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Research Article

On Some Arithmetical Properties of the Genocchi Numbers and Polynomials

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We investigate the properties of the Genocchi functions and the Genocchi polynomials. We obtain the Fourier transform on the Genocchi function. We have the generating function of (h,q)-Genocchi polynomials. We define the Cangul-Ozden-Simsek's type twisted (h,q)-Genocchi polynomials and numbers. We also have the generalized twisted (h,q)-Genocchi numbers attached to the Dirichlet's character χ. Finally, we define zeta functions related to (h,q)-Genocchi polynomials and have the generating function of the generalized (h,q)-Genocchi numbers attached to χ.

1. Introduction

After Carlitz introduced an interesting q-analogue of Frobenius-Euler numbers in [1], q-Bernoulli and q-Euler numbers and polynomials have been studied by several authors. Recently, many authors have an interest in the q-extension of the Genocchi numbers and polynomials(cf. [2–5]). Kim et al. [5] defined the q-Genocchi numbers and the q-Genocchi polynomials. In [3], Kim derived the q-analogs of the Genocchi numbers and polynomials by constructing q-Euler numbers. He also gave some interesting relations between q-Euler and q-Genocchi numbers. The first author et al. [6] obtained the distribution relation for the Genocchi polynomials.

The main aim of this paper is to derive the Fourier transform for the Genocchi function. Recently, Kim [7] investigated the properties of the Euler functions and derived the interesting formula related to the infinite series by using the Fourier transform for the Euler function. In this paper, we investigate some arithmetical properties of the Genocchi functions and the Genocchi polynomials.

In [8], Cangul-Ozden-Simsek constructed new generating functions of the twisted (h,q)-extension of twisted Euler polynomials and numbers attached to the Dirichlet character χ. Cangul et al. [8] also defined the twisted (h,q)-extension of zeta functions, which interpolate the twisted (h,q)-extension of Euler numbers at negative integers. In this paper, we define the Cangul-Ozden-Simsek type twisted (h,q)-Genocchi polynomials and numbers. We have the generating function of (h,q)-Genocchi polynomials. We have the generalized twisted (h,q)-Genocchi numbers attached to the Dirichlet character χ. We define zeta functions related to (h,q)-Genocchi polynomials and we have the generating function of the generalized (h,q)-Genocchi numbers attached to χ.

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|1-q|_p < 1$. We also use the following notations:

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

For d, a fixed positive integer with $(p,d) = 1$, set

$$\begin{gathered} \mathrm{X} = \mathrm{X}_d = \frac{\lim\limits_{\overleftarrow{N}} \mathbb{Z}}{dp^N\mathbb{Z}}, \qquad \mathrm{X}_1 = \mathbb{Z}_p, \\ \mathrm{X}^* = \bigcup_{\substack{0<a<dp, \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \left\{x \in \mathrm{X} \mid x \equiv a \pmod{dp^N}\right\}, \end{gathered} \tag{1.2}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$. The distribution is defined by

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}. \tag{1.3}$$

We say that f is uniformly differential function at a point $a \in \mathbb{Z}_p$, and we write $f \in UD(\mathbb{Z}_p)$, if the difference quotients, $F_f(x,y) = f(x) - f(y)/x - y$ have a limit $f^{'}(a)$ as $(x,y) \to (a,a)$.

For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \tag{1.4}$$

The fermionic p-adic q-measures on $\mathbb{Z}_p$ are defined as

$$\mu_{-q}\left(a+dp^N\mathbb{Z}_p\right)=\frac{(-q)^a}{\left[dp^N\right]_{-q}},\tag{1.5}$$

and the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f)=\int_{\mathbb{Z}_p}f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\frac{1}{\left[p^N\right]_{-q}}\sum_{x=0}^{p^N-1}f(x)(-q)^x,\tag{1.6}$$

for $f\in UD(\mathbb{Z}_p)$. For $f\in UD(\mathbb{Z}_p)$, we note that

$$I_{-1}\left(f_1\right)+I_{-1}(f)=2f(0),\tag{1.7}$$

where $f_1(x)=f(x+1)$. (For details see [1–44].)

