Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2007, Article ID 41541, 22 pages
doi:10.1155/2007/41541

Research Article

Global Asymptotic Behavior of $y_{n+1} = (py_n + y_{n-1})/(r + qy_n + y_{n-1})$

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Received 9 July 2007; Accepted 19 November 2007

Recommended by Elena Braverman

We investigate the global stability character of the equilibrium points and the period-two solutions of $y_{n+1} = (py_n + y_{n-1})/(r + qy_n + y_{n-1})$, $n = 0, 1, \ldots$, with positive parameters and nonnegative initial conditions. We show that every solution of the equation in the title converges to either the zero equilibrium, the positive equilibrium, or the period-two solution, for all values of parameters outside of a specific set defined in the paper. In the case when the equilibrium points and period-two solution coexist, we give a precise description of the basins of attraction of all points. Our results give an affirmative answer to Conjecture 9.5.6 and the complete answer to Open Problem 9.5.7 of Kulenović and Ladas, 2002.

1. Introduction

We investigate the global stability character of the equilibrium points and the period-two solutions of the second order rational difference equation

$$y_{n+1} = \frac{py_n + y_{n-1}}{r + qy_n + y_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the parameters p, q, r are positive and the initial conditions y_{-1}, y_0 are nonnegative real numbers. We also present one conjecture, which together with the established results, gives a complete picture of the nature of solutions of this equation. Our results improve and extend the asymptotic results in [1, Section 9.4]. Equation (1.1) is an important stepping stone in understanding the global dynamics of second-order rational difference equation of the form

$$y_{n+1} = \frac{\alpha + \beta y_n + \gamma y_{n-1}}{A + By_n + Cy_{n-1}}, \quad n = 0,1,\ldots, \tag{1.2}$$

with nonnegative parameters and initial conditions; see [1].

Related nonlinear, second-order, rational difference equations were investigated in [1–6]. Four important special cases of (1.1) were discussed in details in [1, 4, 5] (case $q = 0$), [7] (case $p = 0$), and [6] (case $r = 0$). Major conjectures for the special cases when one or two of the parameters p, q, and r are zero have been resolved in [8, 7, 9] completing the study of the global dynamics of these equations in the hyperbolic case. Finally, the result in [10] provides the answer for the global dynamics of these equations in the nonhyperbolic case.

The study of rational difference equations of order greater than one is quite challenging and rewarding and the results about these equations serve as prototypes in the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one; see Theorems B–F below. The techniques and results about these equations are also useful in analyzing the equations in the mathematical models of various biological systems and other applications; see, for instance, [11–13].

In this paper, we show that every solution of (1.1) converges to either the zero equilibrium, the positive equilibrium, or the period-two solution, for all values of parameters outside of a specific set that will be defined. In the case when the equilibrium points and period-two solution coexist, we give the precise description of the basins of attraction of all three invariant points.

Our results give an affirmative answer to Conjecture 9.5.6 and the complete answer to Open Problem 9.5.7 from [1].

2. Preliminaries

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$. Let $\overline{x} \in I$ be an equilibrium point of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0,1,\ldots, \tag{2.1}$$

that is,

$$\overline{x} = f(\overline{x}, \overline{x}). \tag{2.2}$$

Definition 2.1. (i) The equilibrium $\overline{x}$ of (2.1) is called *locally stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x_0, x_{-1} \in I$ with $|x_0 - \overline{x}| + |x_{-1} - \overline{x}| < \delta$, then

$$|x_n - \overline{x}| < \varepsilon \quad \forall n \geq -1. \tag{2.3}$$

(ii) The equilibrium $\overline{x}$ of (2.1) is called *locally asymptotically stable* if it is locally stable, and if there exists $\gamma > 0$ such that $x_0, x_{-1} \in I$ with $|x_0 - \overline{x}| + |x_{-1} - \overline{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{2.4}$$

(iii) The equilibrium $\overline{x}$ of (2.1) is called a *global attractor* if for every $x_0, x_{-1} \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{2.5}$$

(iv) The equilibrium $\overline{x}$ of (2.1) is called *globally asymptotically stable* if it is locally asymptotically stable and a global attractor.

(v) The equilibrium $\overline{x}$ of (2.1) is called *unstable* if it is not stable.

Let

$$s = \frac{\partial f}{\partial u}(\overline{x},\overline{x}), \qquad t = \frac{\partial f}{\partial v}(\overline{x},\overline{x}) \tag{2.6}$$

denote the partial derivatives of $f(u,v)$ evaluated at an equilibrium $\overline{x}$ of (2.1). Then the equation

$$y_{n+1} = sy_n + ty_{n-1}, \quad n = 0,1,\ldots, \tag{2.7}$$

is called the *linearized equation* associated with (2.1) about the equilibrium point $\overline{x}$.

Theorem A (linearized stability). (a) *If both roots of the quadratic equation*

$$\lambda^2 - s\lambda - t = 0 \tag{2.8}$$

lie in the open unit disk $\{\lambda : |\lambda| < 1\}$, *then the equilibrium* $\overline{x}$ *of (2.1) is locally asymptotically stable.*

(b) *If at least one of the roots of (2.8) has absolute value greater than one, then the equilibrium* $\overline{x}$ *of (2.1) is unstable.*

(c) *A necessary and sufficient condition for both roots of (2.8) to lie in the open unit disk* $\{\lambda : |\lambda| < 1\}$ *is*

$$|s| < 1 - t < 2. \tag{2.9}$$

In this case, the locally asymptotically stable equilibrium $\overline{x}$ *is also called a sink.*

We believe that a semicycle analysis of the solutions of a scalar difference equation is a powerful tool for a detailed understanding of the entire character of solutions and often leads to straightforward proofs of their long-term behavior.

We now give the definitions of positive and negative semicycles of a solution of (2.1) relative to an equilibrium point $\overline{x}$.

A *positive semicycle* of a solution $\{x_n\}$ of (2.1) consists of a "string" of terms $\{x_l, x_{l+1},\ldots,x_m\}$, all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ such that

$$\begin{aligned} &\text{either } l = -1, \quad \text{or} \quad l > -1, \quad x_{l-1} < \overline{x}, \\ &\text{either } m = \infty, \quad \text{or} \quad m < \infty, \quad x_{m+1} < \overline{x}. \end{aligned} \tag{2.10}$$

A *negative semicycle* of a solution $\{x_n\}$ of (2.1) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all less than the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\begin{aligned} &\text{either } l = -1, \quad \text{or} \quad l > -1, \quad x_{l-1} \geq \overline{x}, \\ &\text{either } m = \infty, \quad \text{or} \quad m < \infty, \quad x_{m+1} \geq \overline{x}. \end{aligned} \tag{2.11}$$

The next five results are general convergence theorems for (2.1).

Our first result is an important characterization of the global behavior of solutions of (2.1) when f satisfies specific monotonicity conditions, which was established recently in [10].

Theorem B [10]. *Consider (2.1) and assume that $f : I \times I \to I$, $I \subset R$ is a function which is decreasing in first variable and increasing in second variable. Then for every solution $\{x_n\}_{n=-1}^{\infty}$ of (2.1), the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd indexed terms of the solution do exactly one of the following:*

(i) *they are both monotonically increasing;*

(ii) *they are both monotonically decreasing;*

(iii) *eventually (i.e., for $n \geq N$), one of them is monotonically increasing and the other is monotonically decreasing.*

In particular if f is as in Theorem B and (2.1) does not possess a period-two solution then every bounded solution of this equation converges to an equilibrium.

