2.5$, $q = 3$ both equilibrium points $\widehat{x}_1 = 0.925$ and $\widehat{x}_2 = -0.675$ are stable. Two corresponding trajectories of solutions are shown in Figure 20 with the initial conditions $x_{-1} = 2.1$, $x_0 = 0.2$ and $x_{-1} = -0.2$, $x_0 = -1.4$.

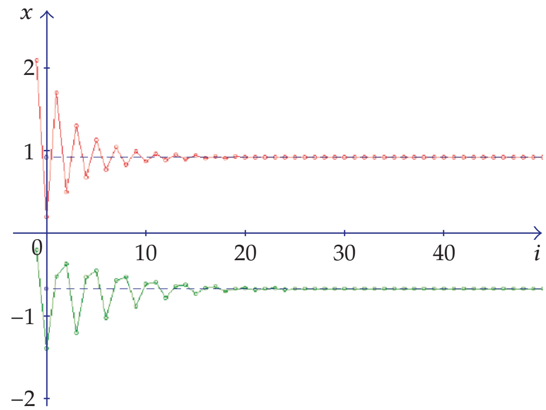

Figure 20: Stable equilibrium points $\hat{x}_1 = 0.925$ and $\hat{x}_2 = -0.675$ for $p = 2.5$, $q = 3$.

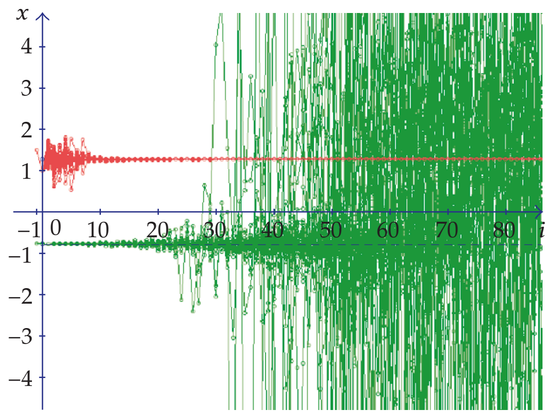

Figure 21: Stable equilibrium points $\hat{x}_1 = 1.281$ and $\hat{x}_2 = -0.781$ for $p = 2$, $q = 1$, $\sigma = 0.7$.

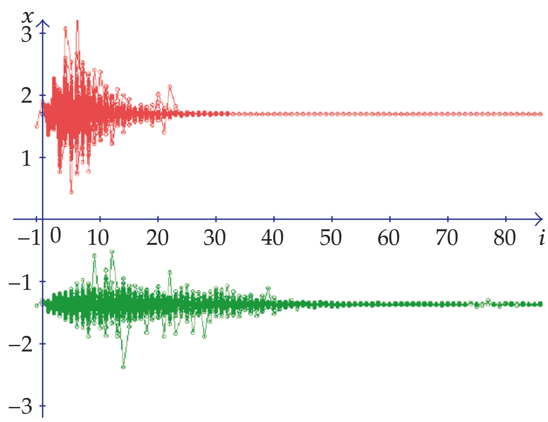

Figure 22: Stable equilibrium points $\hat{x}_1 = 1.703$ and $\hat{x}_2 = -0.37$ for $p = 7$, $q = 2$, $\sigma = 0.7$.

Consider the behavior of the equilibrium points of (4.22) by stochastic perturbations with $\sigma = 0.7$. In Figure 21 trajectories of solutions are shown for $p = 2$, $q = 1$ (the point E in Figure 16) with the initial conditions $x_{-1} = 1.5$, $x_0 = 1$ and $x_{-1} = x_0 = -0.78$. One can see that the equilibrium point $\hat{x}_1 = 1.281$ (red trajectories) is stable and the equilibrium point

$\hat{x}_2 = -0.781$ (green trajectories) is unstable. In Figure 22 trajectories of solutions are shown for $p = 7$, $q = 2$ (the point F in Figure 16) with the initial conditions $x_{-1} = 1.5$, $x_0 = 1.9$ and $x_{-1} = -1.4$, $x_0 = -1.3$. In this case both equilibrium points $\hat{x}_1 = 1.703$ (red trajectories) and $\hat{x}_2 = -1.37$ (green trajectories) are stable.

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