In this paper, we investigate arithmetical properties of the Genocchi functions and the Genocchi polynomials. In Section 2, we derive the Fourier transform on the Genocchi function. In Section 3, we define the Cangul-Ozden-Simsek type twisted (h,q)-Genocchi polynomials and numbers. We have the generating function of (h,q)-Genocchi polynomials. We also have the generalized twisted (h,q)-Genocchi numbers attached to χ. In Section 4, we define zeta functions related to (h,q)-Genocchi polynomials and we have the generating function of the generalized (h,q)-Genocchi numbers attached to χ.

2. Genocchi numbers and functions

The Genocchi numbers are defined as

$$\frac{2t}{e^t+1}=e^{Gt}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}\quad\text{for } |t|<\pi,\tag{2.1}$$

where we use the technique method notation by replacing G^n by G_n, symbolically. From this definition, we can derive the following relation:

$$G_0=0,\qquad (G+1)^n+G_n=\begin{cases}2 & \text{if } n=1,\\ 0 & \text{if } n>1.\end{cases}\tag{2.2}$$

From (2.2), we note that $G_0=0$, $G_1=1$, $G_2=-1,\ldots$, and $G_{2k+1}=0$, $G_{2k}\in\mathbb{Z}$ $(k=1,2,\ldots)$. The Genocchi polynomials $G_n(x)$ are defined as

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad\text{for } x\in\mathbb{R}.\tag{2.3}$$

From (2.1) and (2.3), we can derive

$$G_n(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} G_\ell x^{n-\ell}, \quad \text{where } \binom{n}{\ell} = \frac{n(n-1)\cdots(n-\ell+1)}{\ell!}. \tag{2.4}$$

By (2.1), it is not difficult to show that the recurrence relation for the Genocchi numbers is given by

$$G_0 = 0, \qquad \sum_{\ell=0}^{n} \binom{n}{\ell} G_\ell + G_n = 2\delta_{1,n}, \tag{2.5}$$

where $\delta_{1,n}$ is the Kronecker symbol.

From (2.4) and (2.5), we note that

$$G_n(1) + G_n = 0 \quad \text{for } n \geq 2. \tag{2.6}$$

Thus, we obtain the following lemma.

Lemma 2.1. *For $n (\geq 2) \in \mathbb{N}$, one has $G_n(1) = -G_n$.*

From (2.4), we can easily derive

$$\begin{aligned} \frac{d}{dx} G_n(x) &= \frac{d}{dx} \sum_{\ell=0}^{n} \binom{n}{\ell} G_\ell x^{n-\ell} = \sum_{\ell=0}^{n} \binom{n}{\ell} (n-\ell) G_\ell x^{n-\ell-1} \\ &= n \sum_{\ell=0}^{n} \frac{(n-1)!}{(n-\ell-1)!\ell!} G_\ell x^{n-1-\ell} = n \sum_{\ell=0}^{n} \binom{n-1}{\ell} G_\ell x^{n-1-\ell} = n G_{n-1}(x). \end{aligned} \tag{2.7}$$

By (2.7), we obtain the following proposition.

Proposition 2.2. *For $n \geq 0$, one has*

$$\int_0^x G_n(t)dt = \frac{1}{n+1} G_{n+1}(x). \tag{2.8}$$

From now on, we assume that $G_n(x)$ is the Genocchi function. Let us consider the Fourier transform for the Genocchi function $G_n(x)$ as follows.

For $m \in \mathbb{N}$, the Fourier transform on the Genocchi function is given by

$$G_m(x) = \sum_{n=-\infty}^{\infty} a_n^{(m)} e^{(2n+1)\pi i x}, \quad \left(a_n^{(m)} \in \mathbb{C}\right), \tag{2.9}$$

where

$$a_n^{(m)} = \int_0^1 G_m(x) e^{-(2n+1)\pi i x} dx. \tag{2.10}$$

From (2.8) and (2.10), we note that

$$\begin{aligned} a_n^{(m)} &= \int_0^1 G_m(x) e^{-(2n+1)\pi i x} dx \\ &= \left[\frac{G_{m+1}(x)}{m+1} e^{-(2n+1)\pi i x} \right]_0^1 + \frac{(2n+1)\pi i}{m+1} \int_0^1 G_{m+1}(x) e^{-(2n+1)\pi i x} dx \\ &= \frac{(2n+1)\pi i}{m+1} a_n^{(m+1)}. \end{aligned} \tag{2.11}$$

