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Research Article

Exponential Stability for Impulsive BAM Neural Networks with Time-Varying Delays and Reaction-Diffusion Terms

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Impulsive bidirectional associative memory neural network model with time-varying delays and reaction-diffusion terms is considered. Several sufficient conditions ensuring the existence, uniqueness, and global exponential stability of equilibrium point for the addressed neural network are derived by M-matrix theory, analytic methods, and inequality techniques. Moreover, the exponential convergence rate index is estimated, which depends on the system parameters. The obtained results in this paper are less restrictive than previously known criteria. Two examples are given to show the effectiveness of the obtained results.

1. Introduction

The bidirectional associative memory (BAM) neural network model was first introduced by Kosko [1]. This class of neural networks has been successfully applied to pattern recognition, signal and image processing, artificial intelligence due to its generalization of the single-layer auto-associative Hebbian correlation to two-layer pattern-matched heteroassociative circuits. Some of these applications require that the designed network has a unique stable equilibrium point.

In hardware implementation, time delays occur due to finite switching speed of the amplifiers and communication time [2]. Time delays will affect the stability of designed neural networks and may lead to some complex dynamic behaviors such as periodic oscillation, bifurcation, or chaos [3]. Therefore, study of neural dynamics with consideration of the delayed problem becomes extremely important to manufacture high-quality neural networks. Some results concerning the dynamical behavior of BAM neural networks with delays have been reported, for example, see [2–12] and references therein. The circuits diagram and connection pattern implementing for the delayed BAM neural networks can be found in [8].

Most widely studied and used neural networks can be classified as either continuous or discrete. Recently, there has been a somewhat new category of neural networks which are neither purely continuous-time nor purely discrete-time ones, these are called impulsive neural networks. This third category of neural networks displays a combination of characteristics of both the continuous-time and the discrete systems [13]. Impulses can make unstable systems stable, so they have been widely used in many fields such as physics, chemistry, biology, population dynamics, and industrial robotics. Some results for impulsive neural networks have been given, for example, see [13–22] and references therein.

It is well known that diffusion effect cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields [23], so we must consider that the activations vary in space as well as in time. There have been some works devoted to the investigation of the stability of neural networks with reaction-diffusion terms, which are expressed by partial differential equations, for example, see [23–26] and references therein. To the best of our knowledge, few authors have studied the stability of impulsive BAM neural network model with both time-varying delays and reaction-diffusion terms.

Motivated by the above discussions, the objective of this paper is to give some sufficient conditions ensuring the existence, uniqueness, and global exponential stability of equilibrium point for impulsive BAM neural networks with time-varying delays and reaction-diffusion terms, without assuming the boundedness, monotonicity, and differentiability on these activation functions. Our methods, which do not make use of Lyapunov functional, are simple and valid for the stability analysis of impulsive BAM neural networks with time-varying or constant delays.

2. Model description and preliminaries

In this paper, we consider the following model:

$$\begin{aligned}
\frac{\partial u_i(t,x)}{\partial t} &= \sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i(t,x)}{\partial x_k}\right) - a_i u_i(t,x) \\
&\quad + \sum_{j=1}^{m} c_{ij} f_j\big(v_j(t-\tau_{ij}(t),x)\big) + \alpha_i, \quad t \neq t_k,\ i = 1,\dots,n, \\
\Delta u_i(t_k,x) &= I_k\big(u_i(t_k,x)\big), \quad i = 1,\dots,n,\ k = 1,2,\dots, \\
\frac{\partial v_j(t,x)}{\partial t} &= \sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D^*_{jk}\frac{\partial v_j(t,x)}{\partial x_k}\right) - b_j v_j(t,x) \\
&\quad + \sum_{i=1}^{n} d_{ji} g_i\big(u_i(t-\sigma_{ji}(t),x)\big) + \beta_j, \quad t \neq t_k,\ j = 1,\dots,m, \\
\Delta v_j(t_k,x) &= J_k\big(v_i(t_k,x)\big), \quad j = 1,\dots,m,\ k = 1,2,\dots
\end{aligned} \tag{2.1}$$

for $t > 0$, where $x = (x_1, x_2, \ldots, x_l)^T \in \Omega \subset R^l$, Ω is a bounded compact set with smooth boundary $\partial\Omega$ and $\operatorname{mes}\Omega > 0$ in space R^l; $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$; $v = (v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^m$; $u_i(t,x)$ and $v_j(t,x)$ are the state of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V at time t and in space x, respectively; f_j and g_i denote the activation functions of the jth neurons from F_V and the ith neurons from F_U at time t and in space x, respectively; α_i and β_j are constants, and denote the external inputs on the ith neurons from F_U and the jth neurons from F_V, respectively; $\tau_{ij}(t)$ and $\sigma_{ji}(t)$ correspond to the transmission delays and satisfy $0 \le \tau_{ij}(t) \le \tau_{ij}$ and $0 \le \sigma_{ji}(t) \le \sigma_{ji}$ (τ_{ij} and σ_{ji} are constants); a_i and b_j are positive constants, and denote the rates with which the ith neurons from F_U and the jth neurons from F_V will reset their potentials to the resting state in isolation when disconnected from the networks and external inputs, respectively; c_{ij} and d_{ji} are constants, and denote the connection strengths; smooth functions $D_{ik} = D_{ik}(t,x) \ge 0$ and $D^*_{jk} = D^*_{jk}(t,x) \ge 0$ correspond to the transmission diffusion operator along the ith neurons from F_U and the jth neurons from F_V, respectively. $\Delta u_i(t_k,x) = u_i(t_k^+,x) - u_i(t_k^-,x)$ and $\Delta v_j(t_k,x) = v_j(t_k^+,x) - v_j(t_k^-,x)$ are the impulses at moments t_k and in space x, and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$. The boundary conditions and initial conditions are given by

$$\begin{aligned}
\frac{\partial u_i}{\partial n} &:= \left(\frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, \ldots, \frac{\partial u_i}{\partial x_l}\right)^T = 0, \quad i = 1,2,\ldots,n, \\
\frac{\partial v_j}{\partial n} &:= \left(\frac{\partial v_j}{\partial x_1}, \frac{\partial v_j}{\partial x_2}, \ldots, \frac{\partial v_j}{\partial x_l}\right)^T = 0, \quad j = 1,2,\ldots,m,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
u_i(s,x) &= \phi_{u_i}(s,x), \quad s \in [-\sigma, 0], \quad \sigma = \max_{1\le i\le n, 1\le j\le m} \{\sigma_{ji}\}, \quad i = 1,2,\ldots,n, \\
v_j(s,x) &= \phi_{v_j}(s,x), \quad s \in [-\tau, 0], \quad \tau = \max_{1\le i\le n, 1\le j\le m} \{\tau_{ij}\}, \quad j = 1,2,\ldots,m,
\end{aligned} \tag{2.3}$$

where $\phi_{u_i}(s,x)$, $\phi_{v_j}(s,x)$ ($i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$) denote real-valued continuous functions defined on $[-\sigma,0] \times \Omega$ and $[-\tau,0] \times \Omega$, respectively.

