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Numerical analysis of a linear second-order finite difference scheme for space-fractional Allen–Cahn equations

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Abstract

In this paper, we construct a new linear second-order finite difference scheme with two parameters for space-fractional Allen–Cahn equations. We first prove that the discrete maximum principle holds under reasonable constraints on time step size and coefficient of stabilized term. Secondly, we analyze the maximum-norm error. Thirdly, we can see that the proposed scheme is unconditionally energy-stable by defining the modified energy and selecting the appropriate parameters. Finally, two numerical examples are presented to verify the theoretical results.

Keywords: Space-fractional Allen–Cahn equation; Finite difference method; Maximum principle; Energy stability; Maximum-norm error

1 Introduction

In this paper, we study the finite difference approximations of the following space-fractional Allen–Cahn equation

$$\frac{\partial u}{\partial t} = -\epsilon^2(-\Delta)^{\frac{\alpha}{2}} u - f(u), \quad \mathbf{x} \in \Omega, t \in (0, T], \tag{1.1}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \tag{1.3}$$

where Ω is a bounded regular domain in R^d $(d = 1, 2, 3)$, $\alpha \in (1, 2)$, and $f(u) = u^3 - u$. The fractional Laplacian operator in 1D is defined as

$$-(-\Delta)^{\frac{\alpha}{2}} u = -(-\Delta)_x^{\frac{\alpha}{2}} u := \frac{1}{-2\cos\frac{\pi\alpha}{2}} \left({}_aD_x^\alpha u + {}_xD_b^\alpha u\right),$$

where the left and right Riemann–Liouville fractional derivatives are defined as

$${}_aD_x^\alpha u = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_a^x \frac{u(\xi)}{(x-\xi)^{\alpha-1}} \, d\xi,$$

$$ {}_xD_b^{\alpha}u = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_x^b \frac{u(\xi)}{(\xi-x)^{\alpha-1}}\,d\xi. $$

The fractional Laplacian operators in 2D and 3D can be defined similarly. For example, the 3D operator is defined as

$$ -(-\Delta)^{\frac{\alpha}{2}}u(x,y,z) = \bigl\{\bigl[-(-\Delta)_x^{\frac{\alpha}{2}}\bigr] + \bigl[-(-\Delta)_y^{\frac{\alpha}{2}}\bigr] + \bigl[-(-\Delta)_z^{\frac{\alpha}{2}}\bigr]\bigr\}u(x,y,z). $$

Fractional models are an important tool to describe the various complex mechanical and physical phenomena. They can also describe the subdiffusion and superdiffusion processes. The analytical solutions for most of fractional differential equations are impossible to obtain. Therefore, the numerical solution techniques have attracted much attention; see, e.g., [8, 15, 27]. Recently, researchers pay more attention to the front propagation of reaction-diffusion systems with an anomalous diffusion as super diffusion, i.e., the fractional Allen–Cahn equation. For space-fractional Allen–Cahn equations, Hou et al. [16] considered second-order Crank–Nicolson finite difference scheme and discussed the discrete maximum principle and the nonlinear energy stability. Based on the convex splitting in time and the Fourier spectral method in space, Bu et al. [2] proposed stable second-order numerical schemes for the fractional Cahn–Hilliard and Allen–Cahn equations. Meanwhile, the unique solvability and energy stability of the numerical schemes were proved. The numerical methods for time-fractional Allen–Cahn equations were also studied in [3, 18, 22]. The nonlocal Allen–Cahn equation is similar to the space-fractional Allen–Cahn equation. A detailed convergence analysis for nonlocal Allen–Cahn and nonlocal Cahn–Hilliard equations were provided in [12, 21]. Du et al. [5] proposed two energy-stable linear semi-implicit methods for solving the nonlocal Cahn–Hilliard equation and established the energy stabilities. Guan et al. [14] devised a convex splitting scheme for periodic nonlocal Cahn–Hilliard and established the unconditional unique solvability, energy stability, and stability of the scheme. Guan et al. [13] devised a convex splitting scheme for the nonlocal Cahn–Hilliard and nonlocal Allen–Cahn equations.