Thus, for $m \geq 2$, we have

$$a_n^{(m)} = \frac{m}{(2n+1)\pi i} a_n^{(m-1)} = \frac{m(m-1)}{((2n+1)\pi i)^2} a_n^{(m-2)} = \cdots = \frac{m!}{((2n+1)\pi i)^{m-1}} a_n^{(1)}. \tag{2.12}$$

From (2.4) and (2.10), we derive

$$a_n^{(1)} = \int_0^1 G_1(x) e^{-(2n+1)\pi i x} dx = \int_0^1 e^{-(2n+1)\pi i x} dx = \left[\frac{-e^{-(2n+1)\pi i x}}{(2n+1)\pi i} \right]_0^1 = \frac{2}{(2n+1)\pi i}. \tag{2.13}$$

From (2.12) and (2.13), we can derive

$$a_n^{(m)} = \frac{m!2}{((2n+1)\pi i)^m} \quad (m \in \mathbb{N}), \ a_n^{(0)} = 0. \tag{2.14}$$

By (2.9) and (2.14), we have that $G_0(x) = 0$ and

$$G_m(x) = m!2 \sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\pi i x}}{((2n+1)\pi i)^m}, \quad \text{for } 0 \leq x < 1. \tag{2.15}$$

Therefore, we obtain the following theorem.

Theorem 2.3. *For $m \in \mathbb{Z}_+$, $x \in \mathbb{R}$ with $0 \leq x < 1$, one has*

$$G_m(x) = m!2 \sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\pi i x}}{((2n+1)\pi i)^m}. \tag{2.16}$$

If we take $x = 1$, then we have

$$G_m(1) = -m!2 \sum_{n=-\infty}^{\infty} \frac{1}{((2n+1)\pi i)^m}. \tag{2.17}$$

By (2.17) *and Lemma 2.1, we obtain the following corollary.*

Corollary 2.4. *For $m \in \mathbb{Z}_+$, one has*

$$G_m = m!2\sum_{n=-\infty}^{\infty}\frac{1}{((2n+1)\pi i)^m}. \tag{2.18}$$

From Corollary 2.4, we note that

$$G_{2m} = (2m)!2\sum_{n=-\infty}^{\infty}\frac{1}{((2n+1)\pi i)^{2m}} = (2m)!2\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^{2m}\pi^{2m}(-1)^m}. \tag{2.19}$$

Thus, we have

$$\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^{2m}} = (-1)^m\frac{G_{2m}}{2(2m)!}\pi^{2m}. \tag{2.20}$$

By (2.20)*, we obtain the following corollary.*

Corollary 2.5. *For $m \in \mathbb{Z}_+$, one has*

$$\sum_{n=1}^{\infty}\frac{1}{(2n+1)^{2m}} = (-1)^m\frac{G_{2m}}{4(2m)!}\pi^{2m}. \tag{2.21}$$

3. (h,q)-extension of twisted Genocchi numbers and polynomials

In this section, we will define the (h,q)-extensions of twisted Genocchi numbers and polynomials which are the Cangul-Ozden-Simsek type twisted (h,q)-Genocchi numbers and polynomials, respectively. We will have the generating function of (h,q)-Genocchi polynomials and the generalized twisted (h,q)-Genocchi numbers attached to χ.