Theorem C [1, 14]. *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \longrightarrow [a,b] \tag{2.12}$$

is a continuous function satisfying the following properties:

(a) *$f(x,y)$ is nondecreasing in each of its arguments;*

(b) *f has a unique fixed point $\overline{x} \in [a,b]$.*

Then every solution of (2.1) converges to $\overline{x}$.

Closely related is the following global attractivity result.

Theorem D [12]. *Let*

$$f : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \tag{2.13}$$

be a continuous function satisfying the following properties:

(a) *there exist two numbers L and U, $0 < L < U$ such that*

$$f(L,L) \geq L, \qquad f(U,U) \leq U \tag{2.14}$$

and $f(x,y)$ is nondecreasing in each of its arguments in $[L,U]$;

(b) *f has a unique fixed point $\overline{x} \in [L,U]$.*

Then every solution of (2.1) converges to $\overline{x}$.

Theorem E [1, 6, 14]. *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \longrightarrow [a,b] \tag{2.15}$$

is a continuous function satisfying the following properties:

(a) $f(x,y)$ *is nondecreasing in* $x \in [a,b]$ *for each* $y \in [a,b]$, *and* $f(x,y)$ *is nonincreasing in* $y \in [a,b]$ *for each* $x \in [a,b]$;

(b) *if* $(m,M) \in [a,b] \times [a,b]$ *is a solution of the system*

$$f(m,M) = m, \qquad f(M,m) = M, \tag{2.16}$$

then $m = M$.

Then (2.1) has a unique equilibrium $\overline{x} \in [a,b]$ *and every solution of (2.1) converges to* $\overline{x}$.

Theorem F [15]. *Consider the difference equation*

$$x_{n+1} = f_0(x_n, x_{n-1})x_n + f_1(x_n, x_{n-1})x_{n-1}, \quad n = 0,1,\ldots, \tag{2.17}$$

where f_0 *and* f_1 *are continuous real functions defined on some interval* $I \subset R$. *If there exist constants* $a < 1$ *and* N *such that*

$$|f_0(x_n, x_{n-1})| + |f_1(x_n, x_{n-1})| \le a, \quad n \ge N, \tag{2.18}$$

then the zero equilibrium of (2.17) is global attractor.

We will use a recent general result for competitive systems of difference equations of the form

$$\begin{aligned} x_{n+1} &= f(x_n, y_n), \\ y_{n+1} &= g(x_n, y_n), \end{aligned} \tag{2.19}$$

where f, g are continuous functions and $f(x,y)$ is nondecreasing in x and nonincreasing in y and $g(x,y)$ is nonincreasing in x and nondecreasing in y in some domain A.

We now present some basic notions about competitive maps in plane. Define a partial order $\preceq$ on $\mathbb{R}^2$ so that the positive cone is the fourth quadrant, that is, $(x^1, y^1) \preceq (x^2, y^2)$ if and only if $x^1 \le x^2$ and $y^1 \ge y^2$. A map T on a set $B \subset \mathbb{R}^2$ is a continuous function $T : B \to B$. The map is *smooth* if it is continuously differentiable on B. A set $A \subset B$ is *invariant* for the map T if $T(A) \subset A$. A point $x \in B$ is a *fixed point* of T if $T(x) = x$. The *orbit of* $x \in B$ is a sequence $\{T^\ell(x)\}_{\ell=0}^{\infty}$. A *prime period-two orbit* is an orbit $\{x_\ell\}_{\ell=0}^{\infty}$ for which $x_0 \neq x_1$ and $x_0 = x_2$. The map T is *competitive* if $T(x^1, y^1) \preceq T(x^2, y^2)$ whenever $(x^1, y^1) \preceq (x^2, y^2)$, and *strongly competitive* if $T(x^1, y^1) - T(x^2, y^2)$ is in the interior of the fourth quadrant whenever $(x^1, y^1) \preceq (x^2, y^2)$. The *basin of attraction* of a fixed point $\mathbf{e}$ is the set of all $\mathbf{x} \in B$ such that $T^n(\mathbf{x}) \to \mathbf{e}$ as $n \to \infty$. The interior of a set $\mathcal{R}$ is denoted as $\mathcal{R}^\circ$. Consider a competitive system (2.19), where $f, g : B \to \mathbb{R}$ are continuous functions such that the range of (f, g) is a subset of B. Then one may associate a competitive map T to (2.19) by setting $T = (f, g)$. If T is differentiable, a sufficient condition for T to be strongly competitive is that the Jacobian matrix of T at any $(x,y) \in B$ has the sign configuration

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}. \tag{2.20}$$

If $(x,y) \in B$, we denote with $Q_\ell(x,y)$, $\ell \in \{1,2,3,4\}$, the usual four quadrants relative to (x,y), for example, $Q_1(x,y) = \{ (u,v) \in B : u \geq x, v \geq y \}$. For additional definitions and results, see [16, 17].

A result from [16] we need is the following.

Theorem G . *Let $\mathcal{I}_1$, $\mathcal{I}_2$ be intervals in $\mathbb{R}$ with endpoints a_1, a_2 and b_1, b_2, respectively, with $a_1 < a_2$ and $b_1 < b_2$. Let T be a competitive map on $\mathcal{R} = \mathcal{I}_1 \times \mathcal{I}_2$. Let $\overline{\mathbf{x}} \in \mathcal{R}^\circ$. Suppose that the following hypotheses are satisfied.*

(1) *$\mathcal{R}^\circ$ is an invariant set, and T is strongly competitive on $\mathcal{R}^\circ$.*
(2) *The point $\overline{\mathbf{x}}$ is the only fixed point of T in $(Q_1(\overline{\mathbf{x}}) \cup Q_3(\overline{\mathbf{x}})) \cap \mathcal{R}^\circ$.*
(3) *The map T is continuously differentiable in a neighborhood of $\overline{\mathbf{x}}$, and $\overline{\mathbf{x}}$ is a saddle point.*
(4) *At least one of the following statements is true.*
   (a) *T has no prime period-two orbits in $(Q_1(\overline{\mathbf{x}}) \cup Q_3(\overline{\mathbf{x}})) \cap \mathcal{R}^\circ$.*
   (b) *$\det J_T(\overline{\mathbf{x}}) > 0$ and $T(\mathbf{x}) = \overline{\mathbf{x}}$ only for $\mathbf{x} = \overline{\mathbf{x}}$.*

Then the following statements are true.

(i) *The stable manifold $\mathcal{W}^s(\overline{\mathbf{x}})$ is connected and it is the graph of a continuous increasing curve with endpoints in $\partial\mathcal{R}$. $\mathcal{R}^\circ$ is divided by the closure of $\mathcal{W}^s(\overline{\mathbf{x}})$ into two invariant connected regions $\mathcal{W}_+$ ("below the stable manifold") and $\mathcal{W}_-$ ("above the stable manifold"), where*

$$\begin{aligned}\mathcal{W}_+ &:= \{\mathbf{x} \in \mathcal{R} \setminus \mathcal{W}^s(\overline{\mathbf{x}}) : \text{there exists } \mathbf{x}' \in \mathcal{W}^s(\overline{\mathbf{x}}) \text{ such that } \mathbf{x}' \preceq \mathbf{x}\},\\ \mathcal{W}_- &:= \{\mathbf{x} \in \mathcal{R} \setminus \mathcal{W}^s(\overline{\mathbf{x}}) : \text{there exists } \mathbf{x}' \in \mathcal{W}^s(\overline{\mathbf{x}}) \text{ such that } \mathbf{x} \preceq \mathbf{x}'\}.\end{aligned} \tag{2.21}$$

