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Research Article

The Periodic Character of the Difference Equation $x_{n+1} = f(x_{n-l+1}, x_{n-2k+1})$

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In this paper, we consider the nonlinear difference equation $x_{n+1} = f(x_{n-l+1}, x_{n-2k+1})$, $n = 0, 1, \ldots$, where $k, l \in \{1, 2, \ldots\}$ with $2k \neq l$ and $\gcd(2k, l) = 1$ and the initial values $x_{-\alpha}, x_{-\alpha} + 1, \ldots, x_0 \in (0, +\infty)$ with $\alpha = \max\{l-1, 2k-1\}$. We give sufficient conditions under which every positive solution of this equation converges to a (not necessarily prime) 2-periodic solution, which extends and includes corresponding results obtained in the recent literature.

1. Introduction

In this paper, we consider a nonlinear difference equation and deal with the question of whether every positive solution of this equation converges to a periodic solution. Recently, there has been a lot of interest in studying the global attractivity, the boundedness character, and the periodic nature of nonlinear difference equations (e.g., see [1, 2]). In [3], Grove et al. considered the following difference equation:

$$x_{n+1} = \frac{p + x_{n-(2m+1)}}{1 + x_{n-2r}}, \quad n = 0, 1, \ldots, \tag{E1}$$

where $p \in (0, +\infty)$ and the initial values $x_{-\alpha}, x_{-\alpha+1}, \ldots, x_0 \in (0, +\infty)$ with $\alpha = \max\{2r, 2m+1\}$, and proved that every positive solution of ($E1$) converges to (not necessarily prime) a $2s$-periodic solution with $s = \gcd(m+1, 2r+1)$. In [4], Stević investigated the periodic character of positive solutions of the following difference equation:

$$x_{n+1} = 1 + \frac{x_{n-2s+1}}{x_{n-(2r+1)s+1}}, \quad n = 0, 1, \ldots, \tag{E2}$$

and proved that every positive solution of $(E2)$ converges to (not necessarily prime) a $2s$-periodic solution, which generalized the main result of [5]. Furthermore, Stević [6] studied the periodic character of positive solutions of the following difference equation:

$$x_n = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{n-p_i}}{\sum_{j=1}^{m} \beta_j x_{n-q_j}}, \quad n = 1, 2, \ldots, \tag{E3}$$

where α_i, $i \in \{1, \ldots, k\}$, and β_j, $j \in \{1, \ldots, m\}$, are positive numbers such that $\Sigma_{i=1}^{k} \alpha_i = \Sigma_{j=1}^{m} \beta_j = 1$, and p_i, $i \in \{1, \ldots, k\}$, and q_j, $j \in \{1, \ldots, m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$. For closely related results, see [7, 8].

In this paper, we consider the more general equation

$$x_{n+1} = f\left(x_{n-l+1}, x_{n-2k+1}\right), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $k, l \in \{1, 2, \ldots\}$ with $2k \neq l$ and $\gcd(2k, l) = 1$, the initial values $x_{-\alpha}, x_{-\alpha+1}, \ldots, x_0 \in (0, +\infty)$ with $\alpha = \max\{l-1, 2k-1\}$, and f satisfies the following hypotheses:

(H_1) $f \in C(E \times E, (0, +\infty))$ with $a = \inf_{(u,v) \in E \times E} f(u, v) \in E$, where $E \in \{(0, +\infty), [0, +\infty)\}$;

(H_2) $f(u, v)$ is decreasing in u and increasing in v;

(H_3) there exists a decreasing function $g \in C((a, +\infty), (a, +\infty))$ such that

(i) for any $x > a$, $g(g(x)) = x$ and $x = f(g(x), x)$;

(ii) $\lim_{x \to a^+} g(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = a$.

The main result of this paper is the following theorem.

