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An inverse problem of fourth-order partial differential equation with nonlocal integral condition

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Abstract

In this study, the time-dependent potential coefficient in a higher-order PDE with initial and boundary conditions is numerically constructed for the first time from a nonlocal integral condition. Even though the inverse identification problem investigated in this study is ill-posed, it has a unique solution. For discretizing the direct problem and finding stable and accurate solutions, we employ the Quintic B-spline (QBS) collocation and Tikhonov regularization methods, respectively. The following nonlinear minimization problem is solved using MATLAB. The collected findings demonstrate that accurate and stable solutions can be found.

Keywords: Higher-order PDE; Inverse problem; QBS functions; Collocation technique; Tikhonov regularization; Nonlinear optimization; Stability analysis; Convergence analysis

1 Introduction

With the current advancement and reliance on science and engineering, inverse problems (IPs) are becoming a core component in these fields. Some of the applications include the use of heat equation which has been widely applied in scientific processes such as melting, freezing, and manufacturing, in microwave heating applications, medicine, biology, seismology, movement of liquid in a porous media, desalination of seawater, and many more. Many of the applications are modeled using partial differential equations (PDEs). If the input values or conditions are known, solutions can be obtained to determine the behavior of the system [5]. Some of the necessary inputs that are required include initial and boundary conditions, solution domain's geometry, coefficients, and forcing terms. If some of this information is missing or unknown, in general it will not be possible to determine the behavior of the physical system. However, certain outputs can be experimentally measured and this information, in addition to the input data, can be used to restore the missing input data. This is what is called an inverse problem. IPs are in general ill-posed. In most cases, this means that a small change in the input data can bring about a substantial change in the output solution.

The scope of IPs has been present in several branches of mathematics, engineering, and physics for a long period. Over the past decade, the theory of inverse problems has been considerably developed given partly its significance in applications. However, numerical solutions to these problems require considerable computations. For example, deconvolution in parameter identification, image reconstruction, and seismic exploration all require high performance computers to do the computation in a timely fashion, or to solve the problem more precisely within a specific time interval [36].

Most of the time, the coefficient entries in a PDE model are linked to the physical properties of the system under consideration. In cases where the model is quite simple, these physical properties can be identified through experimentation and the outcomes used to reduce the model to a particular physical system. In cases where the model is sophisticated, it may be difficult or even impossible to measure the physical properties linked to a coefficient in a model equation. In such a case, it may be important to move forward indirectly because of inadequate information, that is, to design and solve the inverse problem for the missing data.

Parameter identification problems tend to involve the use of actual observation or indirect measurement contaminated by noise, to deduce the values of the parameters making up the system under consideration. In most cases, these inverse problems are ill-posed in line with the Hadamard postulate, that is, if the solution does not exist or is not unique, or if it is in violation of the continuous dependence on input data [8]. Over the past few years, many scholars have shown a significant interest in inverse coefficient identification problems. The primary motivation for this study is to determine the unknown potential property of a given region by considering only the data on its boundary. There is also specific attention to such coefficients that describe a physical quantity of a medium.

The existence of additional unknown terms in the inverse problems needs to impose some additional conditions such as nonlocal integral condition as an overdetermination condition. The basic theory and investigation of inverse problems were established in the underlying works of Ivanov [13], Lavrentiev et al. [23], Tikhonov [34], etc. Inverse problems of the higher order PDEs with additional measurements have been studied by few authors. For example, in [10–12], the authors studied IP to recover the timewise potential coefficients in fourth-order pseudo-parabolic equations. In [25, 37], the authors considered an IP to recover an unknown term in fourth- and sixth-order PDEs. Yuldashev [38] established the unique solvability for the solutions for IPs of the fourth-order PDE. Furthermore, Senapati and Jena [33] developed an FEM including the collocation method with septic B-splines for solving the fifth-order boundary value problems. Jena and Gebremedhin [15–17] applied nonic B-spline, decatic B-spline, and octic B-spline collocation techniques, respectively, to approximate the Kuramoto–Sivashinsky equation, Burgers' equation, and heat and advection–diffusion equations. The authors of [7, 19] proposed ninth and tenth step methods, respectively, to approximate a fourth-order differential equation, while the authors of [6, 14, 18] developed six, seven, and eighth step block methods, respectively, for a fifth-order differential equation. Jena et al. [20] approximated the real definite integrals using the mixed quadrature rule with the quasi-singular integral of electromagnetic field problems. Jena et al. [21, 22] used quartic B-spline approach together with the Butcher's fifth-order Runge–Kutta scheme to find the approximate solution of the MRLW equation. Mohanty et al. [29] proposed three integral transforms through modified ADM to approximate analytical solutions of different mathematical models arising in physical problems, while Mohanty and Jena [28] presented the differential transformation method to approximate an ODE. Mohanty et al. [30] adopted five-point ILMM by collocation and interpolation on the basic of power series and its derivatives, respectively, for the approximation of fourth-order ODEs.

Recently, Abbasova et al. [1] investigated of solvability and proved the existence and uniqueness of the classical solution for the IP of the linear equation of motion of a homogeneous elastic beam. In this paper, the time-dependent potential coefficient and temperature are determined numerically for the first time for a higher-order PDE from nonlocal integral condition. This is a completely new inverse problem, which has never been investigated before. We use a QBS collocation method to discretize the direct problem whilst the least-squares objective functional is minimized to obtain a quasi-solution to the inverse problem. The potential coefficient is proved [1] to be unique by the contraction mapping principle for the problem. It should be noted that the fundamental contribution of this work is the proposal of a regularization algorithm to solve the identification problem and its numerical realization. Nevertheless, since the inverse problem under investigation is ill-posed, the Tikhonov regularization method is employed in order to obtain stable numerical results.

The layout of the proposed study is considered as follows: The higher-order PDE form nonlocal integral condition is formulated in Sect. 2. In Sect. 3, the QBS collocation technique is given. The convergence and stability of the proposed method are analyzed in Sects. 4 and 5, respectively. Section 6 proposes a numerical minimization approach for the regularized objective function, whereas Sect. 7 offers numerical experiments. The concluding remarks of proposed work are given in Sect. 8.

2 Mathematical formulation of the inverse problem

As a mathematical model, we consider a fourth-order motion equation of a homogeneous elastic beam (HEB) in an square plate $D_T = (0,1) \times (0,T)$, over the time interval from the initial time $t = 0$ to a given final time $t = T > 0$. The governing equation is given by the following HEB [1]:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} = q(t)u + g(x,t), \quad (x,t) \in D_T, \tag{1}$$

where $u = u(x,t)$ is the temperature, β is a given positive number, $q(t)$ is the potential coefficient, and $g(x,t)$ is transverse force, subject to the initial conditions (ICs)

$$u(x,0) = \zeta(x), \qquad u_t(x,0) = \eta(x), \quad 0 \le x \le 1, \tag{2}$$

where $\zeta(x)$ and $\eta(x)$ are the given initial temperature and its rate of change, respectively. For the boundary conditions, we assume that these are of Dirichlet and Neumann type (BCs)

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad 0 \le t \le T. \tag{3}$$

The direct problem is concerned with the determination of the temperature u satisfying the initial boundary value problem (1)–(3), when $q(t)$ and $g(x,t)$ are known. The inverse problem, on the other hand, is to determine the unknown $q(t)$ satisfying (1)–(3) and the nonlocal integral observation

$$\int_0^1 f(x)u(x,t)\,dx = \kappa(t), \quad 0 \le t \le T, \tag{4}$$

where $f(x)$, $\kappa(t)$ are given functions. Physically, expression (4) represents a space average measurement of the temperature. This inverse formulation is significant to modeling several practical applications related to unknown potential and temperature. For instance, in the various fields of human activity, such as mineral exploration, medicine, seismology, biology, desalination of seawater, movement of liquid in a porous medium, etc. [1].

Denote

$$\tilde{C}^{2,4}\big(D_T = \big\{(x,t) : x \in [0,1], t \in [0,T]\big\}\big) = \big\{u : u \in C^2(D_T), u_{xxxx} \in C(D_T)\big\}.$$

Definition 1 The solution $\{q(t), u\}$ comprising functions $q(t) \in C[0,T]$ and $u \in \tilde{C}^{2,4}(D_T)$ of the IP (1)–(4) satisfies equation (1) in D_T, conditions (2) in $[0,1]$ and conditions (3) in $[0,T]$.

The following theorem is true [1].

Theorem 1 *Let $g(x,t) \in C(D_T)$, $\zeta(x), \eta(x) \in C[0,1], f(x) \in L_2(0,1)$, $\kappa(t) \in C^2[0,T]$, and*

$$\int_0^1 f(x)\zeta(x)\,dx = \kappa(0), \qquad \int_0^1 f(x)\eta(x)\,dx = \kappa'(0)$$

be fulfilled. Then, finding the required solution of equations (1)–(4) is equivalent to the problem of obtaining $q(t) \in C[0,T]$ and $u \in \tilde{C}^{2,4}(D_T)$ from (1)–(3) and satisfying

$$\begin{aligned}\kappa''(t) &+ \int_0^1 f(x)u_{xxxx}\,dx + \beta\int_0^1 f(x)u_{xx}\,dx \\ &= q(t)\kappa(t) + \int_0^1 f(x)g(x,t)\,dx, \quad 0 \le t \le T.\end{aligned} \tag{5}$$

Let the data of equations (1)–(3), (5) satisfy:

1. $\zeta(x) \in C^4[0,1]$, $\zeta^{(5)}(x) \in L_2(0,1)$, $\zeta(0) = \varphi(1) = \zeta''(0) = \zeta''(1) = \zeta^{(4)}(0) = \zeta^{(4)}(1) = 0$,
2. $\eta(x) \in C^4[0,1]$, $\eta'''(x) \in L_2(0,1)$, $\eta(0) = \eta(1) = \eta''(0) = \eta''(1) = 0$,
3. $g(x,t), g_x(x,t), g_{xx}(x,t) \in C(D_T)$, $g_{xxx}(x,t) \in L_2(D_T)$,
   $g(0,t) = g(1,t) = g_{xx}(0,t) = g_{xx}(1,t) = 0$, $0 \le t \le T$,
4. $0 < \beta < \frac{\pi^2}{2}, f(x) \in L_2(0,1)$, $\kappa(t) \in C^2[0,T]$, $\kappa(t) \neq 0$, $0 \le t \le T$.

Theorem 2 *Let the above conditions* (1)–(4) *of Theorem* 1 *and*

$$\int_0^1 f(x)\zeta(x)\,dx = \kappa(0), \qquad \int_0^1 f(x)\eta(x)\,dx = \kappa'(0)$$

be fulfilled. Then, $\{q(t), u\}$ from $C[0,T] \times \tilde{C}^{2,4}(D_T)$ of the IP (1)–(4) contains a unique solution in the ball K.

3 Discretization of the direct problem via quintic spline functions

In mathematical physics, a direct problem is a problem of modeling some physical fields, processes, or phenomena, especially using partial differential equations (PDEs). The aim of solving a direct problem is to obtain the main dependent variable function that describes and governs naturally a physical field or process.