Since the solution $(u_1(t,x),\ldots,u_n(t,x),v_1(t,x),\ldots,v_m(t,x))^T$ of model (2.1) is discontinuous at the point t_k, by theory of impulsive differential equations, we assume that $(u_1(t_k,x),\ldots,u_n(t_k,x),v_1(t_k,x),\ldots,v_m(t_k,x)) \equiv (u_1(t_k-0,x),\ldots,u_n(t_k-0,x),v_1(t_k-0,x),\ldots,v_m(t_k-0,x))^T$. It is clear that, in general, the partial derivatives $\partial u_i(t_k,x)/\partial t$ and $\partial v_j(t_k,x)/\partial t$ do not exist. On the other hand, according to the first and the third equations of model (2.1), there exist the limits $\partial u_i(t_k \mp 0,x)/\partial t$ and $\partial v_j(t_k \mp 0,x)/\partial t$. According to the above convention, we assume $\partial u_i(t_k,x)/\partial t = \partial u_i(t_k-0,x)/\partial t$ and $\partial v_j(t_k,x)/\partial t = \partial v_j(t_k-0,x)/\partial t$.

Throughout this paper, we make the following assumption.

(H) There exist two positive diagonal matrices $G = \operatorname{diag}(G_1, G_2, \ldots, G_n)$ and $F = \operatorname{diag}(F_1, F_2, \ldots, F_m)$ such that

$$|g_i(u_1) - g_i(u_2)| \le G_i |u_1 - u_2|, \qquad |f_j(v_1) - f_j(v_2)| \le F_j |v_1 - v_2| \tag{2.4}$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

For convenience, we introduce two notations. For any $u(t,x) = (u_1(t,x), u_2(t,x), \dots, u_k(t,x))^T \in \mathbb{R}^k$, define

$$||u_i(t,x)||_2 = \left[\int_\Omega |u_i(t,x)|^2 dx\right]^{1/2}, \quad i = 1,2,\dots,k. \tag{2.5}$$

For any $u(t) = (u_1(t), u_2(t), \dots, u_k(t))^T \in \mathbb{R}^k$, define $\|u(t)\| = [\sum_{i=1}^k |u_i(t)|^r]^{1/r}$, $r > 1$.

Definition 2.1. A constant vector $(u_1^*, \dots, u_n^*, v_1^*, \dots, v_m^*)^T$ is said to be an equilibrium of model (2.1) if

$$\begin{gathered} -a_i u_i^* + \sum_{j=1}^m c_{ij} f_j(v_j^*) + \alpha_i = 0, \quad i = 1,2,\dots,n, \\ I_k(u_i^*) = 0, \quad i = 1,2,\dots,n,\ k \in \mathbb{Z}^+, \\ -b_j v_j^* + \sum_{i=1}^n d_{ji} g_i(u_i^*) + \beta_j = 0, \quad j = 1,2,\dots,m, \\ J_k(v_j^*) = 0, \quad j = 1,2,\dots,m,\ k \in \mathbb{Z}^+, \end{gathered} \tag{2.6}$$

where $\mathbb{Z}^+$ denotes the set of all positive integers.

Definition 2.2 (see [3]). A real matrix $A = (a_{ij})_{n\times n}$ is said to be an M-matrix if $a_{ij} \le 0$ (i, $j = 1,2,\dots,n$, $i \ne j$) and successive principle minors of A are positive.

Definition 2.3 (see [27]). A map $H : \mathbb{R}^n \to \mathbb{R}^n$ is a homomorphism of $\mathbb{R}^n$ onto itself if $H \in C^0$, H is one-to-one, H is onto, and the inverse map $H^{-1} \in C^0$.

To prove our result, the following four lemmas are necessary.

Lemma 2.4 (see [3]). *Let Q be $n \times n$ matrix with nonpositive off-diagonal elements, then Q is an M-matrix if and only if one of the following conditions holds.*
(i) *There exists a vector $\xi > 0$ such that $Q\xi > 0$.*
(ii) *There exists a vector $\xi > 0$ such that $\xi^T Q > 0$.*

Lemma 2.5 (see [27]). *If $H(x) \in C^0$ satisfies the following conditions:*
(i) *$H(x)$ is injective on $\mathbb{R}^n$,*
(ii) *$\|H(x)\| \to +\infty$ as $\|x\| \to +\infty$,*
then $H(x)$ is homomorphism of $\mathbb{R}^n$.

Lemma 2.6 (see [28]). *Let $a, b \ge 0$, $p > 1$, then*

$$a^{p-1} b \le \frac{p-1}{p} a^p + \frac{1}{p} b^p. \tag{2.7}$$

Lemma 2.7 (see [29]) (C_p inequality). *Let $a \ge 0$, $b \ge 0$, $p > 1$, then*

$$(a+b)^{1/p} \le a^{1/p} + b^{1/p}. \tag{2.8}$$

3. Existence and uniqueness of equilibria

THEOREM 3.1. *Under assumption (H), if there exist real constants α_{ij}, β_{ij}, α^*_{ji}, β^*_{ji} ($i = 1,2,\dots,n$, $j = 1,2,\dots,m$), and $r > 1$ such that*

$$W = \begin{pmatrix} A-\widetilde{C} & -C^* \\ -D^* & B-\widetilde{D} \end{pmatrix} \tag{3.1}$$

is an M-matrix, and

$$\begin{aligned} I_k(u_i^*) &= 0, \quad i = 1,2,\dots,n,\ k \in \mathbb{Z}^+, \\ J_k(v_j^*) &= 0, \quad j = 1,2,\dots,m,\ k \in \mathbb{Z}^+, \end{aligned} \tag{3.2}$$

then model (2.1) has a unique equilibrium point $(u_1^*,\dots,u_n^*,v_1^*,\dots,v_m^*)^T$, where

$$\begin{gathered} A = \operatorname{diag}(a_1,a_2,\dots,a_n), \\ \widetilde{C} = \operatorname{diag}(\widetilde{c}_1,\dots,\widetilde{c}_n) \quad \text{with } \widetilde{c}_i = \sum_{j=1}^{m} \frac{r-1}{r} |c_{ij}|^{(r-\alpha_{ij})/(r-1)} F_j^{(r-\beta_{ij})/(r-1)}, \\ B = \operatorname{diag}(b_1,b_2,\dots,b_m), \\ \widetilde{D} = \operatorname{diag}(\widetilde{d}_1,\dots,\widetilde{d}_m) \quad \text{with } \widetilde{d}_j = \sum_{i=1}^{n} \frac{r-1}{r} |d_{ji}|^{(r-\alpha^*_{ji})/(r-1)} G_i^{(r-\beta^*_{ji})/(r-1)}, \\ C^* = (c^*_{ij})_{n\times m} \quad \text{with } c^*_{ij} = \frac{1}{r}|c_{ij}|^{\alpha_{ij}} F_j^{\beta_{ij}}, \\ D^* = (d^*_{ji})_{m\times n} \quad \text{with } d^*_{ji} = \frac{1}{r}|d_{ji}|^{\alpha^*_{ji}} G_i^{\beta^*_{ji}}. \end{gathered} \tag{3.3}$$