Most finite difference approximations in the above literature are based on the second-order central difference. There also exist a lot of works on the fourth-order difference approximation of various nonlinear partial differential equations. For incompressible Boussinesq equations, Liu et al. [23] presented a fourth-order finite difference method that is especially suitable for moderate to large Reynolds number flows. Wang et al. [28] established the convergence of a fourth-order finite difference method and provided theoretical results on the stability and accuracy of the method. Cheng et al. [4] proposed a fourth-order finite difference scheme for the Cahn–Hilliard equation. They established the unique solvability, energy stability, and an optimal a priori error estimates in the $\ell^\infty(0,T;\ell^2)\cap\ell^2(0,T;H_h^2)$ norm. Samelson et al. [24] proposed and analyzed a fourth-order finite difference numerical method for the planetary geostrophic equations with inviscid balance equation that are reformulated in an alternate form.

The Allen–Cahn equation was first introduced in 1979 [1]. It can be used to describe the interface evolving of the phase separation process of the crystalline solids. As the equation is nonlinear, many research works were devoted to the numerical solution of the Allen-Cahn equation; see, e.g., [10, 25, 29]. The intrinsic properties of the Allen–Cahn equation is the energy dissipation law and the maximum bound principle(MBP). Therefore, numerical schemes preserve the energy dissipation law and the maximum bound principle attracted the attention of many scholars. Hou et al. [17] constructed a new second-order maximum-principle preserving finite difference scheme for Allen–Cahn equations with periodic boundary conditions. Many classic schemes in the existing literature can be given by this scheme. The proposed scheme is unconditionally energy-stable by choosing proper parameters. Shen and Zhang [26] considered a high-order finite difference scheme for a generalized Allen–Cahn equation coupled with a passive convection for a given incompressible velocity field. They proved that the discrete maximum principle holds under suitable mesh size and time step constraints. Feng et al. [9] constructed a linear second-order finite difference scheme based on the Leap-Frog scheme. The proposed scheme is MBP-preserving and unconditionally energy-stable. Du et al. [6] analyzed first- and second-order exponential time-differencing schemes for solving the nonlocal Allen–Cahn equation, which preserve the discrete maximum principle unconditionally. Du et al. [7] first provided a framework of the Allen–Cahn-type equations satisfying the MBP and studied the MBP-preserving exponential time-differencing (ETD) schemes. Using the exponential integrator method, a fourth-order conditionally MBP-preserving scheme [19] and a third-order unconditionally MBP-preserving scheme [20] for Allen–Cahn euqations were proposed. Feng et al. [11] presented linear second-order stabilized Crank–Nicolson/Adams–Bashforth schemes for the Allen–Cahn and Cahn–Hilliard equations. It is shown that the proposed time discretization schemes are either unconditionally energy stable or conditionally energy stable under some reasonable stability conditions.

The goal of this paper is to construct a linear second-order three-level finite difference scheme with two stabilized terms for the space-fractional Allen–Cahn equation. We first discuss the discrete maximum principle and then analyze the maximum-norm error. We find that our scheme is unconditionally energy-stable by defining the modified energy and selecting the appropriate parameters.

The rest of the paper is introduced as follows. In Sect. 2, we present the finite difference scheme for the problem (1.1)–(1.3). In Sects. 3–5, we analyze the discrete maximum principle, the discrete energy stability, and the error estimate. In the last section, we give two numerical examples to verify the theoretical results.

2 Fully discretized scheme

We will adopt the finite difference approach in [27] to discretize the fractional Laplacian operator $-(-\Delta)^{\frac{\alpha}{2}}$. To begin with, we denote D_h as the discretization matrix of the fractional Laplacian operator. In particular, the discretization matrix of ${}_aD_x^\alpha$ with homogeneous Dirichlet boundary conditions on interval $[0,L]$ in 1D is given by

$$A = \frac{1}{h^\alpha}\begin{bmatrix} \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & & & \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & & \\ \vdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \ddots & \\ \omega_{N-1}^{(\alpha)} & \cdots & \ddots & \ddots & \omega_0^{(\alpha)} \\ \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \cdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} \end{bmatrix}_{N\times N} =: \frac{1}{h^\alpha}M,$$

