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Research Article

On the Solutions of Systems of Difference Equations

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We show that every solution of the following system of difference equations $x_{n+1}^{(1)} = x_n^{(2)}/(x_n^{(2)}-1)$, $x_{n+1}^{(2)} = x_n^{(3)}/(x_n^{(3)}-1),\dots,x_{n+1}^{(k)} = x_n^{(1)}/(x_n^{(1)}-1)$ as well as of the system $x_{n+1}^{(1)} = x_n^{(k)}/(x_n^{(k)}-1)$, $x_{n+1}^{(2)} = x_n^{(1)}/(x_n^{(1)}-1),\dots,x_{n+1}^{(k)} = x_n^{(k-1)}/(x_n^{(k-1)}-1)$ is periodic with period $2k$ if $k \neq 0$ (mod2), and with period k if $k = 0$ (mod 2) where the initial values are nonzero real numbers for $x_0^{(1)}, x_0^{(2)},\dots,x_0^{(k)} \neq 1$.

1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so on [1]. So, recently there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions. (see [1–12] and the references cited therein.)

Papaschinopoulos and Schinas [9, 10] studied the behavior of the positive solutions of the system of two Lyness difference equations

$$x_{n+1} = \frac{by_n + c}{x_{n-1}}, \quad y_{n+1} = \frac{dx_n + e}{y_{n-1}}, \quad n = 0, 1, 2, \dots, \tag{1.1}$$

where b, c, d, e are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive.

In [2] Camouzis and Papaschinopoulos studied the behavior of the positive solutions of the system of two difference equations

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, 2, \dots, \tag{1.2}$$

where the initial values $x_i, y_i, i = -m, -m+1, \dots, 0$ are positive numbers and m is a positive integer.

Moreover, Çinar [3] investigated the periodic nature of the positive solutions of the system of difference equations

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \quad z_{n+1} = \frac{1}{x_{n-1}}, \quad n = 0, 1, 2, \dots, \tag{1.3}$$

where the initial values x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0 are positive real numbers.

Also, Özban [7] investigated the periodic nature of the solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_{n-k}}, \quad y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}, \quad n = 0, 1, 2, \dots, \tag{1.4}$$

where k is a nonnegative integer, m is a positive integer, and the initial values $x_{-m}, x_{-m+1}, \dots, x_0, y_{-m-k}, y_{-m-k+1}, \dots, y_0$ are positive real numbers.

In [12] Irićanin and Stević studied the positive solution of the following two systems of difference equations

$$x^{(1)}_{n+1} = \frac{1 + x^{(2)}_n}{x^{(3)}_{n-1}}, \qquad x^{(2)}_{n+1} = \frac{1 + x^{(3)}_n}{x^{(4)}_{n-1}}, \dots, \qquad x^{(k)}_{n+1} = \frac{1 + x^{(1)}_n}{x^{(2)}_{n-1}}, \tag{1.5}$$

$$x^{(1)}_{n+1} = \frac{1 + x^{(2)}_n + x^{(3)}_{n-1}}{x^{(4)}_{n-2}}, \qquad x^{(2)}_{n+1} = \frac{1 + x^{(3)}_n + x^{(4)}_{n-1}}{x^{(5)}_{n-2}}, \dots, \qquad x^{(k)}_{n+1} = \frac{1 + x^{(1)}_n + x^{(2)}_{n-1}}{x^{(3)}_{n-2}}, \tag{1.6}$$

where $k \in \mathbb{N}$ fixed.

In [11] Papaschinopoulos et al. studied the system of difference equations

$$\begin{gathered}
x_1(n+1) = \frac{a_k x_k(n) + b_k}{x_{k-1}(n-1)}, \\
x_2(n+1) = \frac{a_1 x_1(n) + b_1}{x_k(n-1)}, \\
x_i(n+1) = \frac{a_{i-1} x_{i-1}(n) + b_{i-1}}{x_{i-2}(n-1)}, \quad \text{for } i = 3, 4, \dots, k,
\end{gathered} \tag{1.7}$$

where a_i, b_i (for $i = 1, 2, \dots, k$) are positive constants, $k \geqslant 3$ is an integer, and the initial values $x_i(-1)$, $x_i(0)$ (for $i = 1, 2, \dots, k$) are positive real numbers.