Proof. Define the following map associated with model (2.1):

$$H(x,y) = \begin{pmatrix} -A & 0 \\ 0 & -B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \begin{pmatrix} g(x) \\ f(y) \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{3.4}$$

where

$$\begin{gathered} C = (c_{ij})_{n\times m}, \qquad D = (d_{ji})_{m\times n}, \\ g(x) = (g_1(x_1), g_2(x_2), \dots, g_n(x_n))^T, \\ f(y) = (f_1(y_1), f_2(y_2), \dots, f_m(y_m))^T, \\ \alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)^T, \qquad \beta = (\beta_1,\beta_2,\dots,\beta_m)^T. \end{gathered} \tag{3.5}$$

In the following, we will prove that $H(x,y)$ is a homomorphism.

First, we prove that $H(x,y)$ is an injective map on $\mathbb{R}^{n+m}$.

In fact, if there exist $(x,y)^T$, $(\overline{x},\overline{y})^T \in \mathbb{R}^{n+m}$ and $(x,y)^T \neq (\overline{x},\overline{y})^T$ such that $H(x,y) = H(\overline{x},\overline{y})$, then

$$a_i(x_i - \overline{x}_i) = \sum_{j=1}^{m} c_{ij}\left(f_j(y_j) - f_j(\overline{y}_j)\right), \quad i = 1,2,\ldots,n, \tag{3.6}$$

$$b_j(y_j - \overline{y}_j) = \sum_{i=1}^{n} d_{ji}\left(g_i(x_i) - g_i(\overline{x}_i)\right), \quad j = 1,2,\ldots,m. \tag{3.7}$$

Multiply both sides of (3.6) by $|x_i - \overline{x}_i|^{r-1}$, it follows from assumption (H) and Lemma 2.6 that

$$\begin{aligned} a_i\left|x_i - \overline{x}_i\right|^r &\leq \sum_{j=1}^{m}\left|c_{ij}\right|F_j\left|x_i - \overline{x}_i\right|^{r-1}\left|y_j - \overline{y}_j\right| \\ &\leq \sum_{j=1}^{m}\frac{r-1}{r}\left|c_{ij}\right|^{(r-\alpha_{ij})/(r-1)}F_j^{(r-\beta_{ij})/(r-1)}\left|x_i - \overline{x}_i\right|^r \\ &\quad + \frac{1}{r}\sum_{j=1}^{m}\left|c_{ij}\right|^{\alpha_{ij}}F_j^{\beta_{ij}}\left|y_j - \overline{y}_j\right|^r. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\begin{aligned} b_j\left|y_j - \overline{y}_j\right|^r &\leq \sum_{i=1}^{n}\frac{r-1}{r}\left|d_{ji}\right|^{(r-\alpha^*_{ji})/(r-1)}G_i^{(r-\beta^*_{ji})/(r-1)}\left|y_j - \overline{y}_j\right|^r \\ &\quad + \frac{1}{r}\sum_{i=1}^{n}\left|d_{ji}\right|^{\alpha^*_{ji}}g_i^{\beta^*_{ji}}\left|x_i - \overline{x}_i\right|^r. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we get

$$W\left(\left|x_1 - \overline{x}_1\right|^r,\ldots,\left|x_n - \overline{x}_n\right|^r,\left|y_1 - \overline{y}_1\right|^r,\ldots,\left|y_m - \overline{y}_m\right|^r\right)^T \leq 0. \tag{3.10}$$

Since W is an M-matrix, we get $x_i = \overline{x}_i$, $y_j = \overline{y}_j$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, which is a contradiction. So, $H(x,y)$ is an injective map on $\mathbb{R}^{n+m}$.

Second, we prove that $\|H(x,y)\| \to +\infty$ as $\|(x,y)^T\| \to +\infty$.

Since W is an M-matrix, from Lemma 2.4, we know that there exists a vector $\gamma = (\lambda_1,\ldots,\lambda_n,\lambda_{n+1},\ldots,\lambda_{n+m})^T > 0$ such that $\gamma^T W > 0$, that is,

$$\begin{aligned} &\lambda_i(a_i - \widetilde{c}_i) - \sum_{j=1}^{m}\lambda_{n+j}d^*_{ji} > 0, \quad i = 1,2,\ldots,n, \\ &\lambda_{n+j}(b_j - \widetilde{d}_j) - \sum_{i=1}^{n}\lambda_i c^*_{ij} > 0, \quad j = 1,2,\ldots,m. \end{aligned} \tag{3.11}$$

We can choose a small number δ such that

$$\begin{aligned}
&\lambda_i(a_i-\widetilde{c}_i)-\sum_{j=1}^{m}\lambda_{n+j}d_{ji}^{*}\geq\delta>0,\quad i=1,2,\dots,n,\\
&\lambda_{n+j}(b_j-\widetilde{d}_j)-\sum_{i=1}^{n}\lambda_i c_{ij}^{*}\geq\delta>0,\quad j=1,2,\dots,m.
\end{aligned}\tag{3.12}$$