(ii) *The unstable manifold $\mathcal{W}^u(\overline{\mathbf{x}})$ is connected and it is the graph of a continuous decreasing curve.*
(iii) *For every $x \in \mathcal{W}_+$, $T^n(x)$ eventually enters the interior of the invariant set $\mathcal{Q}_4(\overline{\mathbf{x}}) \cap \mathcal{R}$, and for ever $x \in \mathcal{W}_-$, $T^n(x)$ eventually enters the interior of the invariant set $\mathcal{Q}_2(\overline{\mathbf{x}}) \cap \mathcal{R}$.*
(iv) *Let $\mathbf{m} \in \mathcal{Q}_2(\overline{\mathbf{x}})$ and $\mathbf{M} \in \mathcal{Q}_4(\overline{\mathbf{x}})$ be the endpoints of $\mathcal{W}^u(\overline{\mathbf{x}})$, where $\mathbf{m} \preceq \overline{\mathbf{x}} \preceq \mathbf{M}$. For ever $x \in \mathcal{W}_-$ and every $z \in \mathcal{R}$ such that $\mathbf{m} \preceq z$, there exists $m \in \mathbb{N}$ such that $T^m(x) \preceq z$, and for every $x \in \mathcal{W}_+$ and every $z \in \mathcal{R}$ such that $z \preceq \mathbf{M}$, there exists $m \in \mathbb{N}$ such that $z \preceq T^m(x)$.*

Now we present the local stability analysis of (1.1).

The equilibrium points of (1.1) are zero equilibrium and when

$$p + 1 > r, \tag{2.22}$$

equation (1.1) also possesses the unique positive equilibrium

$$\overline{y} = \frac{p+1-r}{q+1}. \tag{2.23}$$

The linearized equation associated with (1.1) about the zero equilibrium is

$$z_{n+1} - \frac{p}{r} z_n - \frac{1}{r} z_{n-1} = 0, \quad n = 0, 1, \dots. \tag{2.24}$$

The following theorem is a consequence of Theorems A and F.

Theorem 2.2. (a) *Assume that*

$$p+1 \le r. \tag{2.25}$$

Then the zero equilibrium of (1.1) is globally asymptotically stable.

(b) *Assume that*

$$p+1 > r. \tag{2.26}$$

Then the zero equilibrium of (1.1) is unstable. Furthermore the zero equilibrium is a saddle point when

$$1-p < r < 1+p \tag{2.27}$$

and a repeller when

$$r < 1-p. \tag{2.28}$$

The linearized equation associated with (1.1) about its positive equilibrium $\overline{y}$ is

$$z_{n+1} - \frac{p-q+qr}{(p+1)(q+1)} z_n - \frac{q-p+r}{(p+1)(q+1)} z_{n-1} = 0, \quad n = 0,1,\dots. \tag{2.29}$$

The following result is a consequence of Theorem A.

Theorem 2.3. *Assume that (2.22) holds. Then the positive equilibrium of (1.1) is locally asymptotically stable when*

$$q+r < 3p+1+qr+pq, \tag{2.30}$$

and unstable (a saddle point) when

$$q+r > 3p+1+qr+pq, \tag{2.31}$$

and nonhyperbolic, with one root of characteristic equation equal to -1, *when*

$$q+r = 3p+1+qr+pq. \tag{2.32}$$

3. Existence and local stability of period-two cycles

Concerning prime period-two solutions for (1.1), the following result is true.

Theorem 3.1. *Equation (1.1) has a prime period-two solution*

$$\dots,\Phi,\Psi,\Phi,\Psi,\dots \tag{3.1}$$

if and only if (2.31) and

$$p+r < 1 < q \tag{3.2}$$

holds. In this case the values of Φ *and* Ψ *are the positive roots of the quadratic equation*

$$t^2-(1-p-r)t+\frac{p(1-p-r)}{q-1}=0. \tag{3.3}$$

Furthermore when (2.31) holds, this period-two solution is locally asymptotically stable.

Proof. Let

$$\dots,\Phi,\Psi,\Phi,\Psi,\dots \tag{3.4}$$

be a period-two cycle of (1.1). Then

$$\Phi=\frac{p\Psi+\Phi}{r+q\Psi+\Phi},\qquad \Psi=\frac{p\Phi+\Psi}{r+q\Phi+\Psi}. \tag{3.5}$$

To investigate local stability of the period-two solution

$$\dots,\Phi,\Psi,\Phi,\Psi,\dots \tag{3.6}$$

we set

$$u_n=y_{n-1},\quad v_n=y_n,\quad \text{for } n=0,1,\dots \tag{3.7}$$

and write (1.1) in the equivalent form

$$\begin{gathered} u_{n+1}=v_n,\\ v_{n+1}=\frac{pv_n+u_n}{r+qv_n+u_n},\quad n=0,1,\dots. \end{gathered} \tag{3.8}$$

Let T be the function on $(0,\infty)\times(0,\infty)$ defined by

$$T\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}v\\ \dfrac{pv+u}{r+qv+u}\end{pmatrix}. \tag{3.9}$$

Then

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \tag{3.10}$$

is a fixed point of T^2, the second iteration of T. By a simple calculation, we find that

$$T^2\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}g(u,v)\\ h(u,v)\end{pmatrix}, \tag{3.11}$$

where

$$g(u,v)=\frac{pv+u}{r+qv+u},\qquad h(u,v)=\frac{pg(u,v)+v}{r+qg(u,v)+v}. \tag{3.12}$$

Clearly the period-two solution is locally asymptotically stable when the eigenvalues of the Jacobian matrix J_{T^2}, evaluated at $\binom{\Phi}{\Psi}$ lie inside the unit disk.

We have

$$J_{T^2}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} = \begin{pmatrix} \dfrac{\partial g}{\partial u}(\Phi,\Psi) & \dfrac{\partial g}{\partial v}(\Phi,\Psi) \\ \dfrac{\partial h}{\partial u}(\Phi,\Psi) & \dfrac{\partial h}{\partial v}(\Phi,\Psi) \end{pmatrix}, \tag{3.13}$$

where

$$\begin{gathered}
\frac{\partial g}{\partial u} = \frac{r+(q-p)v}{(r+qv+u)^2}, \\
\frac{\partial g}{\partial v} = \frac{pr+(p-q)u}{(r+qv+u)^2}, \\
\frac{\partial h}{\partial u} = \frac{r+(q-p)v}{(r+qv+u)^2}\frac{pr+(p-q)v}{(r+qg(u,v)+v)^2}, \\
\frac{\partial h}{\partial v} = \frac{r+(q-p)g(u,v)+\big(pr+(p-q)v\big)(\partial g/\partial v)}{(r+qg(u,v)+v)^2}.
\end{gathered} \tag{3.14}$$

By evaluating these derivatives at (Φ,Ψ), we obtain

$$\begin{gathered}
\frac{\partial g}{\partial u}(\Phi,\Psi) = \frac{r+(q-p)\Psi}{(r+q\Psi+\Phi)^2}, \\
\frac{\partial g}{\partial v}(\Phi,\Psi) = \frac{pr+(p-q)\Phi}{(r+q\Psi+\Phi)^2}, \\
\frac{\partial h}{\partial u}(\Phi,\Psi) = \frac{r+(q-p)\Psi}{(r+q\Psi+\Phi)^2}\frac{pr+(p-q)\Psi}{(r+q\Phi+\Psi)^2}, \\
\frac{\partial h}{\partial v}(\Phi,\Psi) = \frac{r+(q-p)\Phi+\big(pr+(p-q)\Psi\big)(\partial g/\partial v)(\Phi,\Psi)}{(r+q\Phi+\Psi)^2}.
\end{gathered} \tag{3.15}$$