where

$$\omega_0^{(\alpha)} = \frac{\alpha}{2}g_0^{(\alpha)}, \qquad \omega_k^{(\alpha)} = \frac{\alpha}{2}g_k^{(\alpha)} + \frac{2-\alpha}{2}g_{k-1}^{(\alpha)}, \quad k \ge 1, \tag{2.1}$$

with

$$g_0^{(\alpha)} = 1, \qquad g_k^{(\alpha)} = \left(1 - \frac{1+\alpha}{k}\right) g_{k-1}^{(\alpha)}, \quad k = 1, 2, \ldots. \tag{2.2}$$

Note that the discretization matrix of ${}_xD_b^{\alpha}$ is A^T. Defining

$$D = M + M^T \tag{2.3}$$

produces the discretization matrix of the fractional Laplacian operator in 1D

$$D_h^{(1)} = \frac{1}{-2h^{\alpha}\cos\frac{\pi\alpha}{2}} D. \tag{2.4}$$

Using the Kronecker tensor product notation, we can obtain the corresponding discretization matrix in 2D and 3D

$$D_h^{(2)} = \frac{1}{-2h^{\alpha}\cos\frac{\pi\alpha}{2}} (I \otimes D + D \otimes I), \tag{2.5}$$

$$D_h^{(3)} = \frac{1}{-2h^{\alpha}\cos\frac{\pi\alpha}{2}} (I \otimes I \otimes D + I \otimes D \otimes I + D \otimes I \otimes I), \tag{2.6}$$

where I is the $N \times N$ identity matrix.

Now, we present our numerical scheme for solving the problem (1.1)–(1.3). The second-order three-level linear difference scheme with two stabilized terms is given as follow

$$\begin{aligned} &\frac{U^{n+1} - U^{n-1}}{2\tau} + \left(U^n\right)^{.3} - \frac{U^{n+1} + U^{n-1}}{2} + \gamma\left(U^n\right)^{.2}\left(U^{n+1} + U^{n-1} - 2U^n\right) \\ &\quad + \beta\left(U^{n+1} + U^{n-1} - 2U^n\right) = \frac{\epsilon^2 D_h (U^{n+1} + U^{n-1})}{2}, \end{aligned} \tag{2.7}$$

where τ denotes the time stepsize, U^n represents the vector of numerical solution, and

$$\begin{aligned} &\left(U^n\right)^{.3} := \left(\left(U_1^n\right)^3, \left(U_2^n\right)^3, \ldots, \left(U_N^n\right)^3\right)^T, \\ &\left(U^n\right)^{.2} V^n := \left(\left(U_1^n\right)^2 V_1^n, \left(U_2^n\right)^2 V_2^n, \ldots, \left(U_N^n\right)^2 V_N^n\right)^T. \end{aligned}$$

For the first step, we use the standard Crank–Nicolson scheme

$$\frac{U^1 - U^0}{\tau} + \frac{(U^1)^{.3} - U^1}{2} + \frac{(U^0)^{.3} - U^0}{2} = \frac{\epsilon^2 D_h (U^1 + U^0)}{2}. \tag{2.8}$$

From [16], we have the following lemma.

Lemma 1 *If $D_h^{(d)}$, $d = 1, 2, 3$, is the discrete matrix defined in (2.4)–(2.6). Then $D_h = D_h^{(d)}$ satisfies the following properties*:

- *D_h is symmetric*;
- *D_h is negative definite, i.e., $U^T D_h U < 0$, for any $U \neq 0$, $U \in \mathbf{R}^N$*;
- *The elements of $D_h = (b_{ij})$ satisfy*:

$$b_{ii} = -b < 0 \quad \textit{and} \quad b \geq \max_i \sum_{j \neq i} |b_{ij}|. \tag{2.9}$$

3 Discrete maximum principle

In this section, we will show that the scheme (2.7) preserves the discrete maximum principle.