Let $\widetilde{H}(x,y)=H(x,y)-H(0,0)$, and $\operatorname{sgn}(\theta)$ is the signum function defined as 1 if $\theta>0$, 0 if $\theta=0$, -1 if $\theta<0$. From assumption (H), Lemma 2.6, and (3.12) we have

$$\begin{aligned}
&\sum_{i=1}^{n}\lambda_i|x_i|^{r-1}\operatorname{sgn}(x_i)\widetilde{H}_i(x,y)+\sum_{j=1}^{m}\lambda_{n+j}|y_j|^{r-1}\operatorname{sgn}(y_j)\widetilde{H}_{n+j}(x,y)\\
&\quad\leq-\sum_{i=1}^{n}\lambda_i a_i|x_i|^{r}+\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{m}|c_{ij}|F_j|y_j||x_i|^{r-1}-\sum_{j=1}^{m}\lambda_{n+j}b_j|y_j|^{r}\\
&\qquad+\sum_{j=1}^{m}\lambda_{n+j}\sum_{i=1}^{n}|d_{ji}|G_i|x_i||y_j|^{r-1}\\
&\quad\leq\sum_{i=1}^{n}\lambda_i\Bigg[\Bigg(-a_i+\sum_{j=1}^{m}\frac{r-1}{r}|c_{ij}|^{(r-\alpha_{ij})/(r-1)}F_j^{(r-\beta_{ij})/(r-1)}\Bigg)|x_i|^{r}\\
&\qquad\qquad+\sum_{j=1}^{m}\frac{1}{r}|c_{ij}|^{\alpha_{ij}}F_j^{\beta_{ij}}|y_j|^{r}\Bigg]\\
&\qquad+\sum_{j=1}^{m}\lambda_{n+j}\Bigg[\Bigg(-b_j+\sum_{i=1}^{n}\frac{r-1}{r}|d_{ji}|^{(r-\alpha_{ji}^{*})/(r-1)}G_i^{(r-\beta_{ji}^{*})/(r-1)}\Bigg)|y_j|^{r}\\
&\qquad\qquad+\sum_{i=1}^{n}\frac{1}{r}|d_{ji}|^{\alpha_{ji}^{*}}G_i^{\beta_{ji}^{*}}|x_i|^{r}\Bigg]\\
&\quad=-\sum_{i=1}^{n}\Bigg[\lambda_i(a_i-\widetilde{c}_i)-\sum_{j=1}^{m}\lambda_{n+j}d_{ji}^{*}\Bigg]|x_i|^{r}-\sum_{j=1}^{m}\Bigg[\lambda_{n+j}(b_j-\widetilde{d}_j)-\sum_{i=1}^{n}\lambda_i c_{ij}^{*}\Bigg]|y_j|^{r}\\
&\quad\leq-\delta\|(x,y)^{T}\|^{r}.
\end{aligned}\tag{3.13}$$

From (3.13) we have

$$\begin{aligned}
\delta\|(x,y)^{T}\|^{r}&\leq-\Bigg[\sum_{i=1}^{n}\lambda_i|x_i|^{r-1}\operatorname{sgn}(x_i)\widetilde{H}_i(x,y)+\sum_{j=1}^{m}\lambda_{n+j}|y_j|^{r-1}\operatorname{sgn}(y_j)\widetilde{H}_{n+j}(x,y)\Bigg]\\
&\leq\max_{1\leq i\leq n+m}\{\lambda_i\}\Bigg[\sum_{i=1}^{n}|x_i|^{r-1}|\widetilde{H}_i(x,y)|+\sum_{j=1}^{m}|y_j|^{r-1}|\widetilde{H}_{n+j}(x,y)|\Bigg].
\end{aligned}\tag{3.14}$$

By using Hölder inequality we get

$$||(x,y)^T||^r \le \frac{\max_{1\le i\le n+m}\{\lambda_i\}}{\delta}\left(\sum_{i=1}^{n}|x_i|^r+\sum_{j=1}^{m}|y_j|^r\right)^{(r-1)/r} \times\left(\sum_{i=1}^{n}|\widetilde{H}_i(x,y)|^r+\sum_{j=1}^{m}|\widetilde{H}_{n+j}(x,y)|^r\right)^{1/r}, \tag{3.15}$$

that is,

$$||(x,y)^T|| \le \frac{\max_{1\le i\le n+m}\{\lambda_i\}}{\delta}||\widetilde{H}(x,y)||. \tag{3.16}$$

Therefore, $\|\widetilde{H}(x,y)\|_\infty \to +\infty$ as $\|(x,y)^T\|_\infty \to +\infty$, which directly implies that $\|H(x,y)\| \to +\infty$ as $\|(x,y)^T\| \to +\infty$. From Lemma 2.5 we know that $H(x,y)$ is a homomorphism on $\mathbb{R}^{n+m}$. Thus, equation

$$\begin{aligned} &-a_iu_i+\sum_{j=1}^{m}c_{ij}f_j(v_j)+\alpha_i=0, \quad i=1,2,\dots,n,\\ &-b_jv_j+\sum_{i=1}^{n}d_{ji}g_i(u_i)+\beta_j=0, \quad j=1,2,\dots,m \end{aligned} \tag{3.17}$$

has unique solution $(u_1^*,\dots,u_n^*,v_1^*,\dots,v_m^*)^T$, which is one unique equilibrium point of model (2.1). The proof is completed. □

4. Global exponential stability

Theorem 4.1. *Under assumption (H), if W in Theorem 3.1 is an M-matrix, and $I_k(u_i(t_k,x))$ and $J_k(v_j(t_k,x))$ satisfy*

$$\begin{aligned} I_k(u_i(t_k,x)) &= -\gamma_{ik}(u_i(t_k,x)-u_i^*), \quad 0<\gamma_{ik}<2,\ i=1,2,\dots,n,\ k\in\mathbb{Z}^+,\\ J_k(v_j(t_k,x)) &= -\delta_{jk}(v_j(t_k,x)-v_j^*), \quad 0<\delta_{ik}<2,\ j=1,2,\dots,m,\ k\in\mathbb{Z}^+, \end{aligned} \tag{4.1}$$

then model (2.1) has a unique point $(u_1^*,\dots,u_n^*,v_1^*,\dots,v_m^*)^T$, which is globally exponentially stable.

Proof. From (4.1) we know that $I_k(u_i^*)=0$ and $J_k(v_j^*)=0$ $(i=1,2,\dots,n,\ j=1,2,\dots,m,\ k\in\mathbb{Z}^+)$, so the existence and uniqueness of equilibrium point of (2.1) follow from Theorem 3.1.