Set

$$\begin{gathered}
\mathcal{S} = \frac{\partial g}{\partial u}(\Phi,\Psi)+\frac{\partial h}{\partial v}(\Phi,\Psi), \\
\mathcal{D} = \frac{\partial g}{\partial u}(\Phi,\Psi)\frac{\partial h}{\partial v}(\Phi,\Psi)-\frac{\partial g}{\partial v}(\Phi,\Psi)\frac{\partial h}{\partial u}(\Phi,\Psi).
\end{gathered} \tag{3.16}$$

Then it follows from Theorem A that both eigenvalues of $J_{T^2}\binom{\Phi}{\Psi}$ lie inside the unit disk $\{\lambda : |\lambda| < 1\}$ if and only if

$$|\mathcal{S}| < 1+\mathcal{D} < 2. \tag{3.17}$$

Inequality (3.17) is equivalent to the following three inequalities:

$$\mathcal{D} < 1, \tag{3.18}$$

$$\mathcal{S} < 1 + \mathcal{D}, \tag{3.19}$$

$$-1 - \mathcal{D} < \mathcal{S}. \tag{3.20}$$

First we will establish inequality (3.18). Clearly,

$$\mathcal{D} = \frac{r+(q-p)\Psi}{(r+q\Psi+\Phi)^2}\frac{r+(q-p)\Phi}{(r+q\Phi+\Psi)^2} \tag{3.21}$$

which in view of (3.5) gives

$$\begin{aligned} 0 &< \frac{r+(q-p)\Psi}{(r+q\Psi+\Phi)^2} = \frac{r+(q-p)\Psi}{r+q\Psi+\Phi}\frac{1}{r+q\Psi+\Phi} < \frac{1}{r+q\Psi+\Phi} = \frac{\Phi}{p\Psi+\Phi} < 1, \\ 0 &< \frac{r+(q-p)\Phi}{(r+q\Phi+\Psi)^2} = \frac{r+(q-p)\Phi}{r+q\Phi+\Psi}\frac{1}{r+q\Phi+\Psi} < \frac{1}{r+q\Phi+\Psi} = \frac{\Psi}{p\Phi+\Psi} < 1. \end{aligned} \tag{3.22}$$

This proves (3.18).

Next we prove (3.20). In view of

$$\begin{gathered} S = \frac{r+(q-p)\Psi}{(r+q\Phi+\Psi)^2} + \frac{r+(q-p)\Phi}{(r+q\Psi+\Phi)^2} + \frac{(rp+(p-q)\Psi)(rp+(p-q)\Phi)}{(r+q\Phi+\Psi)^2(r+q\Psi+\Phi)^2}, \\ D = \frac{(r+(q-p)\Psi)(r+(q-p)\Phi)}{(r+q\Phi+\Psi)^2(r+q\Psi+\Phi)^2}, \end{gathered} \tag{3.23}$$

inequality (3.20) is equivalent to

$$\begin{aligned} &\frac{(r+(q-p)\Psi)}{(r+q\Phi+\Psi)^2} + \frac{(r+(q-p)\Phi)}{(r+q\Psi+\Phi)^2} + \frac{(rp+(p-q)\Psi)(rp+(p-q)\Phi)}{(r+q\Phi+\Psi)^2(r+q\Psi+\Phi)^2} \\ &\qquad > -1 - \frac{(r+(q-p)\Psi)(r+(q-p)\Phi)}{(r+q\Phi+\Psi)^2(r+q\Psi+\Phi)^2}, \end{aligned} \tag{3.24}$$

which, in turn, is equivalent to

$$\begin{aligned} &(r+(q-p)\Psi)(r+q\Phi+\Psi)^2 + (r+(q-p)\Phi)(r+q\Psi+\Phi)^2 \\ &\quad + (rp+(p-q)\Psi)(rp+(p-q)\Phi) + (r+q\Phi+\Psi)^2(r+q\Psi+\Phi)^2 \\ &\quad + (r+(q-p)\Psi)(r+(q-p)\Phi) > 0. \end{aligned} \tag{3.25}$$

In view of $q > p$, we have

$$\begin{aligned} &(r+(q-p)\Psi)(r+q\Phi+\Psi)^2 + (r+(q-p)\Phi)(r+q\Psi+\Phi)^2 \\ &\quad + (r+q\Phi+\Psi)^2(r+q\Psi+\Phi)^2 + (r+(q-p)\Psi)(r+(q-p)\Phi) > 0. \end{aligned} \tag{3.26}$$

Thus, we have to show that

$$(rp+(p-q)\Psi)(rp+(p-q)\Phi)+(r+(q-p)\Psi)(r+(q-p)\Phi)>0. \tag{3.27}$$

Expanding the left-hand side of this inequality, we obtain

$$\begin{aligned}&(rp+(p-q)\Psi)(rp+(p-q)\Phi)+(r+(q-p)\Psi)(r+(q-p)\Phi)\\&\quad=(r^2p^2+r^2)+(\Psi+\Phi)r(q-p)(1-p)+2\Psi\Phi(q-p)^2>0.\end{aligned} \tag{3.28}$$

Finally, we prove (3.19). Inequality (3.19) is equivalent to

$$\begin{aligned}&\frac{(r+(q-p)\Psi)}{(r+q\Phi+\Psi)^2}+\frac{(r+(q-p)\Phi)}{(r+q\Psi+\Phi)^2}+\frac{(rp+(p-q)\Psi)(rp+(p-q)\Phi)}{(r+q\Phi+\Psi)^2(r+q\Psi+\Phi)^2}\\&\quad<1+\frac{(r+(q-p)\Psi)(r+(q-p)\Phi)}{(r+q\Phi+\Psi)^2(r+q\Psi+\Phi)^2},\end{aligned} \tag{3.29}$$

which after the expansion and use of

$$\Phi+\Psi=1-p-r, \qquad \Phi\Psi=\frac{p(1-p-r)}{q-1} \tag{3.30}$$

and (3.5) becomes

$$\begin{aligned}&2r^3+r^2(3q+2-p)(1-p-r)+r(q^2+1+2q-2p)\left[(1-p-r)^2-\frac{2p(1-p-r)}{q-1}\right]\\&\quad+4qr(1+q-p)\frac{p(1-p-r)}{q-1}+q(q-p)(q+2)\frac{p(1-p-r)^2}{q-1}\\&\quad+(q-p)(1-p-r)^2\left[(1-p-r)-\frac{3p}{q-1}\right]\\&<\left\{r^2+r(q+1)(1-p-r)+\frac{(q^2+1)p(1-p-r)}{q-1}+q\left[(1-p-r)^2-\frac{2p(1-p-r)}{q-1}\right]\right\}^2\\&\quad+r^2(1-p^2)+r(1+p)(q-p)(1-p-r).\end{aligned} \tag{3.31}$$

The left-hand side LHS of this inequality can be transformed to

$$\text{LHS}=(p-q-r+pq+qr-p^2)(pr-pq-p-qr+2pqr+2p^2-r^2+p^2q+qr^2-1) \tag{3.32}$$

and the right-hand side RHS of this inequality can be factored out as follows:

$$\text{RHS}=(q-p+r-pq-qr+p^2)^2+r^2(1-p^2)+r(1+p)(q-p)(1-p-r). \tag{3.33}$$

Now we have

$$\text{RHS}-\text{LHS}=(1-r-p)(q+r-3p-pq-qr-1)(q+r+p^2-p-pq-qr). \tag{3.34}$$

In view of $p+r<1$ and (2.31), we have $(1-r-p)(q+r-3p-pq-qr-1)>0$ and

$$q+r>3p+1+qr+qp>p+qr+qp \tag{3.35}$$

which implies $q+r-p-pq-qr>0$ and finally

$$q+r+p^2-p-pq-qr>0. \tag{3.36}$$

Thus RHS − LHS > 0 which proves (3.19). □

Theorem 3.1 gives an affirmative answer to [1, Conjecture 9.5.6].