Theorem 1 *Assume the initial value satisfies* $\max_{\mathbf{x}\in\bar{\Omega}} |u_0(\mathbf{x})| \le 1$. *There exist* $\delta > 0$, *the fully discrete scheme* (2.7) *preserves the maximum principle in the sense that* $\|U^n\|_\infty \le 1$ *for all* $n \ge 1$ *provided that the time stepsize satisfies*

$$
\begin{aligned}
&0 < \tau \le \frac{1}{(2+\delta)\beta - 1}, \qquad \gamma \le 0, \qquad \beta \ge \frac{3}{2} - 3\gamma, \qquad \delta \ge \frac{2d\epsilon^2}{\beta h^\alpha},\\
&0 < \tau \le \frac{1}{(2+\delta)\beta + 2\gamma - 1}, \qquad 0 < \gamma \le \frac{1}{2}, \qquad \beta \ge \frac{3}{2} - \gamma, \qquad \delta \ge \frac{2d\epsilon^2}{\beta h^\alpha},\\
&0 < \tau \le \frac{1}{(2+\delta)\beta + 2\gamma - 1}, \qquad \gamma > \frac{1}{2}, \qquad \beta \ge 2\gamma, \qquad \delta \ge \frac{2d\epsilon^2}{\beta h^\alpha},
\end{aligned}
$$

where d *is the dimension number.*

Proof First, it follows from the assumption on u_0 that $\|U^0\|_\infty \le 1$. Then, as in Theorem 1 in [16], when the time stepsize τ satisfies $0 < \tau \le \min\{\frac{1}{2}, \frac{h^\alpha}{2d\epsilon^2}\}$, we have $\|U^1\|_\infty \le 1$. We will prove our theorem by induction. We now assume that the result holds for $n = m - 1$ and $n = m$, i.e., $\|U^{m-1}\|_\infty \le 1$ and $\|U^m\|_\infty \le 1$. Below, we will check that this upper bound is also true for $n = m + 1$. Next, we divide the proof into three cases:

Case I: $\gamma \le 0$

It follows from the scheme (2.7) that

$$
\begin{aligned}
&(1-\tau)U^{m+1} + 2\tau\beta U^{m+1} + 2\tau\gamma\left(U^m\right)^{\cdot 2}U^{m+1} - \tau\epsilon^2 D_h U^{m+1}\\
&\quad = \left(1 + \tau - (2+\delta)\tau\beta\right)U^{m-1} - 2\tau\gamma\left(U^m\right)^{\cdot 2}U^{m-1} + \left(\delta\tau\beta I + \tau\epsilon^2 D_h\right)U^{m-1}\\
&\qquad + \left(4\tau\beta + 3(4\gamma-2)\tau\right)U^m + \tau(4\gamma-2)\left(\left(U^m\right)^{\cdot 3} - 3U^m\right).
\end{aligned}
\tag{3.1}
$$

Suppose $\|U^{m+1}\|_\infty = U_p^{m+1}$. The pth component of (3.1) is

$$
\begin{aligned}
&(1-\tau+2\tau\beta)U_p^{m+1} + 2\gamma\tau\left(U_p^m\right)^2U_p^{m+1} - \tau\epsilon^2\left(\sum_{j=1}^N b_{pj}U_j^{m+1}\right)\\
&\quad = \left(1+\tau-(2+\delta)\tau\beta\right)U_p^{m-1} - 2\tau\gamma\left(U_p^m\right)^2U_p^{m-1} + \delta\tau\beta U_p^{m-1} + \tau\epsilon^2\left(\sum_{j=1}^N b_{pj}U_j^{m-1}\right)\\
&\qquad + \left(4\tau\beta + 3(4\gamma-2)\tau\right)U_p^m + \tau(4\gamma-2)\left(\left(U_p^m\right)^3 - 3U_p^m\right).
\end{aligned}
\tag{3.2}
$$