Theorem 1.1. *Every positive solution of* (1.1) *converges to (not necessarily prime) a* 2*-periodic solution.*

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Without loss of generality, we may assume $l < 2k$ (the proof for the case $l > 2k$ is similar); then

$$\{l, 2l, 3l, \ldots, 2kl\} = \{0, 1, 2, \ldots, 2k-1\} \bmod 2k. \tag{2.1}$$

Lemma 2.1. *Let* $\{x_n\}_{n=-\alpha}^{\infty}$ *be a positive solution of* (1.1). *Then there exists a real number* $L \in (a, +\infty)$ *such that* $L \le x_n \le g(L)$ *for all* $n \ge 1$. *Furthermore, let* $\limsup x_n = M$ *and* $\liminf x_n = m$, *then* $M = g(m)$ *and* $m = g(M)$.

Proof. By (H_1) and (H_2), we have

$$x_i = f\left(x_{i-l}, x_{i-2k}\right) > f\left(x_{i-l} + 1, x_{i-2k}\right) \ge a \quad \text{for every } 1 \le i \le \alpha + 1. \tag{2.2}$$

Then there exists $L \in (a, +\infty)$ with $L < g(L)$ such that

$$L \le x_i \le g(L) \quad \text{for every } 1 \le i \le \alpha + 1. \tag{2.3}$$

It follows from (2.3) and (H_3) that

$$g(L) = f\left(L, g(L)\right) \geq x_{\alpha+2} = f\left(x_{\alpha+2-l}, x_{\alpha+2-2k}\right) \geq f\left(g(L), L\right) = L. \tag{2.4}$$

Inductively, it follows that $L \leq x_n \leq g(L)$ for all $n \geq 1$.

Let $\limsup x_n = M$ and $\liminf x_n = m$, then there exist $A, B, C, D \in [m, M]$ and sequences $t_n \geq 1$ and $r_n \geq 1$ such that

$$\begin{aligned} &\lim_{n\to\infty} x_{t_n} = M, \qquad \lim_{n\to\infty} x_{t_n-l} = A, \qquad \lim_{n\to\infty} x_{t_n-2k} = B,\\ &\lim_{n\to\infty} x_{r_n} = m, \qquad \lim_{n\to\infty} x_{r_n-l} = C, \qquad \lim_{n\to\infty} x_{r_n-2k} = D. \end{aligned} \tag{2.5}$$

Thus by (1.1), (H_2), and (H_3), we have

$$\begin{aligned} f\left(g(M), M\right) = M = f(A, B) \leq f(m, M),\\ f\left(g(m), m\right) = m = f(C, D) \geq f(M, m), \end{aligned} \tag{2.6}$$

from which it follows that $g(M) \geq m$ and $g(m) \leq M$. Since g is decreasing, it follows that

$$m = g\left(g(m)\right) \geq g(M), \qquad M = g\left(g(M)\right) \leq g(m). \tag{2.7}$$

Therefore, $M = g(m)$ and $m = g(M)$. The proof is complete. □

Proof of Theorem 1.1. Let $\{x_n\}_{n=-\alpha}^{\infty}$ be a positive solution of (1.1) with the initial conditions $x_0, x_{-1}, \dots, x_{-\alpha} \in (0, +\infty)$. It follows from Lemma 2.1 that

$$a < \liminf x_n = m = g(M) \leq \limsup x_n = M < +\infty. \tag{2.8}$$

Obviously, every sequence

$$L, g(L), L, g(L), \dots \tag{2.9}$$

is a 2-periodic (not necessarily prime) solution of (1.1), where $L \in \{M, m\}$.