It is well known that all well-defined solutions of the difference equation

$$x_{n+1} = \frac{x_n}{x_n - 1} \tag{1.8}$$

are periodic with period two. Motivated by (1.8), we investigate the periodic character of the following two systems of difference equations:

$$x^{(1)}_{n+1} = \frac{x^{(2)}_n}{x^{(2)}_n - 1}, x^{(2)}_{n+1} = \frac{x^{(3)}_n}{x^{(3)}_n - 1}, \dots, x^{(k)}_{n+1} = \frac{x^{(1)}_n}{x^{(1)}_n - 1}, \tag{1.9}$$

$$x^{(1)}_{n+1} = \frac{x^{(k)}_n}{x^{(k)}_n - 1}, x^{(2)}_{n+1} = \frac{x^{(1)}_n}{x^{(1)}_n - 1}, \dots, x^{(k)}_{n+1} = \frac{x^{(k-1)}_n}{x^{(k-1)}_n - 1} \tag{1.10}$$

which can be considered as a natural generalizations of (1.8).

In order to prove main results of the paper we need an auxiliary result which is contained in the following simple lemma from number theory. Let $GCD(k,l)$ denote the greatest common divisor of the integers k and l.

Lemma 1.1. *Let $k \in \mathbb{N}$ and $GCD(k,2) = 1$, then the numbers $a_l = 2l + 1$ (or $a_l = -2l + 1$) for $l = 0, 1, \ldots, k-1$ satisfy the following property:*

$$a_{l_1} - a_{l_2} \neq 0 \, (\mathrm{mod}\, k) \quad \text{when}\, l_1 \neq l_2. \tag{1.11}$$

Proof. Suppose the contrary, then we have $2(l_1 - l_2) = a_{l_1} - a_{l_2} = ks$ for some $s \in \mathbb{Z} \setminus \{0\}$.

Since $GCD(k,2) = 1$, it follows that k is a divisor of $l_1 - l_2$. On the other hand, since $l_1, l_2 \in \{0, 1, \ldots, k-1\}$, we have $|l_1 - l_2| < k$ which is a contradiction. □

Remark 1.2. From Lemma 1.1 we see that the rests b_l for $l = 0, 1, \ldots, k-1$ of the numbers $a_l = 2l+1$ for $l = 0, 1, \ldots, k-1$, obtained by dividing the numbers a_l by k, are mutually different, they are contained in the set $A = \{0, 1, \ldots, k-1\}$, make a permutation of the ordered set $(0, 1, \ldots, k-1)$, and finally $a_k = 2k+1$ is the first number of the form $2l+1$, $l \in \mathbb{N}$, such that $a_1 - a_0 \equiv 0 \, (\mathrm{mod}\, k)$.

2. The main results

In this section, we formulate and prove the main results in this paper.

Theorem 2.1. *Consider* (1.9) *where $k \geqslant 1$. Then the following statements are true:*

(a) *if $k \neq 0 \, (\mathrm{mod}\, 2)$, then every solution of* (1.9) *is periodic with period $2k$,*

(b) *if $k = 0 \, (\mathrm{mod}\, 2)$, then every solution of* (1.9) *is periodic with period k.*

Proof. First note that the system is cyclic. Hence it is enough to prove that the sequence $(x_n^{(1)})$ satisfies conditions (a) and (b) in the corresponding cases.

Further, note that for every $s \in \mathbb{N}$ system (1.9) is equivalent to a system of ks difference equations of the same form, where

$$x_n^{(i)} = x_n^{(rk+i)}, \quad \forall n \in \mathbb{N},\, i \in \{1, \ldots, k\},\, r = 0, 1, \ldots, s-1. \tag{2.1}$$

On the other hand, we have

$$x_{n+1}^{(1)} = \frac{x_n^{(2)}}{x_n^{(2)} - 1} = \frac{x_{n-1}^{(3)} / (x_{n-1}^{(3)} - 1)}{(x_{n-1}^{(3)} / (x_{n-1}^{(3)} - 1)) - 1} = x_{n-1}^{(3)}. \tag{2.2}$$

(a) Let b_l for $l = 0, 1, \ldots, k-1$ be the rests mentioned in Remark 1.2. Then from (2.2) and Lemma 1.1 we obtain that