If $\beta \ge \frac{1}{2} - \gamma$, we deduce that $(1-\tau+2\tau\beta)U_p^{m+1} + 2\gamma\tau(U_p^m)^2U_p^{m+1}$ and $-\tau\epsilon^2(\sum_{j=1}^N b_{pj}U_j^{m+1})$ are non-positive or non-negative simultaneously. Then, we notice that

$$
\begin{aligned}
&\left|(1-\tau+2\tau\beta)U_p^{m+1} + 2\gamma\tau\left(U_p^m\right)^2U_p^{m+1} - \tau\epsilon^2\left(\sum_{j=1}^N b_{pj}U_j^{m+1}\right)\right|\\
&\quad \ge (1-\tau+2\tau\beta)\left|U_p^{m+1}\right| + 2\gamma\tau\left(U_p^m\right)^2\left|U_p^{m+1}\right|.
\end{aligned}
\tag{3.3}
$$

Taking the absolute value of (3.2) and using (3.3), we see that

$$\begin{aligned}&(1-\tau+2\tau\beta)\left|U_p^{m+1}\right|+2\gamma\tau\left(U_p^m\right)^2\left|U_p^{m+1}\right|\\&\quad\leq\left|\left(1+\tau-(2+\delta)\tau\beta\right)U_p^{m-1}-2\tau\gamma\left(U_p^m\right)^2U_p^{m-1}\right|+\left|\delta\tau\beta U_p^{m-1}+\tau\epsilon^2\left(\sum_{j=1}^N b_{pj}U_j^{m-1}\right)\right|\\&\qquad+\left|\left(4\tau\beta+3(4\gamma-2)\tau\right)U_p^m\right|+\left|\tau(4\gamma-2)\left(\left(U_p^m\right)^3-3U_p^m\right)\right|.\end{aligned}\tag{3.4}$$

If $\tau\leq\frac{1}{(2+\delta)\beta-1}$ and $\beta\geq\frac{1}{2+\delta}$, using $|U_p^{m-1}|\leq\|U^{m-1}\|_\infty\leq1$, we know that

$$\left|\left(1+\tau-(2+\delta)\tau\beta\right)U_p^{m-1}-2\tau\gamma\left(U_p^m\right)^2U_p^{m-1}\right|\leq1+\tau-(2+\delta)\tau\beta-2\tau\gamma\left(U_p^m\right)^2.\tag{3.5}$$

Let $H=\delta\tau\beta I+\tau\epsilon^2D_h$. If $\delta\geq\frac{2d\epsilon^2}{\beta h^\alpha}$, then we know from Theorem 1 in [16] that

$$\|H\|_\infty\leq\delta\tau\beta.\tag{3.6}$$

Consequently, using (3.6) and $\|U^{m-1}\|_\infty\leq1$, we can obtain

$$\left|\delta\tau\beta U_p^{m-1}+\tau\epsilon^2\left(\sum_{j=1}^N b_{pj}U_j^{m-1}\right)\right|\leq\|H\|_\infty\left\|U^{m-1}\right\|_\infty\leq\delta\tau\beta.\tag{3.7}$$

Then, using $|U_p^m|\leq\|U^m\|_\infty\leq1$, if $\beta\geq\frac{3}{2}-3\gamma$, we know that

$$\left|\left(4\tau\beta+3(4\gamma-2)\tau\right)U_p^m\right|\leq4\tau\beta+3(4\gamma-2)\tau.\tag{3.8}$$

Let $g(x)=x^3-3x$. It is easy to see that $|g(x)|\leq2$ for $|x|\leq1$. Since $\gamma\leq0$, we deduce that

$$\left|\tau(4\gamma-2)\left(\left(U_p^m\right)^3-3U_p^m\right)\right|\leq4\tau-8\tau\gamma.\tag{3.9}$$

It follows from (3.4)–(3.9) that

$$\left(1-\tau+2\tau\beta+2\gamma\tau\left(U_p^m\right)^2\right)\left|U_p^{m+1}\right|\leq1-\tau+2\tau\beta-2\tau\gamma\left(U_p^m\right)^2+4\tau\gamma,\tag{3.10}$$

namely,

$$\left|U_p^{m+1}\right|\leq1+\frac{4\tau\gamma(1-(U_p^m)^2)}{1-\tau+2\tau\beta+2\gamma\tau(U_p^m)^2}.\tag{3.11}$$

Since $\gamma\leq0$ and $|U_p^m|\leq\|U^m\|_\infty\leq1$, we can get $|U_p^{m+1}|=\|U^{m+1}\|_\infty\leq1$.