Let $f(x) = e^{xt}$. Then, we have from the definition of the Genocchi numbers and the fermionic p-adic q-integral on $\mathbb{Z}_p$ that

$$\begin{aligned} t\int_{\mathbb{Z}_p} e^{xt}\,d\mu_{-1}(x) &= \sum_{n=0}^{\infty}G_n\frac{t^n}{n!} = t\sum_{n=0}^{\infty}\frac{G_{n+1}}{n+1}\frac{t^n}{n!}, \\ t\int_{\mathbb{Z}_p} e^{(x+y)t}\,d\mu_{-1}(y) &= \sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} = t\sum_{n=0}^{\infty}\frac{G_{n+1}(x)}{n+1}\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

Thus, we obtain

$$\int_{\mathbb{Z}_p} x^n\,d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \qquad \int_{\mathbb{Z}_p}(x+y)^n\,d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}. \tag{3.2}$$

For $f \in UD(\mathbb{Z}_p)$ and $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p} f(x+n)d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) + 2\sum_{\ell=0}^{n-1}(-1)^{n-1+\ell} f(\ell). \tag{3.3}$$

By (3.2) and (3.3), if we take $f(x) = x^k (k \in \mathbb{Z}^+)$, we easily see that

$$\int_{\mathbb{Z}_p} (x+n)^k d\mu_{-1}(x) - \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) = 2\sum_{\ell=0}^{n-1}(-1)^{\ell-1}\ell^k \quad \text{if } n \equiv 0 \pmod 2. \tag{3.4}$$

Thus, we have

$$\frac{G_{k+1}(n)}{k+1} - \frac{G_{k+1}}{k+1} = 2\sum_{\ell=0}^{n-1}(-1)^{\ell-1}\ell^k \quad \text{if } n \equiv 0 \pmod 2 \tag{3.5}$$

If $n \equiv 1 \pmod 2$, then we know that

$$\int_{\mathbb{Z}_p} (x+n)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) = 2\sum_{\ell=0}^{n-1}(-1)^{\ell}\ell^k. \tag{3.6}$$

Thus, we get

$$\frac{G_{k+1}(n)}{k+1} + \frac{G_{k+1}}{k+1} = 2\sum_{\ell=0}^{n-1}(-1)^{\ell}\ell^k \quad \text{if } n \equiv 1 \pmod 2 \tag{3.7}$$

We can consider the generalized Genocchi numbers as follows:

$$\frac{G_{n+1}}{n+1} = \int_X x^n d\mu_{-1}(x), \qquad G_0 = 0, \tag{3.8}$$

where $n \in \mathbb{Z}_+$. Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. From (3.3) and (3.8), we note that

$$t\int_X e^{xt} d\mu_{-1}(x) = \frac{2\sum_{\ell=0}^{d-1}(-1)^{\ell} e^{\ell t}}{e^{dt}+1} t = \sum_{n=0}^{\infty} \frac{G_n}{n!} t^n. \tag{3.9}$$

By (3.8) and (3.9), it is not difficult to show that

$$\begin{aligned}
\frac{G_{n+1}}{n+1} &= \int_X x^n d\mu_{-1}(x) \\
&= d^n \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} \left(\frac{a}{d} + x\right)^n d\mu_{-1}(x) \\
&= d^n \sum_{a=0}^{d-1} (-1)^a \frac{G_{n+1}(a/d)}{n+1}, \\
\int_X (x+y)^n d\mu_{-1}(y) &= d^n \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} \left(\frac{x+a}{d} + y\right)^n d\mu_{-1}(y).
\end{aligned} \tag{3.10}$$

Thus, the distribution relations for the Genocchi numbers and the Genocchi polynomials for $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ are obtained as follows (cf. [6]):

$$\begin{aligned}
\frac{G_{n+1}}{n+1} &= d^n \sum_{a=0}^{d-1} (-1)^a \frac{G_{n+1}(a/d)}{n+1}, \\
\frac{G_{n+1}(x)}{n+1} &= d^n \sum_{a=0}^{d-1} (-1)^a \frac{G_{n+1}((x+a)/d)}{n+1}.
\end{aligned} \tag{3.11}$$

By using the multivariate integral, we can also consider the multiple Genocchi numbers and polynomials.

Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ be indeterminate and let $h \in \mathbb{Z}$. Then, we note that

$$t \int_{\mathbb{Z}_p} q^{hy} e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{q^h e^t + 1} e^{xt}. \tag{3.12}$$

Now, we define the Cangul-Ozden-Simsek type (h,q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$ as follows:

$$\frac{2t}{q^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{3.13}$$

From (3.13), we note that

$$\int_{\mathbb{Z}_p} q^{hy} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1,q}^{(h)}(x)}{n+1}. \tag{3.14}$$

Let C_{p^n} be the space of primitive p^n-th root of unity with

$$C_{p^n} = \left\{ \xi \in \mathbb{C}_p \mid \xi^{p^n} = 1 \right\}, \tag{3.15}$$

and let T_p be the direct limit of C_{p^n}, that is,

$$T_p = \lim_{n\to\infty} C_{p^n} = \bigcup_{n\geq 0} C_{p^n}, \tag{3.16}$$

and then T_p is a p-adic locally constant space.

For $\xi \in T_p$, we define the Cangul-Ozden-Simsek type twisted (h,q)-Genocchi polynomials $G_{n,q,\xi}^{(h)}(x)$ as follows:

$$t\int_{\mathbb{Z}_p} q^{hy}\xi^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{\xi q^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q,\xi}^{(h)}(x)\frac{t^n}{n!}. \tag{3.17}$$

By (3.17), we have

$$\int_{\mathbb{Z}_p} q^{hy}\xi^y (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1,q,\xi}^{(h)}(x)}{n+1}. \tag{3.18}$$

From the result of Cangul et al. [8], we note that

$$E_{n+1,q,\xi}^{(h)}(x) = \frac{G_{n+1,q,\xi}^{(h)}(x)}{n+1}, \tag{3.19}$$

where $E_{n+1,q,\xi}^{(h)}(x)$ is the twisted (h,q)-Euler polynomials.

Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then, we consider the generalized Genocchi numbers attached to χ as follows:

$$\frac{G_{n+1,\chi}}{n+1} = \int_X \chi(x)x^n d\mu_{-1}(x), \qquad G_{0,\chi} = 0, \tag{3.20}$$

where $n \in \mathbb{Z}_+$.

From (3.3) and (3.20), we note that

$$t\int_X e^{xt}\chi(x) d\mu_{-1}(x) = \frac{2\sum_{\ell=0}^{d-1}(-1)^{\ell}\chi(\ell)e^{\ell t}}{e^{dt}+1}t = \sum_{n=0}^{\infty}\frac{G_{n,\chi}}{n!}t^n. \tag{3.21}$$

By (3.20) and (3.21), it is not difficult to show that

$$\begin{aligned}\frac{G_{n+1,\chi}}{n+1} &= \int_X \chi(x)x^n d\mu_{-1}(x)\\ &= d^n\sum_{a=0}^{d-1}(-1)^a\chi(a)\int_{\mathbb{Z}_p}\left(\frac{a}{d}+x\right)^n d\mu_{-1}(x)\\ &= d^n\sum_{a=0}^{d-1}(-1)^a\chi(a)\frac{G_{n+1}(a/d)}{n+1}.\end{aligned} \tag{3.22}$$

Now, we also consider the Cangul-Ozden-Simsek type twisted (h,q)-Genocchi numbers attached to χ as follows.

For $\xi \in T_p$ and $h \in \mathbb{Z}$, we have

$$\begin{aligned}t\int_X \chi(x)\xi^x q^{hx}e^{xt}d\mu_{-1}(x) &= t\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}\xi^a q^{ha}\int_{\mathbb{Z}_p} e^{xdt}\xi^{dx}q^{hdx}d\mu_{-1}(x)\\ &= \frac{2t\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}\xi^a q^{ha}}{q^{hd}\xi^d e^{dt}+1}\\ &= \sum_{n=0}^{\infty}G_{n,q,\xi,\chi}^{(h)}\frac{t^n}{n!}.\end{aligned} \tag{3.23}$$

From (3.23), we have

$$\int_X \chi(x)\xi^x q^{hx}x^n d\mu_{-1}(x) = \frac{G_{n+1,q,\xi,\chi}^{(h)}}{n+1}, \quad \text{for } n \ge 0. \tag{3.24}$$

From the result of Cangul et al. [8], we note that

$$E_{n,q,\xi,\chi}^{(h)} = \frac{G_{n+1,q,\xi,\chi}^{(h)}}{n+1}, \quad \text{for } n \ge 0, \tag{3.25}$$

where $E_{n,q,\xi,\chi}^{(h)}$ are called the generalized twisted (h,q)-Euler numbers attached to χ.