Table 1 The Qs_i, Qs_i', Qs_i'', Qs_i''' and $Qs_i^{(iv)}$

	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
$Qs_i(x)$	0	1	26	66	26	1	0
$Qs_i'(x)$	0	τ_1	$10\tau_1$	0	$-10\tau_1$	$-\tau_1$	0
$Qs_i''(x)$	0	τ_2	$2\tau_2$	$-6\tau_2$	$2\tau_2$	τ_2	0
$Qs_i'''(x)$	0	τ_3	$-2\tau_3$	0	$2\tau_3$	$-\tau_3$	0
$Qs_i^{(iv)}(x)$	0	τ_4	$-2\tau_4$	$6\tau_4$	$-2\tau_4$	τ_4	0

First, the QBS collocation technique is used for solving the proposed problem (1)–(3), when β, $q(t)$, $\zeta(x)$, $\eta(x)$, and $g(x,t)$ are given. For using the QBS method, we divide $[0,1]$ into equally spaced grid points, $x_{i+1}-x_i$, $i=0(1)M$ apart. We denote $u(x_i,t_j)=u_i^j$, $q(t_j)=q^j$, and $g(x_i,t_j)=g_i^j$, where $x_i=i\Delta x$, $t_j=j\Delta t$, $\Delta x=\frac{1}{M}$ and $\Delta t=\frac{T}{N}$ for $i=0(1)M$ and $j=0(1)N$. The $Qs_i(x)$, $i=-2(1)M+2$ are defined as [4, 26]:

$$Qs_i(x)=\frac{1}{(\Delta x)^5}\begin{cases}\mu_{i-3}^5, & [x_{i-3},x_{i-2}),\\ \mu_{i-3}^5-6\mu_{i-2}^5, & [x_{i-2},x_{i-1}),\\ \mu_{i-3}^5-6\mu_{i-2}^5+15\mu_{i-1}^5, & [x_{i-1},x_i),\\ -\mu_{i+3}^5+6\mu_{i+2}^5-15\mu_{i+1}^5, & [x_i,x_{i+1}),\\ -\mu_{i+3}^5+6\mu_{i+2}^5, & [x_{i+1},x_{i+2}),\\ \mu_{i+3}^5, & [x_{i+2},x_{i+3}),\\ 0, & \text{else},\end{cases}\tag{6}$$

where $\mu_i=x-x_i$ and the set of QBS functions $\{Qs_{-2},Qs_{-1},\dots,Qs_{M+2}\}$ form a basis over $[0,1]$. The Qs_i, Qs_i', Qs_i'', Qs_i''' and $Qs_i^{(iv)}$ are defined in Table 1.

Theorem 3 *Let β be a positive number and $g(x,t)$, $\zeta(x)$, $\eta(x)$, $f(x)$, $\kappa(t)$, $q(t)$ given functions. Suppose the QBS method and finite difference scheme are used for space and time discritization, respectively. Then, numerical solution $u(x,t)$ is given in equations* (23), (29), *and* (32).

Proof Assume the expression for approximate solution $u(x,t)$ at (x,t_j) is defined as

$$u(x,t_j)=\sum_{k=-2}^{M+2}C_k^jQs_k(x),\tag{7}$$

where C_k^j are the time-dependent quantities. The variation of the $u_M(x,t)$, over the element, can be defined as

$$u(x,t_j)=\sum_{k=i-2}^{i+2}C_k^jQs_k(x).\tag{8}$$

Using (8), function u with its first four derivatives can be defined as:

$$u_i^j=C_{i+2}^j+26C_{i+1}^j+66C_i^j+26C_{i-1}^j+C_{i-2}^j,\tag{9}$$

$$(u_x)_i^j=\tau_1\big(C_{i+2}^j+10C_{i+1}^j-10C_{i-1}^j-C_{i-2}^j\big),\tag{10}$$

$$(u_{xx})_i^j = \tau_2\left(C_{i+2}^j + 2C_{i+1}^j - 6C_i^j + 2C_{i-1}^j + C_{i-2}^j\right), \tag{11}$$

$$(u_{xxx})_i^j = \tau_3\left(C_{i+2}^j - 2C_{i+1}^j + 2C_{i-1}^j - C_{i-2}^j\right), \tag{12}$$

$$(u_{xxxx})_i^j = \tau_4\left(C_{i+2}^j - 4C_{i+1}^j + 6C_i^j - 4C_{i-1}^j + C_{i-2}^j\right), \tag{13}$$

where

$$\tau_1 = \frac{5}{\Delta x}, \qquad \tau_2 = \frac{20}{(\Delta x)^2}, \qquad \tau_3 = \frac{60}{(\Delta x)^3}, \qquad \tau_4 = \frac{120}{(\Delta x)^4}.$$

Now the discretization of equation (1) yields

$$\begin{aligned}
&\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} + \frac{1}{2}\left((u_{xxxx})_i^{j+1} + (u_{xxxx})_i^j\right) + \frac{\beta}{2}\left((u_{xx})_i^{j+1} + (u_{xx})_i^j\right) \\
&\quad = \frac{1}{2}\left(q_{j+1}u^{j+1} + q_j u^j\right) + \frac{1}{2}\left(g_i^{j+1} + g_i^j\right), \quad i = 0(1)M, j = 0(1)N.
\end{aligned} \tag{14}$$

Simplifying (14) yields

$$\begin{aligned}
&\left(1 - \frac{(\Delta t)^2}{2}q_{j+1}\right)u_i^{j+1} + \frac{\beta(\Delta t)^2}{2}(u_{xx})_i^{j+1} + \frac{(\Delta t)^2}{2}(u_{xxxx})_i^{j+1} \\
&\quad = \left(2 + \frac{(\Delta t)^2}{2}q_j\right)u_i^j - \frac{\beta(\Delta t)^2}{2}(u_{xx})_i^j - \frac{(\Delta t)^2}{2}(u_{xxxx})_i^j + \frac{(\Delta t)^2}{2}\left(g_i^{j+1} + g_i^j\right), \\
&\qquad i = 0(1)M, j = 0(1)N,
\end{aligned} \tag{15}$$

which can be written as

$$\begin{aligned}
&\left(1 - A^{j+1}\right)u_i^{j+1} + \beta B(u_{xx})_i^{j+1} + B(u_{xxxx})_i^{j+1} \\
&\quad = \left(2 + A^j\right)u_i^j - \beta B(u_{xx})_i^j - B(u_{xxxx})_i^j + \frac{(\Delta t)^2}{2}\left(g_i^{j+1} + g_i^j\right), \\
&\qquad i = 0(1)M, j = 0(1)N,
\end{aligned} \tag{16}$$

where

$$A^j = \frac{(\Delta t)^2}{2}q_j, \qquad B = \frac{(\Delta t)^2}{2}.$$

Now, using u, u_{xx}, and u_{xxxx} from equations (9)–(13), we get

$$\begin{aligned}
&\bar{A}^{j+1}C_{i-2}^{j+1} + \bar{B}^{j+1}C_{i-1}^{j+1} + \bar{D}^{j+1}C_i^{j+1} + \bar{B}^{j+1}C_{i+1}^{j+1} + \bar{A}^{j+1}C_{i+2}^{j+1} \\
&\quad = \bar{E}^j C_{i-2}^j + \bar{F}^j C_{i-1}^j + \bar{G}^j C_i^j + \bar{F}^j C_{i+1}^j + \bar{E}^j C_{i+2}^j - C_{i-2}^{j-1} - 26C_{i-1}^{j-1} - 66C_i^{j-1} \\
&\qquad - 26C_{i+1}^{j-1} - C_{i+2}^{j-1} + \frac{(\Delta t)^2}{2}\left(g_i^{j+1} + g_i^j\right), \quad i = 2, \dots, M-2, j = 0(1)N,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
&\bar{A}^j = 1 - A^j\beta B\tau_2 + B\tau_4, \qquad \bar{B}^j = 26 - 26A^j + 2\beta B\tau_2 - 4B\tau_4, \\
&\bar{D} = 66 - 66A^j - 6\beta B\tau_2 + 6B\tau_4, \qquad \bar{E}^j = 2 + A^j - \beta B\tau_2 - B\tau_4, \\
&\bar{F}^j = 52 + 26A^j - 2\beta B\tau_2 + 4B\tau_4, \qquad \bar{G}^j = 132 + 66A^j + 6\beta\tau_2 - 6B\tau_4.
\end{aligned}$$

Now, discretizing the boundary conditions (3), we get

$$\begin{aligned} &C_{-1}^j = -C_1^j - 3C_0^j, \qquad C_{-2}^j = -C_2^j - 12C_0^j, \\ &C_{M+2}^j = 12C_M^j - C_{M-2}^j, \qquad C_{M+1}^j = -3C_M^j - C_{M-1}^j, \quad j = 0(1)N. \end{aligned} \tag{18}$$

For $i = 0$, using equation (18) in (17), we get

$$\begin{aligned} &\left(12\bar{A}^{j+1} - 3\bar{B}^{j+1} + \bar{D}^{j+1}\right)C_0^{j+1} \\ &\quad = \left(12\bar{E}^j - 3\bar{F}^j + \bar{G}^j\right)C_0^j + \frac{(\Delta t)^2}{2}\left(g_0^{j+1} + g_0^j\right), \quad j = 0(1)N. \end{aligned} \tag{19}$$

Now, for $i = 1$, we get

$$\begin{aligned} &\left(-3\bar{A}^{j+1} + \bar{B}^{j+1}\right)C_0^{j+1} + \left(-\bar{A}^{j+1} + \bar{D}^{j+1}\right)C_1^{j+1} + \bar{B}^{j+1}C_2^{j+1} + \bar{A}^{j+1}C_3^{j+1} \\ &\quad = \left(-3\bar{E}^j + \bar{F}^j\right)C_0^j + \left(-\bar{E}^j + \bar{G}^j\right)C_1^j + \bar{F}^jC_2^j + \bar{E}^jC_3^j - 23C_0^{j-1} - 65C_1^{j-1} \\ &\qquad - 26C_2^{j-1} - C_3^{j-1} + \frac{(\Delta t)^2}{2}\left(g_1^{j+1} + g_1^j\right), \quad j = 0(1)N. \end{aligned} \tag{20}$$

Next, for $i = M - 1$, we get

$$\begin{aligned} &\bar{A}^{j+1}C_{M-3}^{j+1} + \bar{B}^{j+1}C_{M-2}^{j+1} + \left(-\bar{A}^{j+1} + \bar{D}^{j+1}\right)C_{M-1}^{j+1} + \left(-3\bar{A}^{j+1} + \bar{B}^{j+1}\right)C_M^{j+1} \\ &\quad = \bar{E}^jC_{M-3}^j + \bar{F}^jC_{M-2}^j + \left(-\bar{E}^j + \bar{G}^j\right)C_{M-1}^j + \left(-3\bar{E}^j + \bar{F}^j\right)C_M^j - C_{M-3}^{j-1} \\ &\qquad - 26C_{M-2}^{j-1} - 65C_{M-1}^{j-1} - 23C_M^{j-1} + \frac{(\Delta t)^2}{2}\left(g_{M-1}^{j+1} + g_{M-1}^j\right), \quad j = 0(1)N. \end{aligned} \tag{21}$$

Finally, for $i = M$, we get

$$\begin{aligned} &\left(12\bar{A}^{j+1} - 3\bar{B}^{j+1} + \bar{D}^{j+1}\right)C_M^{j+1} \\ &\quad = \left(12\bar{E}^j - 3\bar{F}^j + \bar{G}^j\right)C_M^j + \frac{(\Delta t)^2}{2}\left(g_M^{j+1} + g_M^j\right), \quad j = 0(1)N. \end{aligned} \tag{22}$$