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = \cdots = x_{n+1-2(k-1)}^{(2k-1)} = x_{n+1-2k}^{(2k+1)}. \tag{2.3}$$

Using (2.1) for sufficently large s, we obtain that (2.3) is equivalent to (here we use the condition $GCD(k,2) = 1$)

$$x_{n+1}^{(1)} = x_{n-1}^{(b_1)} = x_{n-3}^{(b_2)} = \cdots = x_{n+1-2(k-1)}^{(b_{k-1})} = x_{n+1-2k}^{(1)}. \tag{2.4}$$

From this and since by Lemma 1.1 the numbers $1, b_1, b_2, \ldots, b_{k-1}$ are pairwise different, the result follows in this case.

(b) Let $k = 2s$, for some $s \in \mathbb{N}$. By (2.1) we have

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = \cdots = x_{n+1-2s}^{(2s+1)} = x_{n+1-2s}^{(1)} \tag{2.5}$$

which yields the result. □

Remark 2.2. In order to make the proof of Theorem 2.1 clear to the reader, we explain what happens in the cases $k = 2$ and $k = 3$.

For $k = 2$, system (1.9) is equivalent to the system

$$\begin{aligned} x_{n+1}^{(1)} &= \frac{x_n^{(2)}}{x_n^{(2)} - 1}, \qquad x_{n+1}^{(2)} = \frac{x_n^{(3)}}{x_n^{(3)} - 1}, \qquad x_{n+1}^{(3)} = \frac{x_n^{(4)}}{x_n^{(4)} - 1}, \\ x_{n+1}^{(4)} &= \frac{x_n^{(5)}}{x_n^{(5)} - 1}, \qquad x_{n+1}^{(5)} = \frac{x_n^{(6)}}{x_n^{(6)} - 1}, \qquad x_{n+1}^{(6)} = \frac{x_n^{(1)}}{x_n^{(1)} - 1}, \end{aligned} \tag{2.6}$$

where we consider that

$$x_n^{(1)} = x_n^{(3)} = x_n^{(5)}, \quad x_n^{(2)} = x_n^{(4)} = x_n^{(6)}, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

From this and (2.2), we have

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-1}^{(1)} \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Using again (2.2), we get $x_{n+1}^{(1)} = x_{n-1}^{(1)}$, which means that the sequence $x_n^{(1)}$ is periodic with period equal to 2. If $k = 3$, system (1.9) is equivalent to system (2.6) where we consider that $x_n^{(1)} = x_n^{(4)}$, $x_n^{(2)} = x_n^{(5)}$, and $x_n^{(3)} = x_n^{(6)}$. Using this and (2.2) subsequently, it follows that

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = x_{n-3}^{(2)} = x_{n-5}^{(4)} = x_{n-5}^{(1)}, \tag{2.9}$$

that is, the sequence $x_n^{(1)}$ is periodic with period 6.

Remark 2.3. The fact that every solution of (1.8) is periodic with period two can be considered as the case $k = 1$ in Theorem 2.1, that is, we can take that

$$x_n^{(1)} = x_n^{(2)} = \cdots = x_n^{(k)}, \quad \forall n \in \mathbb{N}. \tag{2.10}$$

Similarly to Theorem 2.1, using Lemma 1.1 with $a_l = -2l + 1$ for $l = 0, 1, \ldots, k - 1$, the following theorem can be proved.

Theorem 2.4. *Consider* (1.10) *where* $k \geqslant 1$. *Then the following statements are true:*

(a) *if* $k \neq 0 \pmod 2$, *then every solution of* (1.10) *is periodic with period* $2k$,

(b) *if* $k = 0 \pmod 2$, *then every solution of* (1.10) *is periodic with period* k.

Proof. First note that the system is cyclic. Hence, it is enough to prove that the sequence $(x_n^{(1)})$ satisfies conditions (a) and (b) in the corresponding cases.