Let $(u_1(t,x),\dots,u_n(t,x),v_1(t,x),\dots,v_m(t,x))^T$ be any solution of model (2.1), then

$$\begin{aligned}\frac{\partial(u_i(t,x)-u_i^*)}{\partial t}&\\&=\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)-a_i(u_i(t,x)-u_i^*)\\&\quad+\sum_{j=1}^{m}c_{ij}(f_j(v_j(t-\tau_{ij}(t),x))-f_j(v_j^*)),\quad t>0,\ t\neq t_k,\ i=1,\dots,n,\ k\in\mathbb{Z}^+,\end{aligned}\tag{4.2}$$

$$\begin{aligned}\frac{\partial(v_j(t,x)-v_j^*)}{\partial t}&\\&=\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{jk}^*\frac{\partial(v_j(t,x)-v_j^*)}{\partial x_k}\right)-b_j(v_j(t,x)-v_j^*)\\&\quad+\sum_{i=1}^{n}d_{ji}(g_i(u_i(t-\sigma_{ji}(t),x))-g_i(u_i^*)),\quad t>0,\ t\neq t_k,\ j=1,\dots,m,\ k\in\mathbb{Z}^+.\end{aligned}\tag{4.3}$$

Multiply both sides of (4.2) by $u_i(t,x)-u_i^*$, and integrate, then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_i(t,x)-u_i^*)^2dx&=\sum_{k=1}^{l}\int_{\Omega}(u_i(t,x)-u_i^*)\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)dx\\&\quad-a_i\int_{\Omega}(u_i(t,x)-u_i^*)^2dx\\&\quad+\sum_{j=1}^{m}c_{ij}\int_{\Omega}(u_i(t,x)-u_i^*)(f_j(v_j(t-\tau_{ij}(t),x))-f_j(v_j^*))dx.\end{aligned}\tag{4.4}$$

From the boundary condition (2.2) and the proof of [22, Theorem 1] we get

$$\sum_{k=1}^{l}\int_{\Omega}(u_i(t,x)-u_i^*)\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)dx=-\sum_{k=1}^{l}\int_{\Omega}D_{ik}\left(\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)^2dx.\tag{4.5}$$

From (4.4), (4.5), assumption (H), and Cauchy integrate inequality we have

$$\begin{aligned}\frac{d\|u_i(t,x)-u_i^*\|_2^2}{dt}&\leq-2a_i\|u_i(t,x)-u_i^*\|_2^2\\&\quad+2\sum_{j=1}^{m}|c_{ij}|F_j\|u_i(t,x)-u_i^*\|_2\|v_j(t-\tau_{ij}(t),x)-v_j^*\|_2.\end{aligned}\tag{4.6}$$

Thus

$$D^+||u_i(t,x)-u_i^*||_2 \le -a_i||u_i(t,x)-u_i^*||_2+\sum_{j=1}^{m}|c_{ij}|F_j||v_j(t-\tau_{ij}(t),x)-v_j^*||_2 \quad (4.7)$$

for $t>0$, $t\neq t_k$, $i=1,\dots,n$, $k\in\mathbb{Z}^+$.

Multiply both sides of (4.3) by $v_j(t,x)-v_j^*$, similarly, we can get

$$D^+||v_j(t,x)-v_j^*||_2 \le -b_j||v_j(t,x)-v_j^*||_2+\sum_{i=1}^{n}|d_{ji}|G_i||u_i(t-\sigma_{ji}(t),x)-u_i^*||_2 \quad (4.8)$$

for $t>0$, $t\neq t_k$, $j=1,\dots,m$, $k\in\mathbb{Z}^+$.

It follows from (4.1) that

$$\begin{aligned}
||u_i(t_k+0,x)-u_i^*||_2 &= |1-\gamma_{ik}|\,||u_i(t_k,x)-u_i^*||_2, \quad i=1,\dots,n,\ k\in\mathbb{Z}^+,\\
||v_j(t_k+0,x)-v_j^*||_2 &= |1-\delta_{jk}|\,||v_j(t_k,x)-v_j^*||_2, \quad i=j,\dots,m,\ k\in\mathbb{Z}^+.
\end{aligned} \quad (4.9)$$

Let us consider functions

$$\begin{aligned}
\rho_i(\theta) &= \lambda_i\left(\frac{\theta}{r}-a_i+\tilde{c}_i\right)+\sum_{j=1}^{m}\lambda_{n+j}c_{ij}^*e^{\tau\theta}, \quad i=1,2,\dots,n,\\
\chi_j(\theta) &= \lambda_{n+j}\left(\frac{\theta}{r}-b_j+\tilde{d}_j\right)+\sum_{i=1}^{n}\lambda_i d_{ji}^*e^{\sigma\theta}, \quad j=1,2,\dots,m.
\end{aligned} \quad (4.10)$$

Since W is an M-matrix, from Lemma 2.4, we know that there exists a vector $\gamma=(\lambda_1,\dots,\lambda_n,\lambda_{n+1},\dots,\lambda_{n+m})^T>0$ such that $W\gamma>0$, that is,

$$\begin{aligned}
\lambda_i(a_i-\tilde{c}_i)-\sum_{j=1}^{m}\lambda_{n+j}c_{ij}^* &> 0, \quad i=1,2,\dots,n,\\
\lambda_{n+j}(b_j-\tilde{d}_j)-\sum_{i=1}^{n}\lambda_i d_{ji}^* &> 0, \quad j=1,2,\dots,m.
\end{aligned} \quad (4.11)$$

From (4.11) and (4.10) we know that $\rho_i(0)<0$, $\chi_j(0)<0$, and $\rho_i(\theta)$ and $\chi_j(\theta)$ are continuous for $\theta\in[0,+\infty)$. Moreover, $\rho_i(\theta),\chi_j(\theta)\to+\infty$ as $\theta\to+\infty$. Since $d\rho_i(\theta)/d\theta>0$, $d\chi_j(\theta)/d\theta>0$, $\rho_i(\theta)$ and $\chi_j(\theta)$ are strictly monotone increasing functions on $[0,+\infty)$. Thus, there exist constants z_i^* and $\tilde{z}_j^*\in(0,+\infty)$ such that

$$\begin{aligned}
\rho_i(z_i^*) &= \lambda_i\left(\frac{z_i^*}{r}-a_i+\tilde{c}_i\right)+\sum_{j=1}^{m}\lambda_{n+j}c_{ij}^*e^{z_i^*\tau}=0, \quad i=1,2,\dots,n,\\
\chi_j(\tilde{z}_j^*) &= \lambda_{n+j}\left(\frac{\tilde{z}_j^*}{r}-b_j+\tilde{d}_j\right)+\sum_{i=1}^{n}\lambda_i d_{ji}^*e^{\tilde{z}_j^*\sigma}=0, \quad j=1,2,\dots,m.
\end{aligned} \quad (4.12)$$

Choosing $0 < \varepsilon < \min\{z_1^*,\dots,z_n^*,\tilde{z}_1^*,\dots,\tilde{z}_m^*\}$, then

$$\lambda_i\left(\frac{\varepsilon}{r} - a_i + \tilde{c}_i\right) + \sum_{j=1}^{m} \lambda_{n+j} c_{ij}^* e^{\varepsilon\tau} < 0, \quad i = 1,2,\dots,n,$$
$$\lambda_{n+j}\left(\frac{\varepsilon}{r} - b_j + \tilde{d}_j\right) + \sum_{i=1}^{n} \lambda_i d_{ji}^* e^{\varepsilon\sigma} < 0, \quad j = 1,2,\dots,m. \tag{4.13}$$