At t_{j+1}, $j = 1, \ldots, N-1$, (19), (20), (17), (21), and (22) can be reformulated as

$$\begin{pmatrix} \hat{p}^{j+1} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \hat{q}^{j+1} & \hat{r}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} & 0 & 0 & \ldots & 0 \\ \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{D}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} & 0 & \cdots & 0 \\ 0 & \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{D}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{D}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} \\ 0 & \ldots & 0 & 0 & \bar{A}^{j+1} & \bar{B}^{j+1} & \hat{r}^{j+1} & \hat{q}^{j+1} \\ 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \hat{p}^{j+1} \end{pmatrix} \begin{pmatrix} C_0^{j+1} \\ C_1^{j+1} \\ C_2^{j+1} \\ \vdots \\ C_{M-2}^{j+1} \\ C_{M-1}^{j+1} \\ C_M^{j+1} \end{pmatrix} = \begin{pmatrix} R_0^j \\ R_1^j \\ R_2^j \\ \vdots \\ R_{M-2}^j \\ R_{M-1}^j \\ R_M^j \end{pmatrix}, \tag{23}$$

where

$$\hat{p}^{j+1} = 12\bar{A}^{j+1} - 3\bar{B}^{j+1} + \bar{D}^{j+1},$$

$$\hat{q}^{j+1} = -3\bar{A}^{j+1} + \bar{B}^{j+1}, \qquad \hat{r}^{j+1} = -\bar{A}^{j+1} + \bar{D}^{j+1},$$

$$R_0^j = \left(12\bar{E}^j - 3\bar{F}^j + \bar{G}^j\right)C_0^j + \frac{(\Delta t)^2}{2}\left(g_0^{j+1} + g_0^j\right), \quad j = 0,\dots,N,$$

$$\begin{aligned} R_1^j &= \left(-3\bar{E}^j + \bar{F}^j\right)C_0^j + \left(-\bar{E}^j + \bar{G}^j\right)C_1^j + \bar{F}^j C_2^j + \bar{E}^j C_3^j - 23C_0^{j-1} \\ &\quad - 65C_1^{j-1} - 26C_2^{j-1} - C_3^{j-1} + \frac{(\Delta t)^2}{2}\left(g_1^{j+1} + g_1^j\right), \quad j = 0(1)N, \end{aligned}$$

$$\begin{aligned} R_i^j &= \bar{E}^j C_{i-2}^j + \bar{F}^j C_{i-1}^j + \bar{G}^j C_i^j + \bar{F}^j C_{i+1}^j + \bar{E}^j C_{i+2}^j \\ &\quad - C_{i-2}^{j-1} - 26C_{i-1}^{j-1} - 66C_i^{j-1} - 26C_{i+1}^{j-1} - C_{i+2}^{j-1} \\ &\quad + \frac{(\Delta t)^2}{2}\left(g_i^{j+1} + g_i^j\right), \quad i = 2,\dots,M-2, j = 0(1)N, \end{aligned}$$

$$\begin{aligned} R_{M-1}^j &= \bar{E}^j C_{M-3}^j + \bar{F}^j C_{M-2}^j + \left(-\bar{E}^j + \bar{G}^j\right)C_{M-1}^j + \left(-3\bar{E}^j + \bar{F}^j\right)C_M^j \\ &\quad - C_{M-3}^{j-1} - 26C_{M-2}^{j-1} - 65C_{M-1}^{j-1} - 23C_M^{j-1} \\ &\quad + \frac{(\Delta t)^2}{2}\left(g_{M-1}^{j+1} + g_{M-1}^j\right), \quad j = 0(1)N, \end{aligned}$$

$$R_M^j = \left(12\bar{E}^j - 3\bar{F}^j + \bar{G}^j\right)C_M^j + \frac{(\Delta t)^2}{2}\left(g_M^{j+1} + g_M^j\right), \quad j = 0(1)N.$$

Now for $j = 0$, using the initial condition (2) in (17), we have

$$\begin{aligned} &\left(1 + \bar{A}^1\right)C_{i-2}^1 + \left(26 + \bar{B}^1\right)C_{i-1}^1 + \left(66 + \bar{D}^1\right)C_i^1 + \left(26 + \bar{B}^1\right)C_{i+1}^1 + \left(1 + \bar{A}^1\right)C_{i+2}^1 \\ &\quad = \bar{E}^0 C_{i-2}^0 + \bar{F}^0 C_{i-1}^0 + \bar{G}^0 C_i^0 + \bar{F}^0 C_{i+1}^0 + \bar{E}^0 C_{i+2}^0 + 2(\Delta t)\eta(x_i) + \frac{(\Delta t)^2}{2}\left(g_i^1 + g_i^0\right), \\ &\quad i = 2,\dots,M-2. \end{aligned} \tag{24}$$

For $i = 0$, using (18) in (24), we have

$$\left(12\bar{A}^1 - 3\bar{B}^1 + \bar{D}^1\right)C_0^1 = \left(12\bar{E}^0 - 3\bar{F}^0 + \bar{G}^0\right)C_0^0 + \frac{(\Delta t)^2}{2}\left(g_0^1 + g_0^0\right). \tag{25}$$

Now, for $i = 1$, we have

$$\begin{aligned} &\left(23 - 3\bar{A}^1 + \bar{B}^1\right)C_0^1 + \left(65 - \bar{A}^1 + \bar{D}^1\right)C_1^1 + \left(26 + \bar{B}^1\right)C_2^1 + \left(1 + \bar{A}^1\right)C_3^1 \\ &\quad = \left(-3\bar{E}^0 + \bar{F}^0\right)C_0^0 + \left(-\bar{E}^0 + \bar{G}^0\right)C_1^0 + \bar{F}^0 C_2^0 + \bar{E}^0 C_3^0 \\ &\qquad + 2(\Delta t)\eta(x_1) + \frac{(\Delta t)^2}{2}\left(g_1^{j+1} + g_1^j\right). \end{aligned} \tag{26}$$

Next, for $i = M-1$, we get

$$\begin{aligned} &\left(1 + \bar{A}^1\right)C_{M-3}^1 + \left(26 + \bar{B}^1\right)C_{M-2}^1 + \left(65 - \bar{A}^1 + \bar{D}^1\right)C_{M-1}^1 + \left(23 - 3\bar{A}^1 + \bar{B}^1\right)C_M^1 \\ &\quad = \bar{E}^0 C_{M-3}^0 + \bar{F}^0 C_{M-2}^0 + \left(-\bar{E}^0 + \bar{G}^0\right)C_{M-1}^0 + \left(-3\bar{E}^0 + \bar{F}^0\right)C_M^0 \\ &\qquad + 2(\Delta t)\eta(x_{M-1}) + \frac{(\Delta t)^2}{2}\left(g_{M-1}^1 + g_{M-1}^0\right). \end{aligned} \tag{27}$$

Finally, for $i = M$, we have

$$\left(12\bar{A}^1 - 3\bar{B}^1 + \bar{D}^1\right)C_M^1 = \left(12\bar{E}^0 - 3\bar{F}^0 + \bar{G}^0\right)C_M^0 + \frac{(\Delta t)^2}{2}\left(g_M^1 + g_M^0\right). \tag{28}$$

At time step t_1, (24)–(28) can be reformulated as

$$\begin{pmatrix} \hat{p}^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 23+\hat{q}^1 & 65+\hat{r}^1 & 26+\bar{B}^1 & 1+\bar{A}^1 & 0 & 0 & \dots & 0 \\ 1+\bar{A}^1 & 26+\bar{B}^1 & 66+\bar{D}^1 & 26+\bar{B}^1 & 1+\bar{A}^1 & 0 & \dots & 0 \\ 0 & 1+\bar{A}^1 & 26+\bar{B}^1 & 66+\bar{D}^1 & 26+\bar{B}^1 & 1+\bar{A}^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1+\bar{A}^1 & 26+\bar{B}^1 & 66+\bar{D}^1 & 26+\bar{B}^1 & 1+\bar{A}^1 \\ 0 & \dots & 0 & 0 & 1+\bar{A}^1 & 26+\bar{B}^1 & 65+\hat{r}^1 & 23+\hat{q}^1 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \hat{p}^1 \end{pmatrix} \times \begin{pmatrix} C_0^1 \\ C_1^1 \\ C_2^1 \\ \vdots \\ C_{M-2}^1 \\ C_{M-1}^1 \\ C_M^1 \end{pmatrix} = \begin{pmatrix} R_0^0 \\ R_1^0 \\ R_2^0 \\ \vdots \\ R_{M-2}^0 \\ R_{M-1}^0 \\ R_M^0 \end{pmatrix}, \tag{29}$$

where

$$\begin{aligned} R_0^0 &= \left(12\bar{E}^0 - 3\bar{F}^0 + \bar{G}^0\right)C_0^0 + \frac{(\Delta t)^2}{2}\left(g_0^1 + g_0^0\right), \\ R_1^0 &= \left(-3\bar{E}^0 + \bar{F}^0\right)C_0^0 + \left(-\bar{E}^0 + \bar{G}^0\right)C_1^0 + \bar{F}^0 C_2^0 \\ &\quad + \bar{E}^0 C_3^0 + 2(\Delta t)\eta(x_1) + \frac{(k)^2}{2}\left(g_1^1 + g_1^0\right), \\ R_i^0 &= \bar{E}^0 C_{i-2}^0 + \bar{F}^0 C_{i-1}^0 + \bar{G}^0 C_i^0 + \bar{F}^0 C_{i+1}^0 + \bar{E}^0 C_{i+2}^0 \\ &\quad + 2(\Delta t)\eta(x_i) + \frac{(\Delta t)^2}{2}\left(g_i^{j+1} + g_i^j\right), \quad i = 2, 3, \dots, M-2, \\ R_{M-1}^0 &= \bar{E}^0 C_{M-3}^0 + \bar{F}^0 C_{M-2}^0 + \left(-\bar{E}^0 + \bar{G}^0\right)C_{M-1}^0 + \left(-3\bar{E}^0 + \bar{F}^0\right)C_M^0 \\ &\quad + 2(\Delta t)\eta(x_{M-1}) + \frac{(\Delta t)^2}{2}\left(g_{M-1}^1 + g_{M-1}^0\right), \\ R_M^0 &= \left(12\bar{E}^0 - 3\bar{F}^0 + \bar{G}^0\right)C_M^0 + \frac{(\Delta t)^2}{2}\left(g_M^1 + g_M^0\right). \end{aligned}$$

Now, we determine the initial vector $(C_{-2}^j, C_{-1}^j, C_0^0, \dots, C_{M+1}^0, C_{M+2}^0)$. For removing C_{-2}^0, C_{-1}^0, C_{M+1}^0, and C_{M+2}^0, we use

$$u_x(0,0) = \zeta_x(x_0), \qquad u_x(1,0) = \zeta_x(x_M), \tag{30}$$

$$u_{xx}(0,0) = 0, \qquad u_{xx}(1,0) = 0. \tag{31}$$

Using u_x and u_{xx} from (10), (11) in equations (30) and (31), we get C^0_{-2}, C^0_{-1}, C^0_{M+1}, and C^0_{M+2}, and removing the unknowns C^0_{-2}, C^0_{-1}, C^0_{M+1}, and C^0_{M+2}, we have an $(M+1)\times(M+1)$ order system as follows:

$$\begin{pmatrix} 54 & 60 & 6 & 0 & 0 & 0 & \cdots & 0 \\ \frac{101}{4} & \frac{135}{2} & \frac{105}{4} & 1 & 0 & 0 & \cdots & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & \cdots & 0 & 0 & 1 & \frac{105}{4} & \frac{135}{2} & \frac{101}{4} \\ 0 & \cdots & 0 & 0 & 0 & 6 & 60 & 54 \end{pmatrix} \begin{pmatrix} C^0_0 \\ C^0_1 \\ C^0_2 \\ C^0_2 \\ \vdots \\ C^0_{M-2} \\ C^0_{M-1} \\ C^0_M \end{pmatrix}$$
$$= \begin{pmatrix} \zeta(x_0) + \frac{3.5}{\tau_1}\zeta_x(x_0) \\ \zeta(x_1) + \frac{1}{8\tau_1}\zeta_x(x_0) \\ \zeta(x_2) \\ \zeta(x_3) \\ \vdots \\ \zeta(x_{M-2}) \\ \zeta(x_{M-1}) - \frac{1}{8\tau_1}\zeta_x(x_M) \\ \zeta(x_M) + \frac{3}{\tau_1}\zeta_x(x_M) \end{pmatrix}. \tag{32}$$