Indeed, similarly to (2.2), we have

$$x_{n+1}^{(1)} = \frac{x_n^{(k)}}{x_n^{(k)} - 1} = \frac{x_{n-1}^{(k-1)} / (x_{n-1}^{(k-1)} - 1)}{(x_{n-1}^{(k-1)} / (x_{n-1}^{(k-1)} - 1)) - 1} = x_{n-1}^{(k-1)}. \tag{2.11}$$

(a) Let b_l for $l = 0, 1, \ldots, k-1$ be the rests mentioned in Remark 1.2. Then from (2.11) and Lemma 1.1 we obtain that

$$x_{n+1}^{(1)} = x_{n-1}^{(k-1)} = x_{n-3}^{(k-3)} = \cdots = x_{n+1-2(k-1)}^{(-2k+3)} = x_{n+1-2k}^{(-2k+1)}. \tag{2.12}$$

Using (2.1) for sufficiently large s, we obtain that (2.12) is equivalent to (here we use the condition $GCD(k, 2) = 1$)

$$x_{n+1}^{(1)} = x_{n-1}^{(b_1)} = x_{n-3}^{(b_2)} = \cdots = x_{n+1-2(k-1)}^{(b_{k-1})} = x_{n+1-2k}^{(1)}. \tag{2.13}$$

From this and since by Lemma 1.1 the numbers $1, b_1, b_2, \ldots, b_{k-1}$ are pairwise different, the result follows in this case.

(b) Let $k = 2s$ for some $s \in \mathbb{N}$. By (2.1) we have

$$x_{n+1}^{(1)} = x_{n-1}^{(k-1)} = x_{n-3}^{(k-3)} = \cdots = x_{n+1-2s}^{(1)} \tag{2.14}$$

which yields the result. □

Corollary 2.5. *Let* $\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\}$ *be solutions of* (1.9) *with the initial values* $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$. *Assume that*

$$x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)} > 1, \tag{2.15}$$

then all solutions of (1.9) *are positive.*

Proof. We consider solutions of (1.9) with the initial values $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$ satisfying (2.15). If $k = 0 \pmod 2$, then from (1.9) and (2.15), we have

$$\text{if } i \text{ is odd, then } x_i^{(m)} = \begin{cases} \dfrac{x_0^{(i+m)}}{x_0^{(i+m)} - 1} & \text{for } (i+m) \le k, \\[2ex] \dfrac{x_0^{(i+m-k)}}{x_0^{(i+m-k)} - 1} & \text{for } (i+m) > k, \end{cases} \tag{2.16}$$

$$\text{if } i \text{ is even, then } x_i^{(m)} = \begin{cases} x_0^{(i+m)} & \text{for } (i+m) \le k, \\ x_0^{(i+m-k)} & \text{for } (i+m) > k, \end{cases}$$

for $i = 1, 2, \ldots, k$ and $m = 1, 2, \ldots, k$.

If $k \neq 0 \pmod 2$, then from (1.9) and (2.15), we have

$$\text{If } i \text{ is odd, then } x_i^{(m)} = \begin{cases} \dfrac{x_0^{(i+m)}}{x_0^{(i+m)} - 1} & \text{for } (i+m) \leq k \text{ or } (i+m) = 2k, \\ \dfrac{x_0^{(i+m-k)}}{x_0^{(i+m-k)} - 1} & \text{for } k < (i+m) < 2k, \\ \dfrac{x_0^{(i+m-2k)}}{x_0^{(i+m-2k)} - 1} & \text{for } 2k < (i+m) < 3k, \end{cases} \tag{2.17}$$

$$\text{If } i \text{ is even, then } x_i^{(m)} = \begin{cases} x_0^{(i+m)} & \text{for } (i+m) \leq k, \\ x_0^{(i+m-k)} & \text{for } k < (i+m) < 2k, \\ x_0^{(i+m-2k)} & \text{for } 2k < (i+m) \leq 3k, \end{cases}$$

for $i = 1, 2, \ldots, 2k$ and $m = 1, 2, \ldots, k$.

From (2.16) and (2.17), all solutions of (1.9) are positive. □

Corollary 2.6. *Let* $\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\}$ *be solutions of* (1.9) *with the initial values* $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$. *Assume that*

$$0 < x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)} < 1, \tag{2.18}$$

then $\{x_{2n}^{(1)}, x_{2n}^{(2)}, \ldots, x_{2n}^{(k)}\}$ *are positive,* $\{x_{2n+1}^{(1)}, x_{2n+1}^{(2)}, \ldots, x_{2n+1}^{(k)}\}$ *are negative for all* $n \geqslant 0$.