By taking a subsequence, we may assume that there exists a sequence $t_n \geq 2kl + 1$ such that

$$\begin{gathered} \lim_{n\to\infty} x_{t_n} = M,\\ \lim_{n\to\infty} x_{t_n-j} = A_j \in \left[g(M), M\right] \quad \text{for } j \in \{1, 2, \dots, 2kl\}. \end{gathered} \tag{2.10}$$

According to (1.1), (2.10), and (H_3), we obtain

$$f\left(g(M), M\right) = M = f\left(A_l, A_{2k}\right) \leq f\left(g(M), M\right), \tag{2.11}$$

from which it follows that

$$A_l = g(M), \qquad A_{2k} = M. \tag{2.12}$$

In a similar fashion, we can obtain

$$f(g(M),M) = M = A_{2k} = f(A_{2k+l}, A_{4k}) \le f(g(M),M),$$
$$f(M,g(M)) = g(M) = A_l = f(A_{2l}, A_{l+2k}) \ge f(M,g(M)), \tag{2.13}$$

from which it follows that

$$A_{4k} = A_{2k} = A_{2l} = M, \qquad A_{2k+l} = A_l = g(M). \tag{2.14}$$

Inductively, we have

$$\begin{gathered} A_{j2k} = M \quad \text{for } j \in \{1,2,\ldots,l\}, \\ A_{jl} = g(M) \quad \text{for } j \in \{1,3,\ldots,2k-1\}, \\ A_{jl} = M \quad \text{for } j \in \{0,2,\ldots,2k\}, \\ A_{jl+r2k} = A_{jl} \quad \text{for } j \in \{0,1,\ldots,2k\},\ r \in \{0,1,\ldots,l\},\ jl + r2k \le 2kl. \end{gathered} \tag{2.15}$$

For every $r \in \{0,1,2,3,\ldots,2k-1\}$, there exist $j_r \in \{0,1,2,3,\ldots,2k-1\}$ and $p_r \in \{0,1,\ldots,l-1\}$ such that $j_r l = 2kp_r + r$, from which, with (2.15), it follows that

$$A_{2k(l-1)+r} = A_{j_r l} = \begin{cases} M & \text{for } r \in \{0,2,4,\ldots,2k-2\}, \\ g(M) & \text{for } r \in \{1,3,\ldots,2k-1\}, \end{cases} \tag{2.16}$$

$$\begin{aligned} &\lim_{n\to\infty} x_{t_n-2k(l-1)-j} = M \quad \text{for } j \in \{0,2,\ldots,2k\}, \\ &\lim_{n\to\infty} x_{t_n-2k(l-1)-j} = g(M) \quad \text{for } j \in \{1,3,\ldots,2k-1\}. \end{aligned} \tag{2.17}$$

In view of (2.17), for any $0 < \varepsilon < M - a$, there exists some $t_\beta \ge 4kl$ such that

$$\begin{gathered} M - \varepsilon < x_{t_\beta-2k(l-1)-j} < M + \varepsilon \quad \text{if } j \in \{0,2,\ldots,2k\}, \\ g(M+\varepsilon) < x_{t_\beta-2k(l-1)-j} < g(M-\varepsilon) \quad \text{if } j \in \{1,3,\ldots,2k-1\}. \end{gathered} \tag{2.18}$$

By (1.1) and (2.18), we have

$$x_{t_\beta-2k(l-1)+1} = f(x_{t_\beta-2k(l-1)-l+1}, x_{t_\beta-2kl+1}) < f(M-\varepsilon, g(M-\varepsilon)) = g(M-\varepsilon). \tag{2.19}$$

Also (1.1), (2.18), and (2.19) imply that

$$x_{t_\beta-2k(l-1)+2} = f(x_{t_\beta-2k(l-1)-l+2}, x_{t_\beta-2kl+2}) > f(g(M-\varepsilon), M-\varepsilon) = M-\varepsilon. \tag{2.20}$$

Inductively, it follows that

$$\begin{gathered} x_{t_\beta-2k(l-1)+2n} > M - \varepsilon \quad \forall n \ge 0, \\ x_{t_\beta-2k(l-1)+2n+1} < g(M-\varepsilon) \quad \forall n \ge 0. \end{gathered} \tag{2.21}$$

Therefore,

$$\lim_{n\to\infty} x_{2n} = M, \qquad \lim_{n\to\infty} x_{2n+1} = g(M) \tag{2.22}$$

or

$$\lim_{n\to\infty} x_{2n} = g(M), \qquad \lim_{n\to\infty} x_{2n+1} = M. \tag{2.23}$$

The proof is complete. □

Remark 2.2. (1) The proofs of Lemma 2.1 and Theorem 1.1 draw on ideas from the proofs of Theorems 2.1 and 2.2 in [6].