□

4 Stability analysis

For the discretized system of (1), the von Neumann stability analysis is performed [2, 27, 31, 35]. For stability, we take $g(x,t) = 0$ and assume local constant $P = k_1$ for a known level, which gives the discretized system of (1) as follows:

$$\begin{aligned} &\tilde{A}C^{j+1}_{i-2} + \tilde{B}C^{j+1}_{i-1} + \tilde{D}C^{j+1}_{i} + \tilde{B}C^{j+1}_{i+1} + \tilde{A}C^{j+1}_{i+2} \\ &\quad = \tilde{E}C^{j}_{i-2} + \tilde{F}C^{j}_{i-1} + \tilde{G}C^{j}_{i} + \tilde{F}C^{j}_{i+1} + \tilde{E}C^{j}_{i+2} \\ &\qquad - C^{j-1}_{i-2} - 26C^{j-1}_{i-1} - 66C^{j-1}_{i} - 26C^{j-1}_{i+1} - C^{j-1}_{i+2}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} \tilde{A} &= 1 - 0.5(\Delta t)^2 k_1 + 0.5\beta(\Delta t)^2\tau_2 + 0.5(\Delta t)^2\tau_4, \\ \tilde{B} &= 26 - 13(\Delta t)^2 k_1 + \beta(\Delta t)^2\tau_2 - 2(\Delta t)^2\tau_4, \\ \tilde{D} &= 66 - 33(\Delta t)^2 k_1 - 3\beta(\Delta t)^2\tau_2 + 3(\Delta t)^2\tau_4, \\ \tilde{E} &= 2 + 0.5(\Delta t)^2 k_1 - 0.5\beta(\Delta t)^2\tau_2 - 0.5(\Delta t)^2\tau_4, \\ \tilde{F} &= 52 + 132(\Delta t)^2 k_1 - \beta(\Delta t)^2 k_1\tau_2 + 2(\Delta t)^2 k_1\tau_4, \\ \tilde{G} &= 132 + 33(\Delta t)^2 k_1 + 3\beta(\Delta t)^2\tau_2 - 3(\Delta t)^2\tau_4. \end{aligned}$$

Now we consider trial solution $C_i^j = \delta^j e^{ki\phi}$ at x_i, where $\phi = \theta h$ and $k = \sqrt{-1}$. Using the trial solution in (33), we get

$$\begin{aligned}&\bigl(2\tilde{A}\cos(2\phi) + 2\tilde{B}\cos(\phi) + \tilde{D}\bigr)\delta^2 - \bigl(2\tilde{E}\cos(2\phi) + 2\tilde{F}\cos(\phi) + \tilde{G}\bigr)\delta \\ &\quad + \bigl(2\cos(2\phi) + 52\cos(\phi) + 66\bigr) = 0,\end{aligned} \tag{34}$$

which can be written as

$$\zeta_1\delta^2 - \zeta_2\delta + \zeta_3 = 0, \tag{35}$$

where

$$\begin{aligned}\zeta_1 &= 2\tilde{A}\cos(2\phi) + 2\tilde{B}\cos(\phi) + \tilde{D}, \\ \zeta_2 &= 2\tilde{E}\cos(2\phi) + 2\tilde{F}\cos(\phi) + \tilde{G}, \\ \zeta_3 &= 2\cos(2\phi) + 52\cos(\phi) + 66.\end{aligned}$$

Applying the Routh–Hurwitz criterion under the transformation $\hat{u} = \frac{1+\rho}{1-\rho}$ in (35), we get

$$(\zeta_1 + \zeta_2 + \zeta_3)\rho^2 + 2(\zeta_1 - \zeta_3)\rho + (\zeta_1 - \zeta_2 + \zeta_3) = 0. \tag{36}$$

The necessary and sufficient conditions for (33) to be stable $|\hat{u}| \le 1$ are

$$\zeta_1 + \zeta_2 + \zeta_3 \ge 0, \qquad \zeta_1 - \zeta_3 \ge 0, \qquad \zeta_1 - \zeta_2 + \zeta_3 \ge 0. \tag{37}$$

Using ζ_1, ζ_2, ζ_3 and simplifying the terms, we get

$$\zeta_1 + \zeta_2 + \zeta_3 = 8\cos^2(\phi) + 208\cos^2\left(\frac{\phi}{2}\right) + 156, \tag{38}$$

$$\begin{aligned}\zeta_1 - \zeta_3 = 2(\Delta t)^2\bigl(&(-k_1 + \tau_4 + \beta\tau_2)\cos^2(\phi) - 26k_1 + 4\tau_4 + 2\beta\tau_2 \\ &+ 3(-k_1 + 3\tau_4 - \beta\tau_2)\bigr),\end{aligned} \tag{39}$$

$$\begin{aligned}\zeta_1 - \zeta_2 + \zeta_3 = 4(\Delta t)^2\bigl(&(-k_1 + \tau_4 + \beta\tau_2)\cos^2(\phi) - 26k_1 + 12\tau_4 - 12\beta\tau_2 \\ &+ 3(-k_1 + \tau_4 - 3\beta\tau_2 - 33)\bigr).\end{aligned} \tag{40}$$

Since $\zeta_1 + \zeta_2 + \zeta_3 \ge 0$, $\zeta_1 - \zeta_3 \ge 0$, and $\zeta_1 - \zeta_2 + \zeta_3 \ge 0$, the discretized system for (1) is unconditionally stable.

Lemma 4.1 *The quintic B-spline satisfies*

$$\left|\sum_{i=-2}^{M+2} Qs_i(x)\right| \le 186, \quad 0 \le x \le 1. \tag{41}$$

Proof We know that

$$\left|\sum_{i=-2}^{M+2} Qs_i(x)\right| \le \sum_{i=-2}^{M+2}\bigl|Qs_i(x)\bigr|$$

at any particular knot x_i, so we have

$$\begin{aligned}\sum_{i=-2}^{M+2}\left|Qs_i(x)\right| &= \left|Qs_{i-2}(x)\right| + \left|Qs_{i-1}(x)\right| + \left|Qs_i(x)\right| + \left|Qs_{i+1}(x)\right| + \left|Qs_{i+2}(x)\right| \\ &= 1 + 26 + 66 + 26 + 1 = 120.\end{aligned}$$

Also in each subinterval $x_{i-1} \le x \le x_i$,

$$\begin{aligned}&\left|Qs_{i-3}(x)\right| \le 1, \qquad \left|Qs_{i-2}(x)\right| \le 26, \qquad \left|Qs_{i-1}(x)\right| \le 66, \\ &\left|Qs_i(x)\right| \le 66, \qquad \left|Qs_{i+1}(x)\right| \le 26, \qquad \left|Qs_{i+2}(x)\right| \le 1.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i=-2}^{M+2}\left|Qs_i(x)\right| &= \left|Qs_{i-3}(x)\right| + \left|Qs_{i-2}(x)\right| + \left|Qs_{i-1}(x)\right| \\ &\quad + \left|Qs_i(x)\right| + \left|Qs_{i+1}(x)\right| + \left|Qs_{i+2}(x)\right| \\ &= 1 + 26 + 66 + 66 + 26 + 1 \le 186.\end{aligned}$$

Thus the proof is complete. □

Theorem 4 *Suppose that $f(x) \in C^5 \in [a,b]$ and $f^5(x) < l^*$ with equally spaced partitioning of the domain with uniform step size (Δx). If the spline $U^*(x)$ uniquely interpolates $f(x)$ at the knots then there exists a constant ρ_n such that*

$$\left\| \left(f(x)\right)^n - \left(U^*\right)^n \right\|_\infty \le \delta_n l^* (\Delta x)^{6-n}. \tag{42}$$

Proof See [3] and [9]. □

5 Convergence analysis

Suppose $U(x,t_j) = \sum_{i=-2}^{M+2} C_i^j Qs_i(x)$ is the quintic B-spline approximation to the exact solution $u(x,t_j)$. Due to computational error, we assume that $U^*(x,t_j) = \sum_{i=-2}^{M+2} C_i^{*j} Qs_i(x)$ is the computed spline approximation to $U(x,t_j)$ where $C_i^{*j} = (C_{-2}^{*j}, C_{-1}^{*j}, C_0^{*j}, \dots, C_{M+1}^{*j}, C_{M+2}^{*j})^T$. So, we estimate the error $\|u_i^j - U_i^{*j}\|_\infty$ and $\|U_i^{*j} - U_i^j\|_\infty$ separately to estimate the error $\|u_i^j - U_i^j\|_\infty$. By putting U_i^{*j} in the simplified form of equation (22), we obtain

$$RC_i^{*(j+1)} = S_i^{*j}. \tag{43}$$

From above equation, we have

$$R\left(C_i^{*(j+1)} - C_i^{j+1}\right) = \left(S_i^{*j} - S_i^j\right). \tag{44}$$

Also equation (16) can be written as

$$U_i^{j+1} + \beta B(U_{xx})_i^{j+1} + B(U_{xxxx})_i^{j+1} + \omega_i^{j+1} = \chi_i^j, \tag{45}$$

where

$$\omega_i^{j+1} = -A^{j+1}U_i^{j+1},$$
$$\chi_i^j = (2 + A^j)U_i^j - \beta B(U_{xx})_i^j - B(U_{xxxx})_i^j + \frac{(\Delta t)^2}{2}(g_i^{j+1} + g_i^j).$$

Applying the triangle inequality and Theorem 4, equation (45) gives

$$\begin{aligned} \left|\chi_i^{*j} - \chi_i^j\right| &= \left|\left(U_i^{*j} + \beta B(U_{xx})_i^{*j} + B(U_{xxxx})_i^{*j} + \omega_i^{*j}\right)\right. \\ &\quad \left.- \left(U_i^j + \beta B(U_{xx})_i^j + B(U_{xxxx})_i^j + \omega_i^j\right)\right| \\ &= \left|\left(U_i^{*j} - U_i^j\right) + \beta B\left((U_{xx})_i^{*j} - (U_{xx})_i^j\right)\right. \\ &\quad \left.+ B\left((U_{xxxx})_i^{*j} - (U_{xxxx})_i^j\right) + \omega_i^{*j} - \omega_i^j\right| \\ &\leq \left|U_i^{*j} - U_i^j\right| + \beta B\left|(U_{xx})_i^{*j} - (U_{xx})_i^j\right| \\ &\quad + B\left|(U_{xxxx})_i^{*j} - (U_{xxxx})_i^j\right| + \left|\omega_i^{*j} - \omega_i^j\right|. \end{aligned}$$

Thus we can write

$$\left\|(S^* - S)_i^j\right\| \leq (1 + \delta)\rho_0 l^*(\Delta x)^6 + \beta B\rho_2 l^*(\Delta x)^4 + B\rho_4 l^*(\Delta x)^2,$$

or

$$\left\|(S^* - S)_i^j\right\| \leq \xi(\Delta x)^2, \tag{46}$$

where $\xi = (1 + \delta)\rho_0 l^*(\Delta x)^6 + \beta B\rho_2 l^*(\Delta x)^4 + B\rho_4 l^*(\Delta x)^2$.