Proof. From (2.16), (2.17), and (2.18), the proof is clear. □

Corollary 2.7. *Let* $\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\}$ *be solutions of* (1.9) *with the initial values* $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$. *Assume that*

$$x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)} < 0, \tag{2.19}$$

then $\{x_{2n}^{(1)}, x_{2n}^{(2)}, \ldots, x_{2n}^{(k)}\}$ *are negative,* $\{x_{2n+1}^{(1)}, x_{2n+1}^{(2)}, \ldots, x_{2n+1}^{(k)}\}$ *are positive for all* $n \geqslant 0$.

Proof. From (2.16), (2.17), and (2.19), the proof is clear. □

Corollary 2.8. *Let* $\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\}$ *be solutions of* (1.9) *with the initial values* $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$, *then the following statements are true (for all* $n \geqslant 0$ *and* $i = 1, 2, \ldots, k$*) :*

(i) *if* $x_0^{(i)} \to \infty$, *then* $\{x_{2n}^{(i)}\} \to \infty$ *and* $\{x_{2n+1}^{(i)}\} \to 1^+$,

(ii) *if* $x_0^{(i)} \to 1^+$, *then* $\{x_{2n}^{(i)}\} \to 1^+$ *and* $\{x_{2n+1}^{(i)}\} \to \infty$,

(iii) *if* $x_0^{(i)} \to 1^-$, *then* $\{x_{2n}^{(i)}\} \to 1^-$ *and* $\{x_{2n+1}^{(i)}\} \to -\infty$,

(iv) *if* $x_0^{(i)} \to 0^+$, *then* $\{x_{2n}^{(i)}\} \to 0^+$ *and* $\{x_{2n+1}^{(i)}\} \to 0^-$,

(v) *if* $x_0^{(i)} \to 0^-$, *then* $\{x_{2n}^{(i)}\} \to 0^-$ *and* $\{x_{2n+1}^{(i)}\} \to 0^+$,

(vi) *if* $x_0^{(i)} \to -\infty$, *then* $\{x_{2n}^{(i)}\} \to -\infty$ *and* $\{x_{2n+1}^{(i)}\} \to 1^-$.

Proof. From (2.16) and (2.17), the proof is clear. □

Corollary 2.9. *Let* $\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\}$ *be solutions of* (1.10) *with the initial values* $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$. *Assume that*

$$x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)} > 1, \tag{2.20}$$

then all solutions of (1.10) *are positive.*

Proof. We consider solutions of (1.10) with the initial values $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$ satisfying (2.20). If $k = 0 \pmod 2$, then from (1.10) and (2.20), we have

$$\text{if } i \text{ is odd, then } x_i^{(m)} = \begin{cases} \dfrac{x_0^{(m-i)}}{x_0^{(m-i)} - 1} & \text{for } 0 < (m-i) < k, \\[2ex] \dfrac{x_0^{(k+m-i)}}{x_0^{(k+m-i)} - 1} & \text{for } (m-i) \le 0, \end{cases} \tag{2.21}$$

$$\text{if } i \text{ is even, then } x_i^{(m)} = \begin{cases} x_0^{(m-i)} & \text{for } 0 < (m-i) < k, \\ x_0^{(k+m-i)} & \text{for } (m-i) \le 0, \end{cases}$$

for $i = 1, 2, \ldots, k$ and $m = 1, 2, \ldots, k$.

If $k \ne 0 \pmod 2$, then from (1.10) and (2.20), we have

$$\text{if } i \text{ is odd, then } x_i^{(m)} = \begin{cases} \dfrac{x_0^{(k+m-i)}}{x_0^{(k+m-i)} - 1} & \text{for } (m-i) \le 0, \\[2ex] \dfrac{x_0^{(m-i)}}{x_0^{(m-i)} - 1} & \text{for } 0 < (m-i) < k, \\[2ex] \dfrac{x_0^{(2k+m-i)}}{x_0^{(2k+m-i)} - 1} & \text{for } -2k < (m-i) \le -k, \end{cases} \tag{2.22}$$

$$\text{if } i \text{ is even, then } x_i^{(m)} = \begin{cases} x_0^{(k+m-i)} & \text{for } (m-i) \le 0, \\ x_0^{(m-i)} & \text{for } 0 < (m-i) < k, \\ x_0^{(m-i+2k)} & \text{for } -2k < (m-i) \le -k, \end{cases}$$

for $i = 1, 2, \ldots, 2k$ and $m = 1, 2, \ldots, k$.