Let

$$U_i(t) = e^{\varepsilon t}\|u_i(t,x) - u_i^*\|_2^r, \quad i = 1,2,\dots,n,$$
$$V_j(t) = e^{\varepsilon t}\|v_j(t,x) - v_j^*\|_2^r, \quad j = 1,2,\dots,m, \tag{4.14}$$

then it follows from (4.7), (4.8), and (4.14) that

$$D^+U_i(t) \le r\left[\left(\frac{\varepsilon}{r} - a_i + \tilde{c}_i\right)U_i(t) + \sum_{j=1}^{m} c_{ij}^* e^{\varepsilon\tau} V_j\big(t - \tau_{ij}(t)\big)\right],$$
$$t > 0,\ t \ne t_k,\ k \in \mathbb{Z}^+,\ i = 1,2,\dots,n,$$
$$D^+V_j(t) \le r\left[\left(\frac{\varepsilon}{r} - b_j + \tilde{d}_j\right)V_j(t) + \sum_{i=1}^{n} d_{ji}^* e^{\varepsilon\sigma} U_i\big(t - \sigma_{ji}(t)\big)\right],$$
$$t > 0,\ t \ne t_k,\ k \in \mathbb{Z}^+,\ j = 1,2,\dots,m. \tag{4.15}$$

From (4.9) and (4.1) we get that

$$U_i(t_k + 0) = |1 - \gamma_{ik}|\, U_i(t_k) \le U_i(t_k), \quad k \in \mathbb{Z}^+,\ i = 1,2,\dots,n,$$
$$V_j(t_k + 0) = |1 - \delta_{jk}|\, V_j(t_k) \le V_j(t_k), \quad k \in \mathbb{Z}^+,\ j = 1,2,\dots,m. \tag{4.16}$$

Let $l_0 = (1+\delta)(\sup_{s\in[-\sigma,0]}\sum_{i=1}^{n}\|\phi_{ui}(s,x) - u_i^*\|_2^r + \sup_{s\in[-\tau,0]}\sum_{j=1}^{m}\|\phi_{vj}(s,x) - v_j^*\|_2^r)/\min_{1\le i\le n+m}\{\lambda_i\}$ (δ is a positive constant), then

$$U_i(s) = e^{\varepsilon s}\|u_i(s,x) - u_i^*\|_2^r \le \|u_i(s,x) - u_i^*\|_2^r = \|\phi_{ui}(s,x) - u_i^*\|_2^r < \lambda_i l_0, \quad -\sigma \le s \le 0,$$
$$V_j(s) = e^{\varepsilon s}\|v_j(s,x) - v_j^*\|_2^r \le \|v_j(s,x) - v_j^*\|_2^r = \|\phi_{vj}(s,x) - v_j^*\|_2^r < \lambda_{n+j} l_0, \quad -\tau \le s \le 0. \tag{4.17}$$

In the following, we will prove

$$U_i(t) < \lambda_i l_0, \quad V_j(t) < \lambda_{n+j} l_0, \quad 0 \le t < t_1,\ i = 1,2,\dots,n,\ j = 1,2,\dots,m. \tag{4.18}$$

If (4.18) is not true, no loss of generality, then there exist some i_0 and $t^* \in [0,t_1)$ such that

$$U_{i_0}(t^*) = \lambda_{i_0} l_0, \qquad D^+U_{i_0}(t^*) \ge 0,$$
$$U_i(t) \le \lambda_i l_0, \quad -\sigma \le t \le t^*,\ i = 1,2,\dots,n,$$
$$V_j(t) \le \lambda_{n+j} l_0, \quad -\tau \le t \le t^*,\ j = 1,2,\dots,m. \tag{4.19}$$

However, from (4.15) and (4.13) we get

$$\begin{aligned} D^{+}U_{i_0}(t^*) &\le r\left[\left(\frac{\varepsilon}{r}-a_{i_0}+\widetilde{c}_{i_0}\right)U_{i_0}(t^*)+\sum_{j=1}^{m}c_{i_0j}^{*}e^{\varepsilon\tau}V_j(t^*-\tau_{i_0j}(t^*))\right] \\ &\le r\left[\left(\frac{\varepsilon}{r}-a_{i_0}+\widetilde{c}_{i_0}\right)\lambda_{i_0}l_0+\sum_{j=1}^{m}c_{i_0j}^{*}e^{\varepsilon\tau}\lambda_{n+j}l_0\right]<0, \end{aligned} \tag{4.20}$$

this is a contradiction, so (4.18) holds.

Suppose that for all $k=1,2,\dots,N$, the inequalities

$$\begin{aligned} U_i(t)&<\lambda_i l_0, \quad t_{N-1}\le t<t_N,\ i=1,2,\dots,n, \\ V_j(t)&<\lambda_{n+j}l_0, \quad t_{N-1}\le t<t_N,\ j=1,2,\dots,m, \end{aligned} \tag{4.21}$$

hold. Then from (4.16) and (4.21) we get

$$\begin{aligned} U_i(t_k+0)&\le U_i(t_k)<\lambda_i l_0, \quad i=1,2,\dots,n, \\ V_j(t_k+0)&\le V_j(t_k)<\lambda_{n+j}l_0, \quad j=1,2,\dots,m. \end{aligned} \tag{4.22}$$

This, together with (4.21), leads to

$$\begin{aligned} U_i(t)&<\lambda_i l_0, \quad t_N-\sigma\le t\le t_N,\ i=1,2,\dots,n, \\ V_j(t)&<\lambda_{n+j}l_0, \quad t_N-\tau\le t\le t_N,\ j=1,2,\dots,m. \end{aligned} \tag{4.23}$$

In the following, we will prove

$$\begin{aligned} U_i(t)&<\lambda_i l_0, \quad t_N\le t<t_{N+1},\ i=1,2,\dots,n, \\ V_j(t)&<\lambda_{n+j}l_0, \quad t_N\le t<t_{N+1},\ j=1,2,\dots,m. \end{aligned} \tag{4.24}$$

If (4.24) is not true, no loss of generality, then there exist some i_1 and $t^{**}\in[t_N,t_{N+1})$ such that

$$\begin{aligned} U_{i_1}(t^{**})&=\lambda_{i_1}l_0, \qquad D^{+}U_{i_1}(t^{**})\ge 0, \\ U_i(t)&\le\lambda_i l_0, \quad t_N-\sigma\le t\le t^{**},\ i=1,2,\dots,n, \\ V_j(t)&\le\lambda_{n+j}l_0, \quad t_N-\tau\le t\le t^{**},\ j=1,2,\dots,m. \end{aligned} \tag{4.25}$$

However, from (4.15), (4.23), and (4.13) we get

$$\begin{aligned} D^{+}U_{i_1}(t^{**}) &\le r\left[\left(\frac{\varepsilon}{r}-a_{i_1}+\widetilde{c}_{i_1}\right)U_{i_1}(t^{**})+\sum_{j=1}^{m}c_{i_1j}^{*}e^{\varepsilon\tau}V_j(t^{**}-\tau_{i_1j}(t^{**}))\right] \\ &\le r\left[\left(\frac{\varepsilon}{r}-a_{i_1}+\widetilde{c}_{i_1}\right)\lambda_{i_1}l_0+\sum_{j=1}^{m}c_{i_1j}^{*}e^{\varepsilon\tau}\lambda_{n+j}l_0\right]<0, \end{aligned} \tag{4.26}$$

this is a contradiction, so (4.24) holds.