Now from (44) we have

$$R(C^* - C)_i^j = (S^* - S)_i^j,$$

which can be written as

$$(C^* - C)_i^j = R^{-1}(S^* - S)_i^j.$$

By using (46), we obtain

$$\left\|(C^* - C)_i^j\right\| \leq \left\|R^{-1}\right\|\xi(\Delta x)^2 \tag{47}$$

and, by the properties of matrices, we have

$$\left\|R^{-1}\right\| \leq \frac{1}{|\nu_i|}, \tag{48}$$

where ν_i is the sum of the ith row of the matrix R. Now substituting the above equation into (47), we have

$$(C^* - C)_i^j \leq \xi_1(\Delta x)^2, \tag{49}$$

where $\xi_1 = \frac{\xi}{|\nu_i|}$ is some finite constant.

Since

$$U_i^{*(j+1)} - U_i^{j+1} = \sum_{i=-2}^{M+2} \left(C_i^{*(j+1)} - C_i^{j+1}\right) Qs_i(x),$$

by taking the norm of both sides and using equations (49) and (41), we have

$$\left\| U_i^{*(j+1)} - U_i^{j+1} \right\| \le 186\xi (\Delta x)^2.$$

Using the triangle inequality, the latter equation yields

$$\begin{aligned}
\left\| u_i^{j+1} - U_i^{j+1} \right\| &= \left\| u_i^{j+1} - U_i^{*(j+1)} + U_i^{*(j+1)} - U_i^{j+1} \right\|_\infty \\
&\le \left\| u_i^{j+1} - U_i^{*(j+1)} \right\|_\infty + \left\| U_i^{*(j+1)} - U_i^{j+1} \right\|_\infty \\
&\le \rho_0 l^* (\Delta x)^6 + 186\xi (\Delta x)^2 \\
&= \gamma (\Delta x)^2,
\end{aligned}$$

where $\gamma = \rho_0 l^* (\Delta x)^4 + 186\xi$. If $U^{j+1}(x,t)$ is the approximate solution of the exact solution $u^{j+1}(x,t)$ then

$$\left\| u_i^{j+1} - U_i^{j+1} \right\| \le \gamma (\Delta x)^2 + \alpha (\Delta t)^2. \tag{50}$$

Hence, the theoretical order of convergence of the proposed scheme is $O(\Delta x^2 + \Delta t^2)$.

6 Inverse problem of HEB equation

We now try to identify the stable and accurate solution for terms $q(t)$ and $u(x,t)$ satisfying (1)–(4). Keep in mind that, since the inverse problem under investigation is ill-posed and very sensitive to noise (small errors in the additional input data cause large errors in the output potential), the solution needs to be regularized. Therefore, the Tikhonov regularization method is employed in order to obtain a stable and accurate solution. Moreover, the total variation regularization algorithm's technique may also be applied [32]. The quasi-solution of the inverse problem (1)–(4) is approximated by the minimizer of the Tikhonov regularization functional, which is the gap between the computed and measured data, namely

$$F(q) = \left\| \int_0^1 f(x) u(x,t)\, dx - \kappa(t) \right\|^2 + \lambda \left\| q(t) \right\|^2, \tag{51}$$

where u solves (1)–(3) for known $q(t)$, and $\lambda \ge 0$ is a parameter of regularization that is introduced to stabilize the approximation solutions. For the discrete form, (51) turns into

$$F(\underline{q}) = \sum_{j=1}^{N} \left[\int_0^1 f(x) u(x,t_j)\, dx - \kappa(t_j) \right]^2 + \lambda \sum_{j=1}^{N} \left(q^j\right)^2. \tag{52}$$

The MATLAB subroutine [24] is utilized to minimize the cost function (52).

To measure the errors in this data, $\kappa(t_j)$ in (52) is replaced by perturbed (noisy) data $\kappa^{\epsilon}(t_j)$ as follows:

$$\kappa^{\epsilon}(t_j) = \kappa(t_j) + \epsilon_j, \quad j = 0, 1, \ldots, N, \tag{53}$$

where ϵ_j are random variables with mean zero and standard deviation

$$\sigma = p \max_{t \in [0,T]} \left|\kappa(t)\right|, \tag{54}$$

where p represents the noise.

7 Numerical experiments

The solutions for $q(t)$ and $u(x,t)$ are constructed in this section for the case of noisy (53) and exact data. We use

$$rmse(q) = \left[\frac{T}{N}\sum_{j=1}^{N}\left(q^{numerical}(t_j) - q^{exact}(t_j)\right)^2\right]^{1/2} \tag{55}$$

for measuring the accuracy. Now, we choose $T = 1$, for simplicity. The lower bound for $q(t)$ is taken as -10^2 while 10^2 is used for the upper bound.

Example 1 First, when the problem proposed in equations (1)–(4) is taken with a smooth potential

$$q(t) = -1 - t, \quad 0 \le t \le 1, \tag{56}$$

the analytical solution is

$$u(x,t) = x^7(1-x)^7 e^{-t}, \quad 0 \le x \le 1, 0 \le t \le 1, \tag{57}$$

the BCs are

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad 0 \le t \le 1, \tag{58}$$

the nonlocal integral condition is

$$\kappa(t) = \int_0^1 f(x)u(x,t)\,dx = \frac{e^{-t}}{51{,}480}, \quad 0 \le t \le 1, \tag{59}$$

where $f(x) = 1$, and the rest of the data are as follows:

$$\begin{aligned} &\zeta(x) = u(x,0) = x^7(1-x)^7, \qquad \eta(x) = u_t(x,0) = -x^7(1-x)^7, \qquad \beta = 1, \\ &g(x,t) = -e^{-t}(-1+x)^3x^3\bigl(840 - 9240x + 33{,}306x^2 - 48{,}314x^3 + (24{,}614+t)x^4 \\ &\qquad - 2(277+2t)x^5 + 2(97+3t)x^6 - 4(2+t)x^7 + (2+t)x^8\bigr). \end{aligned} \tag{60}$$

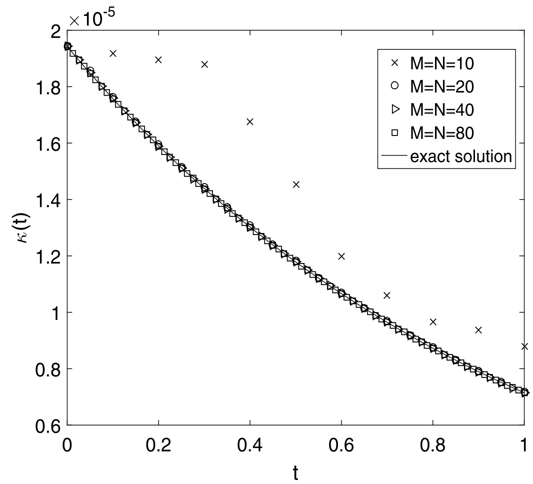

Figure 1 The approximate and exact (59) solutions for $\kappa(t)$ of Example 1

Table 2 The *rmse* error norm for $\kappa(t)$, for the direct problem

$M = N$	10	20	40	80
$rmse(\kappa)$	2.4E–6	3.8E–8	8.6E–9	3.2E–9

It is shown that the criteria of Theorems 1 and 2 are met, indicating that a unique solution is guaranteed. First, when $q(t)$ is supplied by (56), the accuracy of (1)–(3) is tested using the data (57) and (60). The approximate nonlocal integral measurement in (4) is compared to the analytical solution (59) derived using the QBS collocation technique with $M = N \in \{10, 20, 40, 80\}$ in Fig. 1. Figure 2 shows the analytical (57) and estimated $u(x,t)$, as well as absolute errors, for various grid sizes. As the mesh size is reduced, there is a good agreement between the analytical (59) and the estimated $\kappa(t)$, as seen in Table 2.

In the IP (1)–(4), we take the initial guess for $\underline{q}$ as

$$q^0(t_j) = q(0) = -1, \quad j = 1, 2, \ldots, N. \tag{61}$$

When $p = 0$ in (53), we use $M = N = 40$ to begin the analysis for recovering $q(t)$ and $u(x,t)$. The term F in (52) is depicted in Fig. 3(a), where a monotonically decreasing convergence is achieved in 20 iterations for a specified tolerance of $O(10^{-16})$. Figures 3(b) and 3(c) show the exact (56) and approximate $q(t)$ without and with regularization, respectively. These figures show that, for $\lambda = 0$, we get inexact and unstable solutions for $q(t)$ with $rmse(q) = 0.3861$, as predicted from to the ill-posedness issue. As a result, regularization is used to stabilize the answer. It is determined from all chosen λ that $\lambda \in \{10^{-18}, 10^{-17}, 10^{-16}\}$ provides an acceptable and stable accurate estimate for the coefficient $q(t)$, yielding $rmse(q) \in \{0.0437, 0.0394, 0.0560\}$.

Now, as in equation (53), we add $p \in \{0.01\%, 0.1\%\}$ to the nonlocal integral $\kappa(t)$ through (54). In Figs. 4 and 5, the potential $q(t)$ is shown. As noise p is increased, the approximate results start to build up oscillations with $rmse(q) \in \{0.8750, 24.6756\}$, as seen in Figs. 4(a) and 5(a). Figures 4(b) and 5(b) illustrate the reconstructed potential coefficient for a variety of λ, and one can observe that the most accurate solution is achieved for $\lambda \in \{10^{-17}, 10^{-16}\}$,

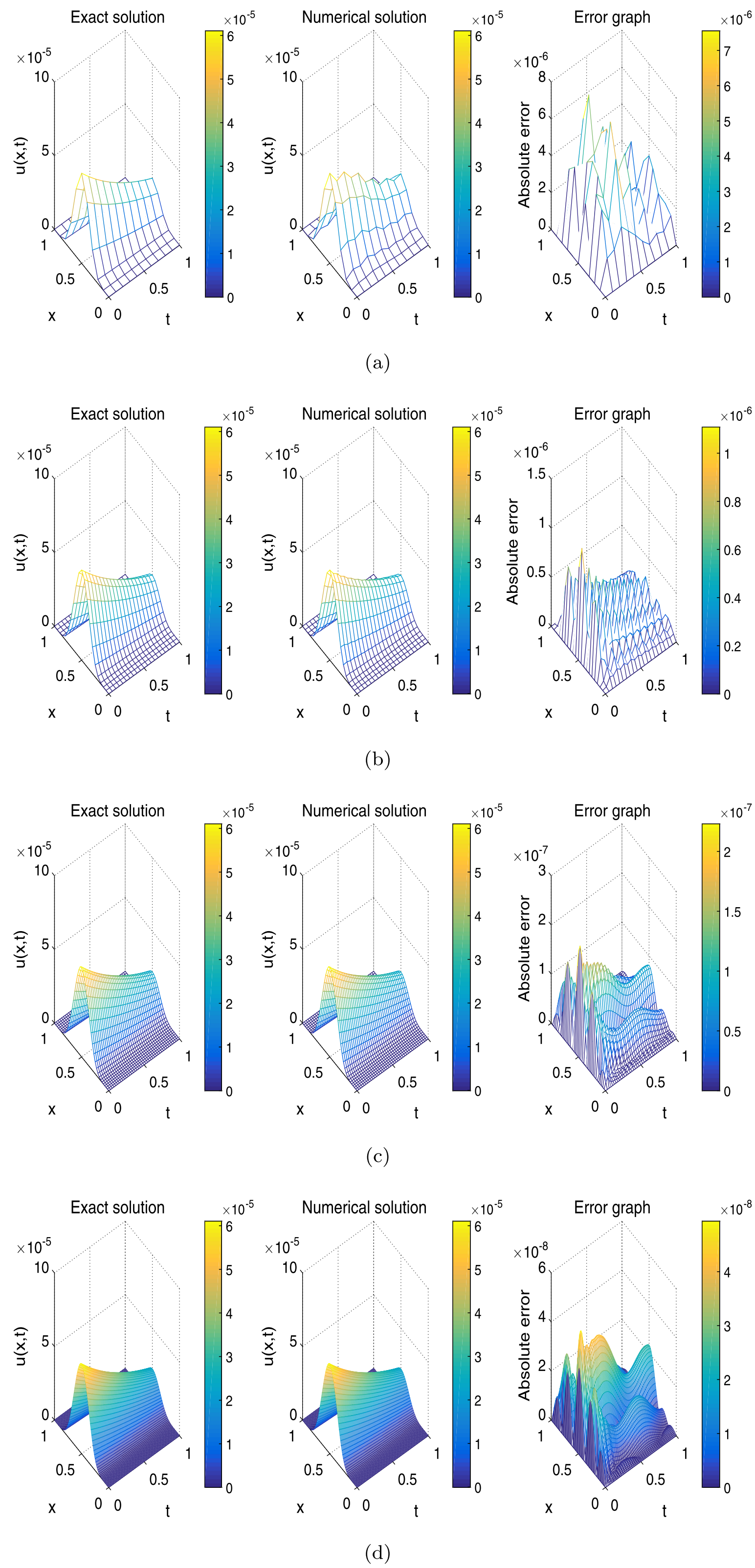

Figure 2 The analytical (57) and approximate $u(x,t)$ with absolute errors for $M = N$: (**a**) 10, (**b**) 20, (**c**) 40, and (**d**) 80, for the direct problem