4. Semicycle analysis and invariant intervals

In this section we list some basic identities for solutions of (1.1).

Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (1.1) and let (Φ,Ψ), (Ψ,Φ) be two prime period-two solutions of (1.1). Then the following identities are true for $n \geq 0$:

$$y_{n+1}-\overline{y}=\frac{(p-q+qr)(y_n-\overline{y})+(q-p+r)(y_{n-1}-\overline{y})}{(q+1)(r+qy_n+y_{n-1})}, \tag{4.1}$$

$$y_{n+1}-y_{n-1}=\frac{y_{n-1}(1-r-y_{n-1})+qy_n(p/q-y_{n-1})}{r+qy_n+y_{n-1}}, \tag{4.2}$$

$$\begin{aligned} y_{n+1}&-\frac{pr}{q-p} \\ &=(q-p-qr)\frac{p(y_n-pr/(q-p))+(y_{n-1}-pr/(q-p))+(pr/(q-p))(p+1)}{(q-p)(r+qy_n+y_{n-1})}, \end{aligned} \tag{4.3}$$

$$y_{n+1}-\frac{pr}{q-p}=(q-p-qr)\frac{py_n+(y_{n-1}-pr/(q-p))+pr/(q-p)}{(q-p)(r+qy_n+y_{n-1})}, \tag{4.4}$$

$$y_{n+1}-\frac{r}{p-q}=\frac{(p(p-q)-qr)(y_n-r/(q-p))+(p-q-r)(y_{n-1}+pr/(p-q))}{(p-q)(r+qy_n+y_{n-1})}, \tag{4.5}$$

$$y_{n+1}-\Phi=\frac{(y_n-\Psi)((p-q)\Phi+pr)+(y_{n-1}-\Phi)(r+(q-p)\Psi)}{(r+q\Psi+\Phi)(r+qy_n+y_{n-1})}, \tag{4.6}$$

$$y_{n+1}-\Psi=\frac{(y_n-\Phi)((p-q)\Psi+pr)+(y_{n-1}-\Psi)(r+(q-p)\Phi)}{(r+q\Phi+\Psi)(r+qy_n+y_{n-1})}. \tag{4.7}$$

Next we establish the following result on the global boundedness of (1.1).

Lemma 4.1. *Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (1.1). Then*

(1)

$$0 \leq y_n \leq \frac{\max\{p,1\}}{\min\{q,r,1\}}=U, \quad n \geq 1. \tag{4.8}$$

The function

$$f(x,y) = \frac{px+y}{r+qx+y} \tag{4.9}$$

is bounded, that is, $0 \leq f(x,y) \leq U$ *for* $x, y \geq 0$.

(2) *If (2.22) and*

$$p-q+qr \geq 0, \qquad q-p+r \geq 0 \tag{4.10}$$

hold, then

$$y_n \geq L = \min\{y_{-1}, y_0, \overline{y}\}, \tag{4.11}$$

where $\overline{y}$ *is the positive equilibrium.*

(3) *If (2.22) and*

$$p-q > r \tag{4.12}$$

hold, then the interval $[r/(p-q), U]$ *is an invariant and attractive interval for all solutions except for the zero equilibrium.*

(4) *If (2.22),*

$$q-p > qr, \tag{4.13}$$

and (2.30) or (2.31) are satisfied, then the interval $[pr/(q-p), U]$ *is an invariant and attractive interval except for the zero equilibrium.*

Proof. (1) The proof follows from the following inequality

$$0 \leq y_{n+1} = \frac{py_n + y_{n-1}}{r+qy_n+y_{n-1}} \leq \frac{\max\{p,1\}(y_n+y_{n-1})}{\min\{q,r,1\}(1+y_n+y_{n-1})} \leq \frac{\max\{p,1\}}{\min\{q,r,1\}} = U, \tag{4.14}$$

and the proof for $f(x,y) \leq U$ is obtained in a similar way.

(2) If $L \geq \overline{y}$, then $y_{-1}, y_0 \geq \overline{y}$, which in view of (4.1) implies that $y_n \geq \overline{y}$ for $n = 0, 1, \ldots$. Suppose that $L < \overline{y}$. Then (4.1) implies the following identity:

$$\begin{aligned} y_{n+1} &- \overline{y} - \frac{r}{r+qy_n+y_{n-1}}K \\ &= \frac{p-q+qr}{(q+1)(r+qy_n+y_{n-1})}(y_n - \overline{y} - K) + \frac{q-p+r}{(q+1)(r+qy_n+y_{n-1})}(y_{n-1} - \overline{y} - K), \end{aligned} \tag{4.15}$$

where K is an arbitrary constant. Let $K = L - \overline{y} < 0$. Then

$$\begin{aligned} y_{n+1} &- \overline{y} - \frac{r}{r+qy_n+y_{n-1}}(L - \overline{y}) \\ &= \frac{p-q+qr}{(q+1)(r+qy_n+y_{n-1})}(y_n - L) + \frac{q-p+r}{(q+1)(r+qy_n+y_{n-1})}(y_{n-1} - L). \end{aligned} \tag{4.16}$$

Since $y_{-1} \geq L$ and $y_0 \geq L$, we have

$$y_1 - \overline{y} - \frac{r}{r + q y_0 + y_{-1}} (L - \overline{y}) \geq 0 \tag{4.17}$$

which implies

$$y_1 - \overline{y} \geq \frac{r}{r + q y_0 + y_{-1}} (L - \overline{y}) \geq L - \overline{y} \tag{4.18}$$

and so $y_1 > L$. Since $y_0, y_1 \geq L$, then

$$y_2 - \overline{y} - \frac{r}{r + q y_1 + y_0} (L - \overline{y}) \geq 0 \tag{4.19}$$

which implies

$$y_2 - \overline{y} \geq \frac{r}{r + q y_1 + y_0} (L - \overline{y}) > L - \overline{y} \tag{4.20}$$

and so $y_2 > L$. By using induction, the proof is completed.

(3) If $y_n \geq r/(p-q)$ for some $n \geq 0$, then by (4.5) $y_{n+1} \geq r/(p-q)$, and so $y_k \geq r/(p-q)$, $k \geq n$.

Suppose that $y_{n-1}, y_n \leq r/(p-q)$ for every n. Then

$$r + q y_n + y_{n-1} \leq r + \frac{qr}{p-q} + \frac{r}{p-q} = \frac{r(p+1)}{p-q}. \tag{4.21}$$

Hence

$$1 < \frac{p-q}{r} \leq \frac{p+1}{r + q y_n + y_{n-1}}. \tag{4.22}$$

Define $m_N = \min\{y_{2N}, y_{2N-1}\}$, $N = 0, 1, \ldots$. Let $K \in R$. Then (1.1) has the generalized identity

$$y_{n+1} - \frac{p+1}{r + q y_n + y_{n-1}} K = \frac{p}{r + q y_n + y_{n-1}} (y_n - K) + \frac{1}{r + q y_n + y_{n-1}} (y_{n-1} - K) \tag{4.23}$$

for $n = 0, 1, \ldots$. Clearly, $y_1, y_2 > 0$ and so $y_n > 0$ for every $n \geq 1$, which implies that $m_N > 0$ for $N = 1, 2, \ldots$. By (4.23) with $K = m_N$ and $n = 2N$, we get that

$$y_{2N+1} - \frac{p+1}{r + q y_{2N} + y_{2N-1}} m_N = \frac{p(y_{2N} - m_N)}{r + q y_{2N} + y_{2N-1}} + \frac{y_{2N-1} - m_N}{r + q y_{2N} + y_{2N-1}} \geq 0 \tag{4.24}$$

and so by (4.22),

$$y_{2N+1} \geq \frac{p+1}{r + q y_{2N} + y_{2N-1}} m_N \geq \frac{p-q}{r} m_N. \tag{4.25}$$