4. Zeta functions related to the Genocchi polynomials

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let $F(t,x)$ be the generating function of (h,q)-Genocchi polynomials defined as follows:

$$F(t,x) = \frac{2t}{q^h e^t+1}e^{xt} = \sum_{n=0}^{\infty}G_{n,q}^{(h)}(x)\frac{t^n}{n!}, \tag{4.1}$$

where $|t + h\ \log q| < \pi$.

Then, we note that

$$F(t,x) = 2t\sum_{n=0}^{\infty}(-1)^n q^{nh} e^{(n+x)t}. \tag{4.2}$$

By (4.1) and (4.2), we easily see that

$$G_{k,q}^{(h)}(x) = \frac{d^k}{dt^k}F(t,x)|_{t=0} = 2k\sum_{n=0}^{\infty}(-1)^n q^{nh}(n+x)^{k-1}, \tag{4.3}$$

for $k \in \mathbb{N}$. Therefore, we obtain the following proposition.

Proposition 4.1. *For $k \in \mathbb{N}$, one has*

$$\frac{G_{k,q}^{(h)}(x)}{k} = 2\sum_{n=0}^{\infty}(-1)^n q^{nh}(n+x)^{k-1}. \tag{4.4}$$

From Proposition 4.1, we can derive the Genocchi zeta function which interpolates Genocchi polynomials related to (h,q)-Genocchi polynomials at negative integers.

For $s \in \mathbb{C}$, we define the Hurwitz-type Genocchi zeta functions related to (h,q)-Genocchi polynomials and numbers as follows.

Definition 4.2. For $s \in \mathbb{N}$, one has

$$\zeta_G(s,x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n q^{nh}}{(n+x)^s}, \qquad \zeta_G(s) = 2\sum_{n=1}^{\infty}\frac{(-1)^n q^{nh}}{n^s}. \tag{4.5}$$

By Proposition 4.1 and Definition 4.2, we obtain the following theorem.

Theorem 4.3. *For $k \in \mathbb{N}$, one has*

$$\zeta_G(1-k,x) = \frac{G_{k,q}^{(h)}(x)}{k}, \qquad \zeta_G(1-k) = \frac{G_{k,q}^{(h)}}{k}. \tag{4.6}$$

The generating function of the generalized (h,q)-Genocchi numbers attached to χ is given by

$$F_{\chi}^{(h)}(t) = \sum_{a=0}^{d-1}\frac{2t(-1)^a\chi(a)e^{at}q^{ha}}{q^{hd}e^{dt}+1} = \sum_{n=0}^{\infty}G_{n,\chi,q}^{(h)}\frac{t^n}{n!}, \tag{4.7}$$

where $|h\ \log q + t| < \pi/d$, $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Therefore, we have

$$F_{\chi}^{(h)}(t) = 2t\sum_{n=0}^{\infty}(-1)^n\chi(n)q^{hn}e^{nt}, \tag{4.8}$$

where χ is a nontrivial character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. From (4.8), it follows that

$$G_{k,\chi,q}^{(h)} = \frac{d^k}{dt^k} F_{\chi}^{(h)}(t)|_{t=0} = 2k \sum_{n=1}^{\infty} (-1)^n \chi(n) q^{hn} n^{k-1}. \tag{4.9}$$

This is equivalent to

$$\frac{G_{k,\chi,q}^{(h)}}{k} = 2\sum_{n=1}^{\infty} (-1)^n \chi(n) q^{hn} n^{k-1}. \tag{4.10}$$

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, let χ be a primitive Dirichlet character with conductor d. Then, we define

$$L_q^{(h)}(s, \chi) = 2\sum_{n=1}^{\infty} \frac{(-1)^n \chi(n) q^{hn}}{n^s} \quad (s \in \mathbb{C}), \; h \in \mathbb{Z}. \tag{4.11}$$

From (4.10) and (4.11), we obtain the following theorem.

Theorem 4.4. *For $k \in \mathbb{N}$, one has*

$$L_q^{(h)}(1-k, \chi) = \frac{G_{k,\chi,q}^{(h)}}{k}. \tag{4.12}$$

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