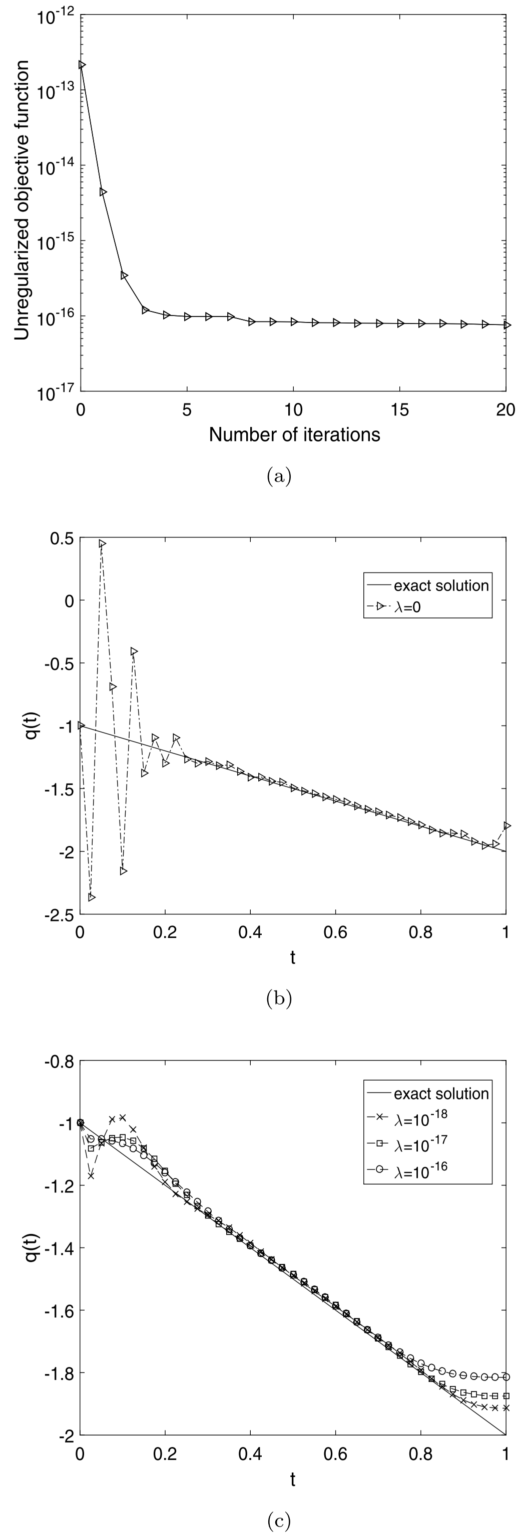

Figure 3 (**a**) The function F in (52) and the exact (56) and approximate $q(t)$, for $p = 0$ with (**b**) $\lambda = 0$ and (**c**) $\lambda = \{10^{-18}, 10^{-17}, 10^{-16}\}$, for Example 1

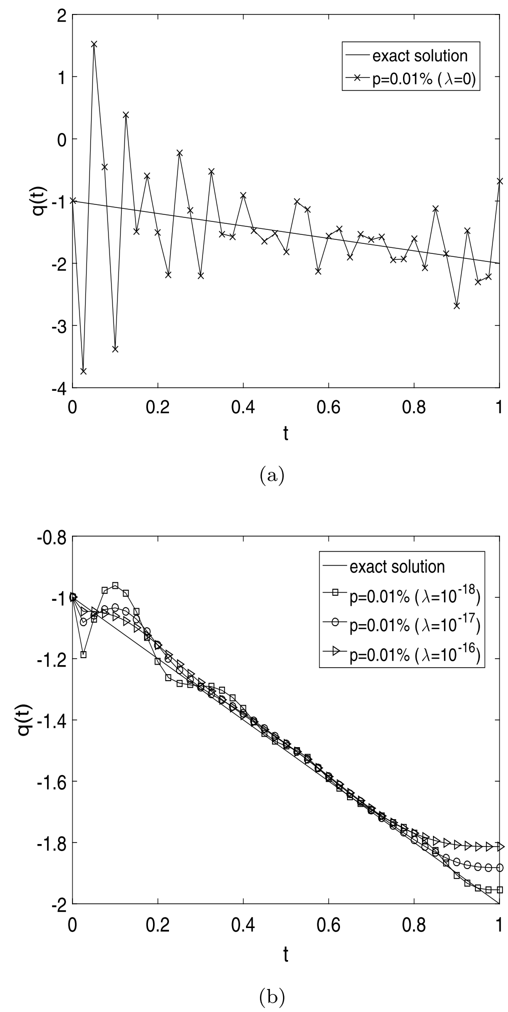

Figure 4 The exact (56) and approximate $q(t)$, for $p = 0.01\%$ with (**a**) $\lambda = 0$ and (**b**) $\lambda = \{10^{-18}, 10^{-17}, 10^{-16}\}$, for Example 1

yielding $rmse(q) \in \{0.0406, 0.0825\}$; see Table 3 for additional details. The absolute error norms between the exact (57) and estimated solutions u are shown in Fig. 6, where the influence of $\lambda > 0$ in minimizing the unstable behavior of the reconstructed u can be seen.

Example 2 Now, we consider the problem proposed in equations (1)–(4), with a nonlinear potential coefficient $q(t)$. Therefore, it is a critical test for the proposed technique of regularization for the governing equation

$$u_{tt} + u_{xxxx} + \beta u_{xx} = q(t)u + g(x,t), \quad 0 \le x \le 1, 0 \le t \le 1, \tag{62}$$

subject to the ICs

$$\begin{aligned} \zeta(x) &= u(x,0) = 7x - 10x^3 + 3x^5, \\ \eta(x) &= u_t(x,0) = -7x + 10x^3 - 3x^5, \quad 0 \le x \le 1, \end{aligned} \tag{63}$$

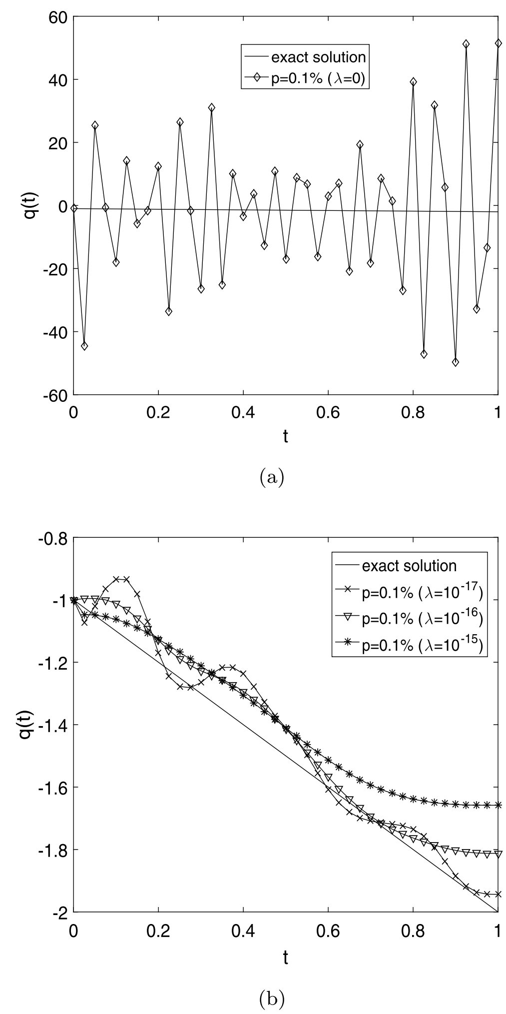

Figure 5 The exact (56) and approximate $q(t)$, for $p = 0.1\%$ with (**a**) $\lambda = 0$ and (**b**) $\lambda = \{10^{-17}, 10^{-16}, 10^{-15}\}$, for Example 1

Table 3 The *rmse* values (55), for $p \in \{0, 0.01\%, 0.1\%\}$ with $\lambda = 0, 10^{-19}, 10^{-18}, 10^{-17}$, and 10^{-16} of Example 1

λ	$p = 0$	$p = 0.01\%$	$p = 0.1\%$
0	0.3861	0.8750	24.6756
10^{-19}	0.0767	0.0853	0.1012
10^{-18}	0.0437	0.0513	0.0733
10^{-17}	0.0394	0.0406	0.0848
10^{-16}	0.0560	0.0578	0.0825

BCs

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad 0 \le t \le 1, \tag{64}$$

and nonlocal integral condition

$$\kappa(t) = \int_0^1 f(x)u(x,t)\,dx = \frac{3e^{-t}}{2}, \quad 0 \le t \le 1, \tag{65}$$

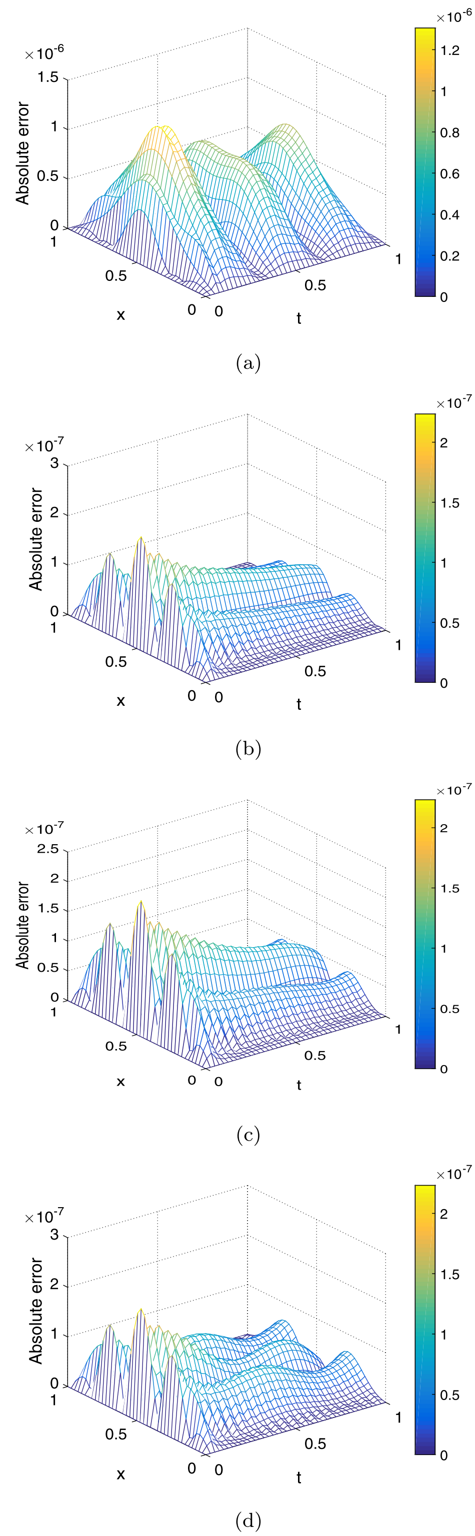

Figure 6 The errors between the exact (57) and approximate $u(x,t)$ with λ being: (**a**) 0, (**b**) 10^{-17}, (**c**) 10^{-16}, and (**d**) 10^{-15}, with $p = 0.1\%$ noise, for Example 1