Case II: $0<\gamma\leq\frac{1}{2}$

The scheme is the same as (3.1). If $\tau\leq\frac{1}{(2+\delta)\beta+2\gamma-1}$ and $\beta\geq\frac{1}{2}$, we know that

$$\left|\left(1+\tau-(2+\delta)\tau\beta\right)U_p^{m-1}-2\tau\gamma\left(U_p^m\right)^2U_p^{m-1}\right|\leq1+\tau-(2+\delta)\tau\beta-2\tau\gamma\left(U_p^m\right)^2.\tag{3.12}$$

Then, we reestimate (3.8) as

$$\left|\left(4\tau\beta+3(4\gamma-2)\tau\right)U_p^m\right|\leq(4\tau\beta+12\tau\gamma-6\tau)\left|U_p^m\right|,\quad\beta\geq\frac{3}{2}-3\gamma.\tag{3.13}$$

Combining (3.4), (3.7), (3.9), (3.12) with (3.13) yields

$$\begin{aligned}(1-\tau+2\tau\beta)\left|U_p^{m+1}\right|+2\gamma\tau\left(U_p^m\right)^2\left|U_p^{m+1}\right| &\le 1+5\tau-2\tau\beta-2\tau\gamma\left(U_p^m\right)^2-8\tau\gamma\\ &\quad+(4\tau\beta+12\tau\gamma-6\tau)\left|U_p^m\right|.\end{aligned}$$

Assume that $\|U^{m+1}\|_\infty > 1$, then we have

$$4\tau\gamma\left|U_p^m\right|^2-(4\tau\beta+12\tau\gamma-6\tau)\left|U_p^m\right|-6\tau+4\tau\beta+8\tau\gamma<0.$$

Let

$$h(x)=4\tau\gamma x^2-(4\tau\beta+12\tau\gamma-6\tau)x-6\tau+4\tau\beta+8\tau\gamma.$$

It is easy to see that $h(1)=0$. And if $\beta\ge\frac{3}{2}-\gamma$, we can get $\frac{4\tau\beta+12\tau\gamma-6\tau}{8\tau\gamma}\ge 1$, which contradicts $\|U^m\|_\infty\le 1$. Thus, we have $\|U^{m+1}\|_\infty\le 1$.

Case III: $\gamma>\frac{1}{2}$

We rewrite (3.1) as

$$\begin{aligned}&(1-\tau)U^{m+1}+2\tau\beta U^{m+1}+2\tau\gamma\left(U^m\right)^{.2}U^{m+1}-\tau\epsilon^2D_hU^{m+1}\\ &\quad=\left(1+\tau-(2+\delta)\tau\beta\right)U^{m-1}-2\tau\gamma\left(U^m\right)^{.2}U^{m-1}+\left(\delta\tau\beta I+\tau\epsilon^2D_h\right)U^{m-1}\\ &\qquad+4\tau\beta U^m+\tau(4\gamma-2)\left(U^m\right)^{.3}.\end{aligned}$$

Using the same technique as in Case I, if $\beta\ge\frac{1}{2}$, $\delta\ge\frac{2d\epsilon^2}{\beta h^\alpha}$ and $\tau\le\frac{1}{(2+\delta)\beta+2\gamma-1}$, we have

$$\begin{aligned}&(1-\tau+2\tau\beta)\left|U_p^{m+1}\right|+2\gamma\tau\left(U_p^m\right)^2\left|U_p^{m+1}\right|\\ &\quad\le 1+\tau-2\tau\beta-2\tau\gamma\left(U_p^m\right)^2+\left|4\tau\beta U_p^m+\tau(4\gamma-2)\left(U_p^m\right)^3\right|.\end{aligned}\tag{3.14}$$