By the mathematical induction, we can conclude that

$$\begin{aligned} &U_i(t) < \lambda_i l_0, \quad t_{N-1} \le t < t_N,\ N = 1,2,\ldots,\ i = 1,2,\ldots,n, \\ &V_j(t) < \lambda_{n+j} l_0, \quad t_{N-1} \le t < t_N,\ N = 1,2,\ldots,\ j = 1,2,\ldots,m. \end{aligned} \tag{4.27}$$

This implies that

$$\begin{aligned} &U_i(t) < \lambda_i l_0, \quad i = 1,2,\ldots,n, \\ &V_j(t) < \lambda_{n+j} l_0, \quad j = 1,2,\ldots,m \end{aligned} \tag{4.28}$$

for any $t > 0$. That is,

$$\begin{aligned} e^{\varepsilon t}\|u_i(t,x) - u_i^*\|_2^r \le{}& \sup_{s\in[-\sigma,0]} \sum_{i=1}^{n} \|\phi_{ui}(s,x) - u_i^*\|_2^r \\ &+ \sup_{s\in[-\tau,0]} \sum_{j=1}^{m} \|\phi_{vj}(s,x) - v_j^*\|_2^r, \quad i = 1,2,\ldots,n, \\ e^{\varepsilon t}\|v_j(t,x) - v_j^*\|_2^r \le{}& \sup_{s\in[-\sigma,0]} \sum_{i=1}^{n} \|\phi_{ui}(s,x) - u_i^*\|_2^r \\ &+ \sup_{s\in[-\tau,0]} \sum_{j=1}^{m} \|\phi_{vj}(s,x) - v_j^*\|_2^r, \quad j = 1,2,\ldots,m \end{aligned} \tag{4.29}$$

for any $t > 0$. Let $M = n^{1/r} + m^{1/r}$, from (4.29) and Lemma 2.7, we get that

$$\begin{aligned} &\left(\sum_{i=1}^{n} \|u_i(t,x) - u_i^*\|_2^r\right)^{1/r} + \left(\sum_{j=1}^{m} \|v_j(t,x) - v_j^*\|_2^r\right)^{1/r} \\ &\quad \le M\left(\left[\sup_{s\in[-\sigma,0]} \sum_{i=1}^{n} \|\phi_{ui}(s,x) - u_i^*\|_2^r\right]^{1/r} + \left[\sup_{s\in[-\tau,0]} \sum_{j=1}^{m} \|\phi_{vj}(s,x) - v_j^*\|_2^r\right]^{1/r}\right) e^{(-\varepsilon/r)t} \end{aligned} \tag{4.30}$$

for all $t > 0$. Therefore, the unique point of model (2.1) is globally exponentially stable, and the exponential convergence rate index ε/r comes from (4.12). The proof is completed. □

Corollary 4.2. *Under assumption (H) and condition (4.1), if*

$$W_1 = \begin{pmatrix} A & -C^* \\ -D^* & B \end{pmatrix} \tag{4.31}$$

is an M-matrix, then model (2.1) has a unique equilibrium point $(u_1^*,\ldots,u_n^*,v_1^*,\ldots,v_m^*)^T$, which is globally exponentially stable, where $A = \operatorname{diag}(a_1,a_2,\ldots,a_n)$, $B = \operatorname{diag}(b_1,b_2,\ldots,b_m)$, $C^* = (F_j|c_{ij}|)_{n\times m}$, $D^* = (G_i|d_{ji}|)_{m\times n}$.

Proof. Take $\alpha_{ij} = \beta_{ij} = \alpha_{ji}^* = \beta_{ji}^* = 1$, and let $r \to 1^+$, then W turns to W_1. The proof is completed. □

Remark 4.3. As the smooth operators $D_{ik} = 0$, $D^*_{jk} = 0$ ($i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $k = 1,2,\ldots,l$), model (2.1) becomes the following impulsive BAM neural networks with time-varying delays:

$$\begin{aligned}
\frac{du_i(t)}{dt} &= -a_i u_i(t) + \sum_{j=1}^{m} c_{ij} f_j(v_j(t-\tau_{ij}(t))) + \alpha_i, \quad t > 0,\ t \neq t_k,\ i = 1,\ldots,n, \\
&\qquad \Delta u_i(t_k) = I_k(u_i(t_k)), \quad i = 1,\ldots,n,\ k = 1,2,\ldots, \\
\frac{dv_j(t)}{dt} &= -b_j v_j(t) + \sum_{i=1}^{n} d_{ji} g_i(u_i(t-\sigma_{ji}(t))) + \beta_j, \quad t > 0,\ t \neq t_k,\ j = 1,\ldots,m, \\
&\qquad \Delta v_j(t_k) = J_k(v_i(t_k)), \quad j = 1,\ldots,m,\ k = 1,2,\ldots.
\end{aligned} \tag{4.32}$$

For this model, we have the following results.

Corollary 4.4. *Under assumption (H), if W in Theorem 3.1 is an M-matrix, and impulsive operators $I_k(u_i(t_k))$ and $J_k(v_j(t_k))$ satisfy*

$$\begin{aligned}
I_k(u_i(t_k)) &= -\gamma_{ik}(u_i(t_k) - u_i^*), \quad 0 < \gamma_{ik} < 2,\ i = 1,2,\ldots,n,\ k \in \mathbb{Z}^+, \\
J_k(v_j(t_k)) &= -\delta_{jk}(v_j(t_k) - v_j^*), \quad 0 < \delta_{ik} < 2,\ j = 1,2,\ldots,m,\ k \in \mathbb{Z}^+,
\end{aligned} \tag{4.33}$$

then model (4.32) has a unique equilibrium point $(u_1^*,\ldots,u_n^*,v_1^*,\ldots,v_m^*)^T$, which is globally exponentially stable.