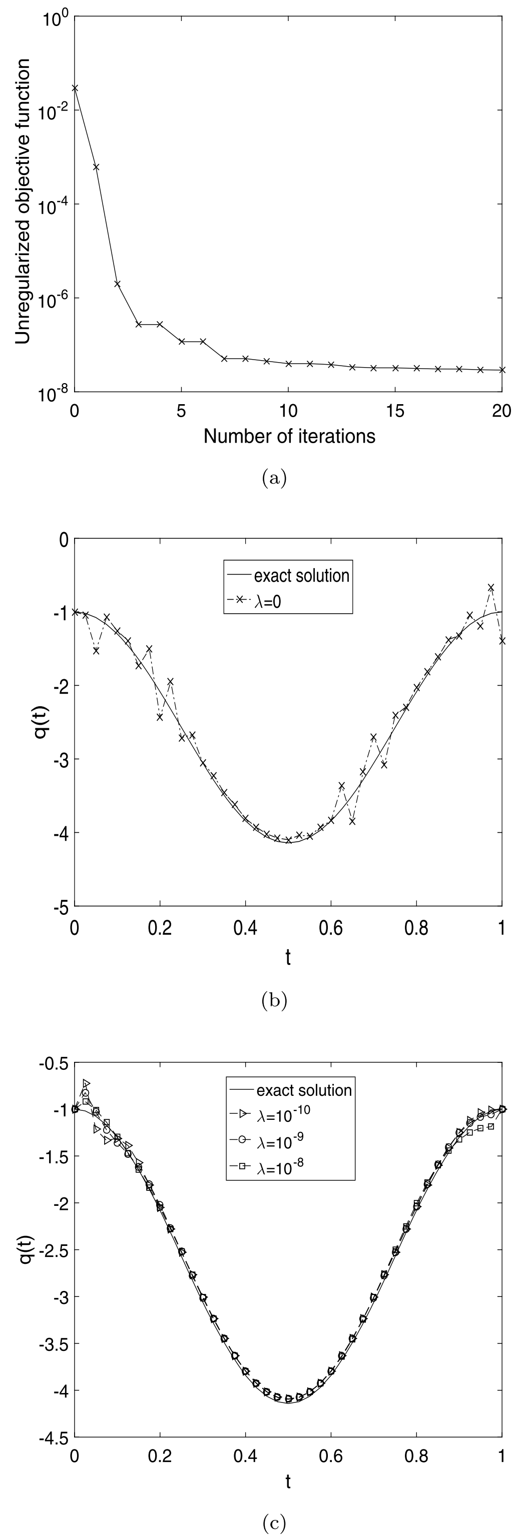

Figure 7 (**a**) The function F in (52) and the exact (68) and approximate $q(t)$, with $p = 0$, and (**b**) without and (**c**) with regularization, for Example 2

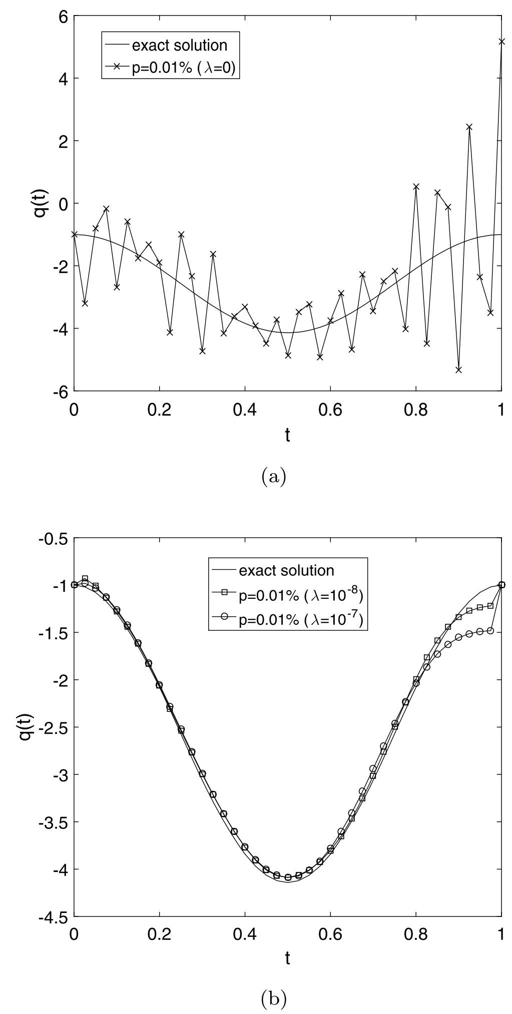

Figure 8 The exact (68) and approximate $q(t)$, for $p = 0.01\%$ with (**a**) $\lambda = 0$ and (**b**) $\lambda = 10^{-8}, 10^{-7}$, for Example 2

where $f(x) = 1$ and

$$g(x,t) = e^{-t}x\big(314 + 40x^2 + 6x^4 + \pi\left(7 - 10x^2 + 3x^4\right)\sin^2(\pi t)\big). \tag{66}$$

The exact solution is

$$u(x,t) = \left(3x^5 - 10x^3 + 7x\right)e^{-t}, \tag{67}$$

$$q(t) = -1 - \pi \sin^2(\pi t). \tag{68}$$

Then, with this input data, the conditions of Theorems 1 and 2 are fulfilled and so the solution is unique. The initial guess for $\underline{q}$ for this example has been chosen as

$$q^0(t_j) = q(0) = -1, \quad j = 1, 2, \ldots, N. \tag{69}$$

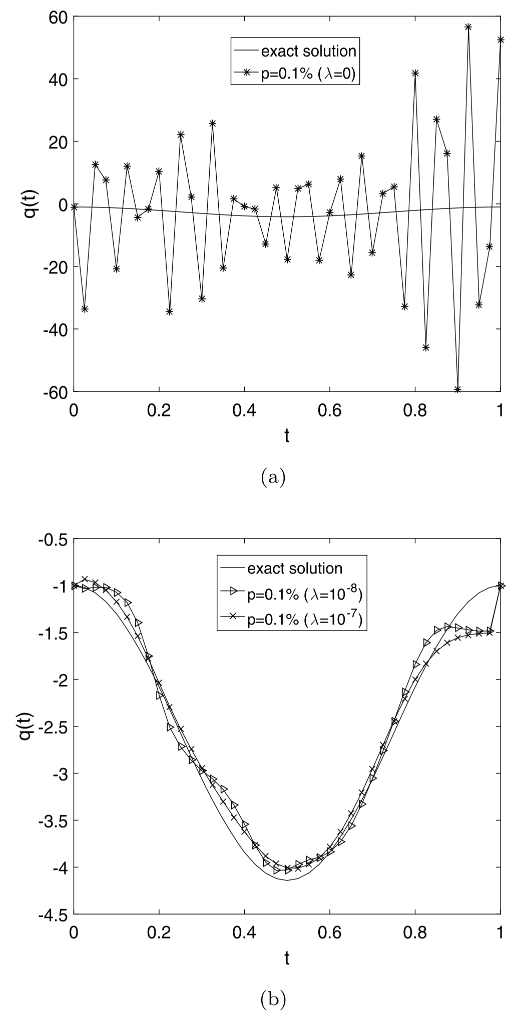

Figure 9 The exact (68) and approximate $q(t)$, for $p = 0.1\%$ with (**a**) $\lambda = 0$ and (**b**) $\lambda = 10^{-8}, 10^{-7}$, for Example 2

As in Example 1, we start with $h = k = 0.025$ and pick the situation where $p = 0$ in the data $\kappa(t)$, as in (54). Figure 7(a) illustrates F from (52), which shows how a monotonically declining convergence is attained in 20 iterations with a specified tolerance of $O(10^{-8})$. Figures 7(b) and 7(c) show the computed potential $q(t)$ without and with regularization, respectively, yielding $rmse(q) \in \{0.1916, 0.0731, 0.0549, 0.0591\}$, as shown in Table 4. Despite the fact that no random noise is simulated using perturbed data (53), numerical noise arises owing to the discrepancy between the QBS collocation solution with a fixed mesh and the actual values of (63)–(68). As a result, regularization is required to reestablish the stability in the $q(t)$ solution. It is determined that $\lambda = 10^{-10}$ produces a more stable potential coefficient solution.

Now we will test the solution's stability using noisy data. We use (53) for $\kappa(t)$ and include $p \in \{0.01\%, 0.1\%\}$ for replicating the input noisy data. In Figs. 8 and 9, the potential $q(t)$ is shown. Figures 8(a) and 9(a) demonstrate the determined potential $q(t)$, where the unstable results are obtained if $\lambda = 0$, with $rmse(q) = 1.7669$ and 24.5476. To stabilize $q(t)$, we use regularization with $\lambda \in \{10^{-9}, 10^{-8}, 10^{-7}\}$ for $p = 0.01\%$ noise, resulting in $rmse(q) \in \{0.0693, 0.0688, 0.1369\}$, and $\lambda \in \{10^{-8}, 10^{-7}, 10^{-6}\}$ for $p = 0.1\%$ noise, resulting