From (2.21) and (2.22), all solutions of (1.10) are positive. □

Corollary 2.10. *Let* $\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\}$ *be solutions of* (1.10) *with the initial values* $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$. *Assume that*

$$0 < x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)} < 1, \tag{2.23}$$

then $\{x_{2n}^{(1)}, x_{2n}^{(2)}, \ldots, x_{2n}^{(k)}\}$ *are positive,* $\{x_{2n+1}^{(1)}, x_{2n+1}^{(2)}, \ldots, x_{2n+1}^{(k)}\}$ *are negative for all* $n \geqslant 0$.

Table 1

i	1	2	3	4	5	6	7	8	9	10	11	12
$x_i^{(1)}$	$\frac{q}{q-1}$	r	$\frac{p}{p-1}$	q	$\frac{r}{r-1}$	p	$\frac{q}{q-1}$	r	$\frac{p}{p-1}$	q	$\frac{r}{r-1}$	p
$x_i^{(2)}$	$\frac{r}{r-1}$	p	$\frac{q}{q-1}$	r	$\frac{p}{p-1}$	q	$\frac{r}{r-1}$	p	$\frac{q}{q-1}$	r	$\frac{p}{p-1}$	q
$x_i^{(3)}$	$\frac{p}{p-1}$	q	$\frac{r}{r-1}$	p	$\frac{q}{q-1}$	r	$\frac{p}{p-1}$	q	$\frac{r}{r-1}$	p	$\frac{q}{q-1}$	r

Proof. From (2.21), (2.22) and (2.23), the proof is clear. □

Corollary 2.11. *Let* $\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\}$ *be solutions of* (1.10) *with the initial values* $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$. *Assume that*

$$x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)} < 0, \tag{2.24}$$

then $\{x_{2n}^{(1)}, x_{2n}^{(2)}, \ldots, x_{2n}^{(k)}\}$ *are negative,* $\{x_{2n+1}^{(1)}, x_{2n+1}^{(2)}, \ldots, x_{2n+1}^{(k)}\}$ *are positive for all* $n \geqslant 0$.

Proof. From (2.21), (2.22) and (2.24), the proof is clear. □

Corollary 2.12. *Let* $\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}\}$ *be solutions of* (1.10) *with the initial values* $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}$, *then following statements are true (for all* $n \geqslant 0$ *and* $i = 1, 2, \ldots, k$) :

(i) *if* $x_0^{(i)} \to \infty$, *then* $\{x_{2n}^{(i)}\} \to \infty$ *and* $\{x_{2n+1}^{(i)}\} \to 1^+$,

(ii) *if* $x_0^{(i)} \to 1^+$, *then* $\{x_{2n}^{(i)}\} \to 1^+$ *and* $\{x_{2n+1}^{(i)}\} \to \infty$,

(iii) *if* $x_0^{(i)} \to 1^-$, *then* $\{x_{2n}^{(i)}\} \to 1^-$ *and* $\{x_{2n+1}^{(i)}\} \to -\infty$,

(iv) *if* $x_0^{(i)} \to 0^+$, *then* $\{x_{2n}^{(i)}\} \to 0^+$ *and* $\{x_{2n+1}^{(i)}\} \to 0^-$,

(v) *if* $x_0^{(i)} \to 0^-$, *then* $\{x_{2n}^{(i)}\} \to 0^-$ *and* $\{x_{2n+1}^{(i)}\} \to 0^+$,

(vi) *if* $x_0^{(i)} \to -\infty$, *then* $\{x_{2n}^{(i)}\} \to -\infty$ *and* $\{x_{2n+1}^{(i)}\} \to 1^-$.

Proof. From (2.21), (2.22), and (2.24), the proof is clear. □

Example 2.13. Let $k = 3$. Then the solutions of (1.9), with the initial values $x_0^{(1)} = p$, $x_0^{(2)} = q$ and $x_0^{(3)} = r$, in its invertal of periodicity can be represented by Table 1.

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