Also

$$y_{2N+2} - \frac{p+1}{r+qy_{2N+1}+y_{2N}}m_N = \frac{p(y_{2N+1}-m_N)}{r+qy_{2N+1}+y_{2N}} + \frac{y_{2N}-m_N}{r+qy_{2N+1}+y_{2N-1}} \geq 0 \tag{4.26}$$

and so by (4.22),

$$y_{2N+2} \geq \frac{p+1}{r+qy_{2N+1}+y_{2N}}m_N \geq \frac{p-q}{r}m_N. \tag{4.27}$$

Thus $m_{N+1} \geq ((p-q)/r)m_N > m_N$ which implies $m_{n+1} \geq ((p-q)/r)m_n \geq ((p-q)/r)^{n+1-N}m_N$ for $n \geq N$ and so by (4.22), $m_n \to \infty$ as $n \to \infty$, which is a contradiction.

(4) The following cases are possible.

Case 1. There exists N such that $y_{N-1}, y_N \in [pr/(q-p), U]$. By (4.3), $y_n \in [pr/(q-p), U]$ for every $n \geq N-1$, which proves our claim.

Case 2. $y_n \in [0, pr/(q-p)]$ for every $n \geq -1$. Observe that the condition (4.13) implies that

$$\frac{p}{q} > \frac{pr}{q-p}, \qquad 1-r > \frac{pr}{q-p}. \tag{4.28}$$

By using (4.2) and (4.28), we obtain

$$\begin{aligned} y_{n+1} - y_{n-1} &= \frac{y_{n-1}(1-r-y_{n-1}) + qy_n(p/q - y_{n-1})}{r+qy_n+y_{n-1}} \\ &> \frac{y_{n-1}(pr/(q-p) - y_{n-1}) + qy_n(pr/(q-p) - y_{n-1})}{r+qy_n+y_{n-1}} > 0, \end{aligned} \tag{4.29}$$

which implies that $y_{n+1} > y_{n-1}$ provided that $y_{n-1} \leq pr/(q-p)$. In this case, every solution $\{y_n\}$ of (1.1) has two increasing, bounded subsequences. Consequently, every solution converges to either a positive limit or period-two solution which belongs to the interval $(0, pr/(q-p)]$. If a solution converges to a limit, this limit would be necessarily an equilibrium of (1.1), which is impossible. If (2.30) is satisfied, then a solution cannot converge to a period-two solution and the proof is complete. If (2.31) is satisfied, then the solution converges either to an equilibrium or to the period-two solution. The convergence of the equilibrium has been ruled out. If the solution converges to a period-two solution (Φ, Ψ) or (Ψ, Φ), then

$$\Phi + \Psi = 1 - p - r < \frac{2pr}{q-p}, \tag{4.30}$$

which implies

$$qr < q - p < \frac{2pr}{q-p} \tag{4.31}$$

and $q+r<2p+r+(p+r)q$. Using (2.31), we obtain

$$3p+1+(p+r)q<q+r<2p+r+(p+r)q, \tag{4.32}$$

which leads to $p+1<r$ which contradicts with (2.22).

Case 3. There exists N such that $y_N \in [pr/(q-p),U]$ and no two subsequent terms are in $[pr/(q-p),U]$. By (4.4), we have that $y_{N+2k} \in [pr/(q-p),U]$, $k=1,2,\ldots$. Assume for the sake of simplicity that $\{y_{2n}\}_{n\geq K} \subset [pr/(q-p),U]$. Then $y_{2K-1} \leq pr/(q-p)$ and by Case 2 the sequence $\{y_{2n-1}\}_{n\geq K}$ is an increasing sequence in $[0,pr/(q-p)]$. Consequently, $\{y_{2n-1}\}_{n\geq K}$ is convergent and so

$$\lim_{n\to\infty} y_{2n-1} = L \leq \frac{pr}{q-p}. \tag{4.33}$$

Equation (1.1) implies

$$y_{2n+1} = \frac{py_{2n}+y_{2n-1}}{r+qy_{2n}+y_{2n-1}}, \tag{4.34}$$

$$y_{2n} = \frac{y_{2n-1}-ry_{2n+1}-y_{2n-1}y_{2n+1}}{qy_{2n+1}-p}. \tag{4.35}$$

In view of (4.28)

$$\lim_{n\to\infty} y_{2n-1} = L \leq \frac{pr}{q-p} < \frac{p}{q}, \tag{4.36}$$

which shows that $qL-p<0$ and $1-r-L>0$. Taking limit in (4.34), we obtain

$$\lim_{n\to\infty} y_{2n} = \frac{L(1-r-L)}{qL-p} < 0, \tag{4.37}$$

which is a contradiction. Thus the only possible case is Case 1. □

5. Global attractivity and global stability of the positive equilibrium

By using the monotonic character of the function (4.9) in each of the invariant intervals together with the appropriate convergence theorem (from among Theorems B, C, D, E, and F), we can obtain some convergence results for the solutions with initial conditions in the invariant intervals.

Case 5.1 ($p=q$). In this case the function $f(x,y)$ is increasing in both of its arguments x and y.

Theorem 5.2. *Assume that*

$$p=q. \tag{5.1}$$

Then every solution of (1.1) converges to the equilibrium $\overline{y}$. The equilibrium $\overline{y}$ is globally asymptotically stable.

Proof. In view of Lemma 4.1, we see that the function $f(x,y)$ is increasing in both of its arguments in an invariant interval $[L,U]$ and that the positive equilibrium of (1.1) is unique in that interval. Let us check Theorem D(a). Indeed

$$f(L,L)-L=\frac{(p+1)L}{r+(q+1)L}-L=L(q+1)\frac{\overline{x}-L}{r+(q+1)L}>0 \tag{5.2}$$

because $L<\overline{x}$. Similarly, in view of $U>\overline{x}$,

$$f(U,U)-U=\frac{(p+1)U}{r+(q+1)U}-U=U(q+1)\frac{\overline{x}-U}{r+(q+1)U}<0. \tag{5.3}$$

The result now follows by employing Theorem D. Clearly, when $p=q$, condition (2.22) implies (2.30) and so $\overline{y}$ is globally asymptotically stable. □

Case 5.3 ($p>q$). In this case the function $f(x,y)$ is always increasing in x and it is increasing in y for $x<r/(p-q)$ and decreasing in y for $x>r/(p-q)$.

Theorem 5.4. *Assume that $r\geq p-q>0$. Then (1.1) possesses an invariant interval $[L,r/(p-q)]$. The equilibrium $\overline{y}$ is globally asymptotically stable.*

Proof. Observe that the conditions on parameters imply (4.10) and so by Lemma 4.1 every solution has a lower bound L. We want to show that $[L,r/(p-q)]$ is an invariant interval for f. Take $x,y\in[L,r/(p-q)]$, then by using the increasing character of f, we have

$$f(x,y)\leq f\left(\frac{r}{p-q},\frac{r}{p-q}\right)=1\leq\frac{r}{p-q}. \tag{5.4}$$

Clearly, the positive equilibrium of (1.1) is unique in that interval.