Since $\beta\ge\frac{1}{2}$ and $\gamma>\frac{1}{2}$, using $|U_p^m|\le\|U^m\|_\infty\le 1$, it is easy to obtain that

$$\left|4\tau\beta U_p^m+\tau(4\gamma-2)\left(U_p^m\right)^3\right|\le 4\tau\beta\left|U_p^m\right|+\tau(4\gamma-2).\tag{3.15}$$

Following (3.14) and (3.15) immediately yields

$$\begin{aligned}&(1-\tau+2\tau\beta)\left|U_p^{m+1}\right|+2\gamma\tau\left(U_p^m\right)^2\left|U_p^{m+1}\right|\\ &\quad\le 1-\tau-2\tau\beta+4\tau\gamma-2\tau\gamma\left(U_p^m\right)^2+4\tau\beta\left|U_p^m\right|.\end{aligned}\tag{3.16}$$

Assume that $\|U^{m+1}\|_\infty>1$, then (3.16) becomes

$$\gamma\left|U_p^m\right|^2-\beta\left|U_p^m\right|+\beta-\gamma<0.$$

Let $z(x)=\gamma x^2-\beta x+\beta-\gamma$. If $\beta\ge 2\gamma$, we have $z(x)\ge 0$, which contradicts $\|U^m\|_\infty\le 1$. Thus, we have $\|U^{m+1}\|_\infty\le 1$.

This completes the proof of Theorem 1. □

4 Discrete energy stability

In this section, we will discuss the discrete energy stability. We define the modified discrete energy:

$$\begin{aligned} E_h(U^n) &= \frac{1}{4}\sum_{i=1}^{N}\big(1-(U_i^n)^2\big)\big(1-(U_i^{n-1})^2\big) - \frac{\epsilon^2}{4}\big((U^n)^T D_h U^n + (U^{n-1})^T D_h U^{n-1}\big) \\ &\quad + \frac{\beta}{2}\sum_{i=1}^{N}\big(U_i^n - U_i^{n-1}\big)^2, \end{aligned}$$

where D_h is given by (2.4)–(2.6) for one to three space dimensions, respectively.

Theorem 2 *The scheme* (2.7) *with* $\gamma = \frac{1}{2}$ *is unconditionally energy stable, namely,*

$$E_h(U^{n+1}) \le E_h(U^n), \quad n = 1, 2, \ldots.$$

Proof Choosing $\gamma = \frac{1}{2}$ in (2.7), then the scheme becomes

$$\begin{aligned} &\frac{U^{n+1}-U^{n-1}}{2\tau} + \frac{(U^n)^{.2}U^{n+1} + (U^n)^{.2}U^{n-1} - U^{n+1} - U^{n-1}}{2} \\ &\quad + \beta\big(U^{n+1} + U^{n-1} - 2U^n\big) = \frac{\epsilon^2 D_h(U^{n+1}+U^{n-1})}{2}. \end{aligned} \tag{4.1}$$

Taking L^2 inner product of (4.1) with $(U^{n+1}-U^{n-1})^T$, we have

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{N}\Big(\big[(U_i^n)^2-1\big]\big[(U_i^{n+1})^2-(U_i^{n-1})^2\big] + \frac{1}{\tau}\big(U_i^{n+1}-U_i^{n-1}\big)^2\Big) \\ &\quad + \beta\sum_{i=1}^{N}\big(U_i^{n+1}+U_i^{n-1}-2U_i^n\big)\big(U_i^{n+1}-U_i^{n-1}\big) - \frac{\epsilon^2}{2}\big(U^{n+1}-U^{n-1}\big)^T D_h\big(U^{n+1}+U^{n-1}\big) \\ &\quad = 0. \end{aligned} \tag{4.2}$$

From Lemma 1, we know that

$$\big(U^{n+1}-U^{n-1}\big)^T D_h\big(U^{n+1}+U^{n-1}\big) = \big(U^{n+1}\big)^T D_h U^{n+1} - \big(U^{n-1}\big)^T D_h U^{n-1}. \tag{4.3}$$