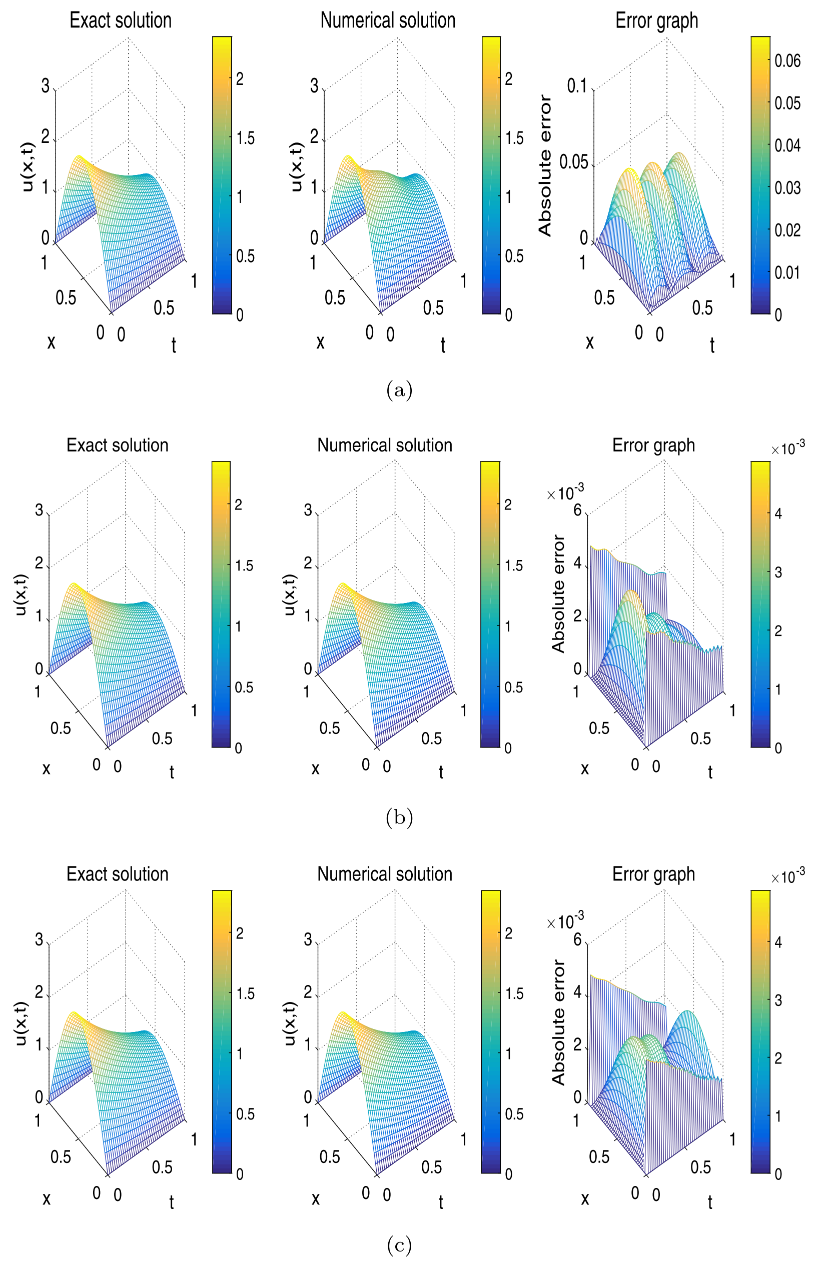

Figure 10 The exact (67) and approximate $u(x,t)$, and absolute errors with $p = 0.1\%$ and λ (**a**) 0, (**b**) 10^{-8}, and (**c**) 10^{-7}, for Example 2

in $rmse(q) \in \{0.1939, 0.1678, 0.3608\}$. Figures 8(b) and 9(b) demonstrate the reconstructed potential for various λ, with the best solution achieved for $\lambda = 10^{-8}$ and 10^{-7}, respectively. The influence of $\lambda > 0$ in lowering the unstable behavior of the reconstructed $u(x,t)$ can be identified in Fig. 10, which shows the precise (67) and approximated $u(x,t)$ with absolute error norms. We refer to Table 4 for further information on the *rmse* values (55) and the minimum value of F (52) at the last iteration. For the stable reconstruction of $q(t)$, identical results may be derived.

Table 4 The *rmse* and the least value of (52) for $p \in \{0, 0.01\%, 0.1\%\}$, with $\lambda = 10^{-10}, 10^{-9}, 10^{-8}, 10^{-7}$, and 10^{-6} at the last iteration for Example 2

p	λ	$rmse(q)$	Minimum values of (52)
0	0	0.1916	2.9E–8
	10^{-10}	0.0731	3.6E–7
	10^{-9}	0.0549	2.2E–6
	10^{-8}	0.0591	1.9E–5
0.01%	0	1.7669	1.9E–7
	10^{-9}	0.0693	3.5E–6
	10^{-8}	0.0688	2.1E–5
	10^{-7}	0.1369	1.8E–4
0.1%	0	24.5476	2.4E–6
	10^{-8}	0.1939	1.5E–4
	10^{-7}	0.1678	3.1E–4
	10^{-6}	0.3608	1.6E–3

8 Conclusions

The reconstruction problem of the potential $q(t)$ along with $u(x,t)$ from the nonlocal integral condition in a higher-order PDE has been solved numerically. The QBS collocation technique has been applied for discretizing the direct problem. The solution has been stabilized using the Tikhonov regularization method. From the obtained results, it has been deduced that stable accurate approximations for $q(t)$ have been obtained for $\lambda \in \{10^{-18}, 10^{-17}, 10^{-16}\}$, when the noise $p = 0$, and for $\lambda \in \{10^{-17}, 10^{-16}\}$, when $p \in \{0.01\%, 0.1\%\}$. For a nonlinear potential coefficient $q(t)$, it has been observed that stable accurate approximations for $q(t)$ have been obtained for $\lambda = 10^{-10}$, when the noise $p = 0$, and for $\lambda \in \{10^{-8}, 10^{-7}\}$, when $p \in \{0.01\%, 0.1\%\}$. The stability has been analyzed, demonstrating that the present technique is unconditionally stable for the discretized system of the higher-order equation of motion of a homogeneous elastic beam.

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References

1. Abbasova, K.E., Mehraliyev, Y.T., Azizbayov, E.I.: Inverse boundary-value problem for linearized equation of motion of a homogeneous elastic beam. Int. J. Appl. Comput. Math. **33**, 157–170 (2020)
2. Caglar, H.N., Caglar, S.H., Twizell, E.H.: The numerical solution of fifth-order boundary value problems with sixth-degree B-spline functions. Appl. Math. Lett. **12**, 25–30 (1999)
3. De Boor, C.: On the convergence of odd-degree spline interpolation. J. Approx. Theory **1**(4), 452–463 (1968)
4. Dhiman, N., Tamsir, M.: Re-modified quintic B-spline collocation method for the solution of Kuramoto–Sivashinsky type equations. Multidiscip. Model. Mater. Struct. (2018). https://doi.org/10.1108/MMMS-06-2018-0111
5. DuChateau, P., Zachmann, D.: Applied Partial Differential Equations. Harper & Row, New York (1989)
6. Gebremedhin, G.S., Jena, S.R.: Approximate solution of ordinary differential equation via hybrid block approach. Int. J. Emerg. Technol. **10**, 201–211 (2019)
7. Gebremedhin, G.S., Jena, S.R.: Approximate solution of a fourth order ordinary differential equation via tenth step block method. Int. J. Comput. Sci. Math. **11**, 253–262 (2020)
8. Hadamard, J.: Lectures on Cauchy's Problem in Linear Partial Differential Equations. Yale University Press, New Haven (2003)
9. Hall, C.A.: On error bounds for spline interpolation. J. Approx. Theory **1**(2), 209–218 (1968)
10. Huntul, M., Tamsir, M.: Identifying an unknown potential term in the fourth-order Boussinesq–Love equation from mass measurement. Eng. Comput. (2021). https://doi.org/10.1108/EC-12-2020-0757
11. Huntul, M.J., Tamisr, M., Ahmadini, A.: An inverse problem of determining the time-dependent potential in the higher-order Boussinesq–Love equation from boundary data. Eng. Comput. (2021). https://doi.org/10.1108/EC-08-2020-0459
12. Huntul, M.J., Tamsir, M., Dhiman, N.: An inverse problem of identifying the time-dependent potential in a fourth-order pseudo-parabolic equation from additional condition. Numer. Methods Partial Differ. Equ. (2021). https://doi.org/10.1002/num.22778
13. Ivanov, V.K.: On linear problems which are not well-posed. Dokl. Akad. Nauk SSSR **145**, 270–272 (1962)
14. Jena, S.R., Gebremedhin, G.S.: Approximate solution of a fifth order ordinary differential equation with block method. Int. J. Comput. Sci. Math. **12**, 413–426 (2020)
15. Jena, S.R., Gebremedhin, G.S.: Numerical treatment of Kuramoto–Sivashinsky equation on B-spline collocation. Arab J. Basic Appl. Sci. **28**, 283–291 (2021)
16. Jena, S.R., Gebremedhin, G.S.: Decatic B-spline collocation scheme for approximate solution of Burgers' equation. Numer. Methods Partial Differ. Equ. (2021). https://doi.org/10.1002/num.22747
17. Jena, S.R., Gebremedhin, G.S.: Computational technique for heat and advection–diffusion equations. Soft Comput. **25**, 1139–1150 (2021)
18. Jena, S.R., Mohanty, M.: Numerical treatment of ODE (fifth order). Int. J. Emerg. Technol. **10**, 191–196 (2019)
19. Jena, S.R., Mohanty, M., Mishra, S.K.: Ninth step block method for numerical solution of a fourth order ordinary differential equation. Adv. Model. Anal. A **55**, 47–56 (2018)
20. Jena, S.R., Nayak, D., Acharya, M.M.: Application of mixed quadrature rule on electromagnetic field problems. Comput. Math. Model. **28**, 267–277 (2017)
21. Jena, S.R., Senapati, A., Gebremedhin, G.S.: Numerical study of solitions in BFRK scheme. Int. J. Mech. Control **21**, 163–175 (2020)
22. Jena, S.R., Senapati, A., Gebremedhin, G.S.: Approximate solution of MRLW equation in B-spline environment. Math. Sci. **14**, 345–357 (2020)
23. Lavrentiev, M.M., Romanov, V.G., Vasiliev, V.G.: Multidimensional Inverse Problems for Differential Equations. Lecture Notes in Mathematics. Springer, Berlin (1970)
24. Mathworks: Documentation optimization toolbox-least squares algorithms, 2019. Available at www.mathworks.com
25. Megraliev, Y.T., Alizade, F.K.: Inverse boundary value problem for a Boussinesq type equation of fourth order with nonlocal time integral conditions of the second kind. Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauk. **26**, 503–514 (2016)
26. Mittal, R.C., Arora, G.: Quintic B-spline collocation method for numerical solution of the Kuramoto–Sivashinsky equation. Commun. Nonlinear Sci. Numer. Simul. **15**, 2798–2808 (2010)
27. Mittal, R.C., Jain, R.K.: B-splines methods with redefined basis functions for solving fourth order parabolic partial differential equations. Appl. Math. Comput. **217**, 9741–9755 (2011)
28. Mohanty, M., Jena, S.R.: Differential transformation method for approximate solution of ordinary differential equation. Adv. Model. Anal. B **61**, 135–138 (2018)
29. Mohanty, M., Jena, S.R., Mishra, S.K.: Mathematical modelling in engineering with integral transforms via modified Adomian decomposition method. Math. Model. Eng. Probl. **8**, 409–417 (2021)
30. Mohanty, M., Jena, S.R., Mishra, S.K.: Approximate solution of fourth order differential equation. Adv. Math. **10**, 621–628 (2021)
31. O'Brien, G.G., Hyman, M.A., Kaplan, S.: A study of the numerical solution of partial differential equations. J. Math. Phys. **29**, 223–251 (1950)
32. Rodriguez, P.: Total variation regularization algorithms for images corrupted with different noise models: a review. J. Electr. Comput. Eng. **2013**, Article ID 217021, 18 pages (2013)
33. Senapati, A., Jena, S.R.: A computational scheme for fifth order boundary value problems. Int. J. Inf. Technol. **14**, 1397–1404 (2022). https://doi.org/10.1007/s41870-022-00871-7
34. Tikhonov, A.N.: On the stability of inverse problems. Dokl. Akad. Nauk SSSR **39**, 195–198 (1943)
35. Vichnevetsky, R.: Stability charts in the numerical approximation of partial differential equations: a review. Math. Comput. Simul. **21**, 170–177 (1979)
36. Wang, Y., Yang, C., Yagola, A.: Optimization and Regularization for Computational Inverse Problems and Applications. Springer, Berlin (2011)
37. Yang, H.: An inverse problem for the sixth-order linear Boussinesq-type equation. UPB Sci. Bull., Ser. A **82**, 27–36 (2020)
38. Yuldashev, T.K.: Inverse boundary-value problem for an integro-differential Boussinesq-type equation with degenerate kernel. J. Math. Sci. **250**, 847–858 (2020)