First, we show that the equilibrium is locally stable. Indeed, conditions $p+1>r$ and $r\geq p-q>0$ imply

$$q+r<q+p+1<2p+1<3p+1+pq+qr. \tag{5.5}$$

Second, by using the identity (4.1), we obtain

$$y_{n+1}-\overline{y}=\frac{p-q+qr}{(q+1)(r+qy_n+y_{n-1})}(y_n-\overline{y})+\frac{q-p+r}{(q+1)(r+qy_n+y_{n-1})}(y_{n-1}-\overline{y}), \tag{5.6}$$

for $n=0,1,\ldots$. Set $e_n=y_n-\overline{y}$, then we get the following "linearized equation":

$$e_{n+1}=f_0e_n+f_1e_{n-1},\quad n=0,1,\ldots, \tag{5.7}$$

where

$$f_0=\frac{p-q+qr}{(q+1)(r+qy_n+y_{n-1})},\qquad f_1=\frac{q-p+r}{(q+1)(r+qy_n+y_{n-1})}. \tag{5.8}$$

Now, by using the inequality (4.8), we obtain

$$|f_0|+|f_1| = \frac{r}{r+qy_n+y_{n-1}} \le \frac{r}{r+(q+1)L} = a < 1. \tag{5.9}$$

Thus all conditions of Theorem F are satisfied and we conclude that the zero equilibrium of (5.7) is a global attractor and so is the globally asymptotically stable. Consequently the equilibrium $\overline{y}$ is globally asymptotically stable. This shows that the interval $[L, r/(p-q)]$ is an attracting interval for (1.1). □

Theorem 5.5. *Assume that $r < p-q$. Assume that either*

$$p \le r+1 \tag{5.10}$$

or

$$0 < p-r-1 \le \frac{4q}{1-q}. \tag{5.11}$$

Then every solution of (1.1) converges to the equilibrium $\overline{y}$.

Proof. It follows from Lemma 4.1 that $[r/(p-q), U]$ is an invariant and attracting interval for the solution of (1.1). We can also show that this interval is an invariant interval for the function f. Take $x, y \in [r/(p-q), U]$, then by using the monotonic character of f and the condition $r < p-q$, we obtain

$$f(x,y) \ge f\left(\frac{r}{p-q}, U\right) = 1 \ge \frac{r}{p-q}. \tag{5.12}$$

Lemma 4.1 implies that $f(x,y) \le U$ for all $x, y \ge 0$. The function $f(x,y)$ is increasing in x and decreasing in y in this interval. In order to employ Theorem F, we have to show that the only solution of the system $M = f(M,m)$, $m = f(m,M)$, is a positive equilibrium $M = m = \overline{y}$. This system of equations is equivalent to

$$\begin{aligned} Mm &= (p-r)M - qM^2 + m, \\ Mm &= (p-r)m - qm^2 + M, \end{aligned} \tag{5.13}$$

which implies

$$(M-m)\big(p-r-1-q(M+m)\big) = 0. \tag{5.14}$$

If $p-1 \le r$, then in view of $M, m \in [r/(p-q), U]$, we see that $(p-r-1-q(M+m)) < 0$ and so $M = m$. If $p-1 > r$ then $M = ((p-r-1)/q) - m$ and substituting in (5.14) we obtain

$$q(1-q)m^2 + (p-r-1)(q-1)m + p-r-1 = 0. \tag{5.15}$$

Likewise, one can show that M satisfies the same equation. If we want to have $M = m$, we must assume that (5.15) cannot have two different real solutions, which is equivalent to the condition that its discriminant is nonpositive. Thus we obtain condition (5.11). □

Based on extensive simulations and the fact that in the special case $r = 0$ the corresponding result holds, we pose the following.

Conjecture 5.6. *Condition (5.11) can be replaced by (2.30).*

Case 5.7 ($p < q$). In this case the function $f(x, y)$ is always increasing in y and it is increasing in x for $y < pr/(q - p)$ and decreasing in x for $y > pr/(q - p)$.

Theorem 5.8. *Assume that $qr \geq q - p > 0$. Then (1.1) possesses an invariant interval $[L, pr/(q - p)]$. The equilibrium $\overline{y}$ is globally asymptotically stable.*

Proof. Observe that the conditions on the parameters imply (4.10) and so by Lemma 4.1 every solution has a lower bound L. We want to show that $[L, pr/(q - p)]$ is an invariant interval for f. Take $x, y \in [L, pr/(q - p)]$, then by using the increasing character of f and the condition $qr \geq q - p > 0$, we have

$$f(x,y) \leq f\left(\frac{pr}{q-p}, \frac{pr}{q-p}\right) = \frac{p}{q} \leq \frac{pr}{q-p}. \tag{5.16}$$

Lemma 4.1 implies that $f(x, y) \geq L$ for all $x, y \geq 0$. Clearly, the positive equilibrium of (1.1) is unique in that interval.

First, we show that the equilibrium is locally stable. Indeed, the conditions $p + 1 > r$ and $qr \geq q - p > 0$ imply

$$q + r < q + p + 1 = q - p + 2p + 1 \leq 2p + 1 + qr < 3p + 1 + pq + qr. \tag{5.17}$$

Second, by using the identity (4.1) we obtain the "linearized equation" (5.7) and the inequality (5.9). Thus all conditions of Theorem F are satisfied and we conclude that the zero equilibrium of (5.7) is global attractor and so it is globally asymptotically stable. Consequently the equilibrium $\overline{y}$ is globally asymptotically stable. This shows that the interval $[L, pr/(q - p)]$ is an attracting interval for (1.1). □

The next result holds in the case when $q - p > qr$.

Theorem 5.9. (a) *Assume that $q - p > qr$ and (2.30). Then every solution of (1.1) with initial conditions in the invariant interval*

$$\left[\frac{pr}{q-p}, U\right] \tag{5.18}$$

converges to the equilibrium $\overline{y}$. The equilibrium $\overline{y}$ is globally asymptotically stable.

(b) *Assume that $q - p > qr$, (2.31) and (3.2) are satisfied. Then every solution of (1.1) converges to either the equilibrium or period-two solutions.*

Proof. Lemma 4.1(3) implies that $[pr/(q - p), U]$ is an attracting interval for all solutions of (1.1). We want to show that $[pr/(q - p), U]$ is an invariant interval for f. Clearly $f(x, y) \leq U$ for all $x, y \geq 0$. Take $x, y \in [pr/(q - p), U]$, then by using the monotonic character of f and the condition $q - p > qr$ we obtain

$$f(x,y) \geq f\left(U, \frac{pr}{q-p}\right) = \frac{p}{q} > \frac{pr}{q-p}. \tag{5.19}$$

The results now follow by employing Theorems B and 3.1. □

Remark 5.10. More precise information about the basins of attraction of the equilibrium and period-two solutions will be given in Section 6.

6. Attractivity of period-two solutions

In this section we will consider the problem of attractivity of period-two solutions. We show that when period-two solutions exist, they will attract all solutions except for those that start on the stable manifold of the equilibrium. Precisely we will prove the following result.