Corollary 4.5 (see [18]). *Under assumption (H) and condition (4.33), when $\tau_{ij}(t),\sigma_{ji}(t)$ ($i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$) are constants, model (4.32) has a unique equilibrium point $(u_1^*,\ldots,u_n^*,v_1^*,\ldots,v_m^*)^T$, which is globally exponentially stable, if*

$$a_i > F_i \sum_{j=1}^{m} |d_{ji}|, \quad b_j > G_j \sum_{i=1}^{n} |c_{ij}|, \quad i = 1,2,\ldots,n,\ j = 1,2,\ldots,m. \tag{4.34}$$

Proof. If condition (4.34) holds, then matrix W_1 in Corollary 4.2 is column diagonally dominant, so W_1 is an M-matrix. The proof is completed. □

Remark 4.6. In [18, 21], the globally exponential stability for impulsive BAM neural networks with constant delays was investigated by constructing a suitable Lyapunov functional. In [20], authors have considered the impulsive BAM neural networks with distributed delays, several sufficient criteria checking the globally exponential stability were obtained by constructing a suitable Lyapunov functional. It should be noted that our methods, which do not make use of Lyapunov functional, are simple and valid for the stability analysis of impulsive BAM neural networks with constant delays, time-varying or distributed delays. It may be difficult to apply the Lyapunov approach in [18, 21, 20] to discuss the exponential stability of model (4.32) and model (2.1).

Remark 4.7. In [3, 5, 8–10, 12, 15], the boundedness of the activation functions was required. In [4, 6, 7, 10, 26], the monotonicity of the activation functions was needed. However, the boundedness and monotonicity of the activation functions have been removed in this paper.

5. Examples

Example 5.1. Consider the following impulsive BAM neural networks with fixed delays:

$$\begin{aligned}
\frac{du(t)}{dt} &= -5u(t)+6f(v(t-3))+11, \quad t>0,\ t\neq t_k,\\
\Delta u(t_k) &= -\gamma_{1k}(u(t_k)-1), \quad k=1,2,\ldots,\\
\frac{dv(t)}{dt} &= -3v(t)-g_i(u_i(t-1))+2, \quad t>0,\ t\neq t_k,\\
\Delta v_j(t_k) &= -\delta_{1k}(v(t_k)-1), \quad k=1,2,\ldots,
\end{aligned} \tag{5.1}$$

where $f(y)=g(y)=-|y|$, and $t_1<t_2<\cdots$ is strictly increasing sequence such that $\lim_{k\to\infty}t_k=+\infty$, $\gamma_{1k}=1+(1/2)\sin(2+k)$, $\delta_{1k}=1+(6/7)\cos(9+k^2)$, $k\in\mathbb{Z}^+$.

Since $b_1=3<|c_{11}|=6$, conditions (4.34) are not satisfied, which means that the theorem in [18] is not applicable to ascertain the stability of neural networks (5.1). However, it is easy to check that (5.1) satisfies all conditions of Corollary 4.4 in this paper. Hence, model (5.1) has a unique equilibrium point, which is globally exponentially stable. In fact, the unique equilibrium $(1,1)^T$ is a unique stable equilibrium point. From (4.12) we can estimate the exponential convergence rate index which is 0.2008.

Example 5.2. Consider the following impulsive BAM neural networks with both time-varying delays and reaction-diffusion terms:

$$\begin{aligned}
\frac{\partial u_i(t,x)}{\partial t} &= \frac{\partial}{\partial x_k}\left(t^2x^6\frac{\partial u_i(t,x)}{\partial x_k}\right)-a_iu_i(t,x)+\sum_{j=1}^{2}c_{ij}f_j(v_j(t-\tau_{ij}(t),x))+\alpha_i, \quad t\neq t_k,\\
&\qquad \Delta u_i(t_k,x)=I_k(u_i(t_k,x)),\\
\frac{\partial v_j(t,x)}{\partial t} &= \frac{\partial}{\partial x_k}\left(t^4x^2\frac{\partial v_j(t,x)}{\partial x_k}\right)-b_jv_j(t,x)+\sum_{i=1}^{2}d_{ji}g_i(u_i(t-\sigma_{ji}(t),x))+\beta_j, \quad t\neq t_k,\\
&\qquad \Delta v_j(t_k,x)=J_k(v_i(t_k,x))
\end{aligned} \tag{5.2}$$

for $i, j = 1,2$, $k \in \mathbb{Z}^+$, where

$$
\begin{gathered}
f_i(r) = g_j(r) = |r+1| + |r-1|, \quad \tau_{ij}(t) = \sigma_{ji}(t) = |\sin((i+j)t)|, \quad i,j = 1,2, \\
\Delta u_1(t_k, x) = -\left(1 + \frac{1}{2}\sin(7+k^2)\right) u_1(t_k, x), \\
\Delta u_2(t_k, x) = -(1+\cos(3-k))(u_2(t_k, x) - 1), \\
\Delta v_1(t_k, x) = -|2\sin(1-5k)|(v_1(t_k, x) - 1), \\
\Delta v_2(t_k, x) = -\left(1 - \frac{1}{3}\cos(11k)\right)(v_2(t_k, x) - 2), \\
a_1 = 7, \quad a_2 = 12, \quad c_{11} = 2, \quad c_{12} = 3, \quad c_{21} = 1.5, \quad c_{22} = 2.5, \quad \alpha_1 = -16, \quad \alpha_2 = -1, \\
b_1 = 4, \quad b_2 = 36, \quad d_{11} = 2, \quad d_{12} = 2, \quad d_{21} = 2, \quad d_{22} = 3.5, \quad \beta_1 = -4, \quad \beta_2 = 61.
\end{gathered}
\tag{5.3}
$$

Since f_j and g_i are not monotone increasing functions, the conditions of two theorems in [26] are not satisfied, which means that the theorems in [26] are not applicable to ascertain the stability of neural networks (5.2). However, It is easy to check that (5.2) satisfies all conditions of Corollary 4.2 in this paper. Hence, model (5.2) has a unique equilibrium point, which is globally exponentially stable. In fact, the unique equilibrium $(0,1,1,2)^T$ is a unique stable equilibrium point. From (4.12) we can estimate the exponential convergence rate index which is 0.1374.

6. Conclusions

In this paper, several easily checked sufficient criteria ensuring the existence, uniqueness, and global exponential stability of equilibrium point have been given for impulsive bidirectional associative memory neural networks with both time-varying delays and reaction-diffusion terms. In particular, the estimate of convergence rate index has been also provided. Some existing results are improved and extended. Two examples have been given to show that obtained results are less restrictive than previously known criteria. The method is simpler and more effective for stability analysis of neural networks with time-varying delays.

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