(2) Consider the nonlinear difference equation

$$x_{n+1} = f\left(x_{n-ls+1}, x_{n-2ks+1}\right), \quad n = 0, 1, \ldots, \tag{2.24}$$

where $s, k, l \in \{1, 2, \ldots\}$ with $2k \neq l$ and $\gcd(2k, l) = 1$, the initial values $x_{-\alpha}, x_{-\alpha+1}, \ldots, x_0 \in (0, +\infty)$ with $\alpha = \max\{ls - 1, 2ks - 1\}$, and f satisfies (H_1)–(H_3). Let $y^i_{n+1} = x_{ns+i+1}$ for every $0 \le i \le s-1$ and $n = 0, 1, 2, \ldots$, then (2.24) reduces to the equation

$$y^i_{n+1} = f\left(y^i_{n-l+1}, y^i_{n-2k+1}\right), \quad 0 \le i \le s-1, \ n = 0, 1, 2, \ldots. \tag{2.25}$$

It follows from Theorem 1.1 that for any $0 \le i \le s-1$, every positive solution of the equation $y^i_{n+1} = f(y^i_{n-l+1}, y^i_{n-2k+1})$ converges to (not necessarily prime) a 2-periodic solution. Thus every positive solution of (2.24) converges to (not necessarily prime) a $2s$-periodic solution.

3. Examples

To illustrate the applicability of Theorem 1.1, we present the following examples.

Example 3.1. Consider the equation

$$x_{n+1} = \frac{p + \sum_{i=1}^{m+1} x^i_{n-2k+1}}{\sum_{i=0}^{m} x^i_{n-2k+1} + x_{n-l+1}}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where $m, k, l \in \{1, 2, \ldots\}$ with $2k \neq l$ and $\gcd(2k, l) = 1$ and the initial values $x_{-\alpha}, x_{-\alpha+1}, \ldots, x_0 \in (0, +\infty)$ with $\alpha = \max\{l-1, 2k-1\}$, $0 < p \le 1$. Let $E = [0, +\infty)$ and

$$f(x, y) = \frac{p + \sum_{i=1}^{m+1} y^i}{\sum_{i=0}^{m} y^i + x} \quad (x \ge 0, \ y \ge 0), \qquad g(x) = \frac{p}{x} \quad (x > 0). \tag{3.2}$$

It is easy to verify that (H_1)–(H_3) hold for (3.1). It follows from Theorem 1.1 that every solution of (3.1) converges to (not necessarily prime) a 2-periodic solution.

Example 3.2. Consider the equation

$$x_{n+1} = 1 + \frac{x_{n-2k+1}^{m+1}}{\sum_{i=1}^{m} x_{n-2k+1}^{i} + x_{n-l+1}}, \quad n = 0, 1, \ldots, \tag{3.3}$$

where $m, k, l \in \{1, 2, \ldots\}$ with $2k \neq l$ and $\gcd(2k, l) = 1$ and the initial values $x_{-\alpha}, x_{-\alpha+1}, \ldots, x_0 \in (0, +\infty)$ with $\alpha = \max\{l-1, 2k-1\}$. Let $E = (0, +\infty)$ and

$$f(x, y) = 1 + \frac{y^{m+1}}{\sum_{i=1}^{m} y^{i} + x} \quad (x > 0,\ y > 0), \qquad g(x) = \frac{x}{x-1} \quad (x > 1). \tag{3.4}$$

It is easy to verify that (H_1)–(H_3) hold for (3.3). It follows from Theorem 1.1 that every solution of (3.3) converges to (not necessarily prime) a 2-periodic solution.

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