Theorem 6.1. *Consider (1.1) where (2.31) and (3.2) are satisfied. Let* (Φ, Ψ) *and* (Ψ, Φ) *be the prime period-two solutions of (1.1) for which* $\Phi < \Psi$. *Then the global stable manifold* $W^s(\overline{y}, \overline{y})$ *is the graph of a smooth increasing function with endpoints on the boundary of* $B = (0, \infty) \times (0, \infty)$, *and is such that every solution with an initial point below* $W^s((\overline{y}, \overline{y}))$ *converges to* (Ψ, Φ), *while every solution with an initial point in B above* $W^s((\overline{y}, \overline{y}))$ *converges to* (Φ, Ψ). *Consequently, except for solutions with an initial point in* $W^s((\overline{y}, \overline{y}))$, *every solution converges to one of the two period-two solutions.*

Proof. First we show that the map T^2 (second iteration of map T) leaves the box $B_{p,q,r} = [pr/(q-p), U]^2$ invariant. Assume that $pr/(q-p) \leq u$, $v \leq U$. Then clearly $g(u,v) = f(v,u) \leq U$ and g is increasing in the first variable and decreasing in the second. In view of (3.2), we have

$$g(u,v) \geq g\left(\frac{pr}{q-p}, v\right) = \frac{p}{q} > \frac{pr}{q-p}. \tag{6.1}$$

The second component $h(u,v)$ of T^2 is decreasing in the first variable and increasing in the second. The inequality $h(u,v) \leq U$ follows from the simple fact that $h(u,v) = f(f(u,v),v)$ and the fact that f is bounded by U. In view of (6.1) and (3.2), we have

$$h(u,v) \geq h\left(u, \frac{pr}{q-p}\right) = \frac{p}{q} > \frac{pr}{q-p}. \tag{6.2}$$

Next we notice that the map T^2 is competitive in the box $B_{p,q,r}$. This is clear from the expressions for $\partial g/\partial u$, $\partial g/\partial v$, $\partial h/\partial u$, and $\partial h/\partial u$.

The fixed points of T^2 in B satisfy $T^2(u,v) = (u,v)$, that is,

$$u = \frac{pv + u}{r + qv + u}, \qquad v = \frac{pu + v}{r + qu + v}, \tag{6.3}$$

which are exactly the equations satisfied by period-two solutions of (1.1). Hence the fixed points of T^2 in $B_{p,q,r}$ are (Φ, Ψ), (Ψ, Φ), and $(\overline{y}, \overline{y})$, where Φ and Ψ may be chosen so that

$\Phi < \overline{y} < \Psi$. A consequence of this and of the fact that T^2 is strongly competitive is that $B' = [\Phi, \Psi] \times [\Phi, \Psi]$ is an invariant box, with a unique fixed point in its interior, namely, $(\overline{y}, \overline{y})$. This can be seen from the fact that points (x, y) in B' satisfy $(\Phi, \Psi) \le (x, y) \le (\Psi, \Phi)$, hence $(\Phi, \Psi) = T^2(\Phi, \Psi) < T^2(x, y) < T^2(\Psi, \Phi) = (\Psi, \Phi)$. Furthermore, $(B')^\circ$ is invariant as well since T^2 is strongly competitive on B'. The same conclusion follows from (4.6) and (4.7).

A straightforward calculation gives that the determinant of the Jacobian matrix of T^2 at $(\overline{y}, \overline{y})$ satisfies

$$\begin{aligned} \det J_{T^2}(\overline{y}, \overline{y}) &= \begin{vmatrix} \dfrac{r+q-p}{(p+1)(q+1)} & \dfrac{p-q+qr}{(p+1)(q+1)} \\ \dfrac{(r+q-p)(p-q+qr)}{(p+1)^2(q+1)^2} & \dfrac{(r+q-p)(p+1)(q+1)+(p-q+qr)^2}{(p+1)^2(q+1)^2} \end{vmatrix} \\ &= \frac{(r+p-q)^2}{(p+1)^2(q+1)^2} > 0. \end{aligned} \tag{6.4}$$

In addition, we have that the only point in B mapped by T^2 to the fixed point $(\overline{y}, \overline{y})$ is the fixed point itself. To see this, note that the equation $T^2(u, v) = (\overline{y}, \overline{y})$ may be written as

$$\begin{aligned} \frac{pv+u}{r+qv+u} &= \overline{y}, \\ \frac{p\overline{y}+v}{r+q\overline{y}+v} &= \overline{y}. \end{aligned} \tag{6.5}$$

Straightforward algebraic manipulations show that (6.5) implies $u = v = \overline{y}$.

The proof follows from Theorem G. In particular, orbits with initial point in $Q_2(\overline{y}, \overline{y})^\circ$ (resp., in $Q_4(\overline{y}, \overline{y})^\circ$) converge to (Φ, Ψ) (resp., to (Ψ, Φ)). □

Theorem 6.1 gives a complete answer to [1, Open Problem 9.5.7].

References

[1] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2002.

[2] V. L. Kocić, G. Ladas, and I. W. Rodrigues, "On rational recursive sequences," *Journal of Mathematical Analysis and Applications*, vol. 173, no. 1, pp. 127–157, 1993.

[3] C. H. Gibbons, M. R. S. Kulenović, and G. Ladas, "On the recursive sequence $x_{n+1} = \alpha + \beta x_{n-1}/\gamma + x_n$," *Mathematical Sciences Research Hot-Line*, vol. 4, no. 2, pp. 1–11, 2000.

[4] M. R. S. Kulenović, G. Ladas, and N. R. Prokup, "On the recursive sequence $x_{n+1} = \alpha x_n + \beta x_{n-1}/A + x_n$," *Journal of Difference Equations and Applications*, vol. 6, no. 5, pp. 563–576, 2000.

[5] M. R. S. Kulenović, G. Ladas, and N. R. Prokup, "A rational difference equation," *Computers & Mathematics with Applications*, vol. 41, no. 5-6, pp. 671–678, 2001.

[6] M. R. S. Kulenović, G. Ladas, and W. S. Sizer, "On the recursive sequence $x_{n+1} = \alpha x_n + \beta x_{n-1}/\gamma x_n + \delta x_{n-1}$," *Mathematical Sciences Research Hot-Line*, vol. 2, no. 5, pp. 1–16, 1998.

[7] M. R. S. Kulenović and O. Merino, "Convergence to a period-two solution for a class of second order rational difference equations," in *Proceedings of the 10th International Conference on Difference Equations*, pp. 344–353, World Scientific, Munich, Germany, July 2007.

[8] M. R. S. Kulenović and O. Merino, "Global attractivity of the equilibrium of $x_{n+1} = px_n + x_{n-1}/qx_n + x_{n-1}$ for $q < p$," *Journal of Difference Equations and Applications*, vol. 12, no. 1, pp. 101–108, 2006.

[9] R. D. Nussbaum, "Global stability, two conjectures and Maple," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 5, pp. 1064–1090, 2007.

[10] E. Camouzis and G. Ladas, "When does local asymptotic stability imply global attractivity in rational equations?" *Journal of Difference Equations and Applications*, vol. 12, no. 8, pp. 863–885, 2006.

[11] G. A. Enciso and E. D. Sontag, "Global attractivity, I/O monotone small-gain theorems, and biological delay systems," *Discrete and Continuous Dynamical Systems. Series A*, vol. 14, no. 3, pp. 549–578, 2006.

[12] M. R. S. Kulenović and A.-A. Yakubu, "Compensatory versus overcompensatory dynamics in density-dependent Leslie models," *Journal of Difference Equations and Applications*, vol. 10, no. 13–15, pp. 1251–1265, 2004.

[13] H. L. Smith, "The discrete dynamics of monotonically decomposable maps," *Journal of Mathematical Biology*, vol. 53, no. 4, pp. 747–758, 2006.

[14] M. R. S. Kulenović and O. Merino, "A global attractivity result for maps with invariant boxes," *Discrete and Continuous Dynamical Systems. Series B*, vol. 6, no. 1, pp. 97–110, 2006.

[15] E. J. Janowski and M. R. S. Kulenović, "Attractivity and global stability for linearizable difference equations".

[16] M. R. S. Kulenović and O. Merino, "Competitive-exclusion versus competitive-coexistence for systems in the plane," *Discrete and Continuous Dynamical Systems. Series B*, vol. 6, no. 5, pp. 1141–1156, 2006.

[17] H. L. Smith, "Planar competitive and cooperative difference equations," *Journal of Difference Equations and Applications*, vol. 3, no. 5-6, pp. 335–357, 1998.

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