Thus, it follows from (4.2)–(4.3) that

$$\begin{aligned} &E_h(U^{n+1}) - E_h(U^n) \\ &\quad = \frac{1}{4}\sum_{i=1}^{N}\big[1-(U_i^n)^2\big]\big[(U_i^{n-1})^2-(U_i^{n+1})^2\big] - \frac{\epsilon^2}{4}\big[(U^{n+1})^T D_h U^{n+1} - (U^{n-1})^T D_h U^{n-1}\big] \\ &\quad\quad + \frac{\beta}{2}\sum_{i=1}^{N}\big[(U_i^{n+1}-U_i^n)^2 - (U_i^n-U_i^{n-1})^2\big] \\ &\quad = -\frac{1}{4\tau}\sum_{i=1}^{N}\big(U_i^{n+1}-U_i^{n-1}\big)^2, \end{aligned}$$

this completes the proof of the theorem. □

5 Maximum-norm error estimate

In this section, we analyze the maximum-norm error estimate for the fully discrete scheme (2.7) based on Theorem 1. Let $C(\epsilon,\gamma,\beta,T)$ be a constant, which depends on ϵ, γ, β, T and regularity of exact solution but is independent of h and τ. Similarly, we can define the $C(\epsilon)$, $C(\epsilon,T)$ and $C(\epsilon,\gamma,\beta)$.

Theorem 3 *Let u be the exact solution of* (1.1) *and U^n be the solution of* (2.7), *respectively. Assume that all the conditions in Theorem* 1 *are valid, then we have*

$$\left\|\boldsymbol{u}^n - U^n\right\|_\infty \le C(\epsilon,\gamma,\beta,T)\left(\tau^2+h^2\right), \quad n = 2,3,\dots,T/\tau, \tag{5.1}$$

where $\boldsymbol{u}^n = (u_1^n, u_2^n, \dots, u_N^n)^T$.

Proof We discretize (1.1) in space and time, respectively, to get

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n-1}}{2\tau} + \left(\boldsymbol{u}^n\right)^{.3} - \frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^{n-1}}{2} = \frac{\epsilon^2 D_h(\boldsymbol{u}^{n+1}+\boldsymbol{u}^{n-1})}{2} + \boldsymbol{\rho}^n, \tag{5.2}$$

where

$$\left\|\boldsymbol{\rho}^n\right\|_\infty \le C(\epsilon)\left(\tau^2+h^2\right).$$

Letting $e^n = \boldsymbol{u}^n - U^n$ and subtracting (2.7) from (5.2), we obtain that

$$\begin{aligned}
&(1-\tau)e^{n+1} + 2\tau\beta e^{n+1} + 2\tau\gamma\left(U^n\right)^{.2}e^{n+1} - \tau\epsilon^2 D_h e^{n+1} \\
&\quad = \left(\frac{1}{2}+\tau\right)e^{n-1} - \left(2\tau\beta e^{n-1} + 2\tau\gamma\left(U^n\right)^{.2}e^{n-1}\right) + \left(\frac{1}{2}I + \tau\epsilon^2 D_h\right)e^{n-1} \\
&\qquad + \left(4\tau\beta e^n + 4\tau\gamma\left(U^n\right)^{.2}e^n\right) - 2\tau\left(\left(\boldsymbol{u}^n\right)^{.3} - \left(U^n\right)^{.3}\right) + 2\tau\beta\left(\boldsymbol{u}^{n+1}+\boldsymbol{u}^{n-1}-2\boldsymbol{u}^n\right) \\
&\qquad + 2\tau\gamma\left(\left(U^n\right)^{.2}\left(\boldsymbol{u}^{n+1}+\boldsymbol{u}^{n-1}-2\boldsymbol{u}^n\right)\right) - 2\tau\boldsymbol{\rho}^n =: \sum_{i=1}^{8} Q_i.
\end{aligned} \tag{5.3}$$

If $\tau \le 1$ and $\beta > \max\{0,-\gamma\}$, similar to (3.3), we estimate the left-hand of (5.3) as

$$\left\|(1-\tau)e^{n+1} + 2\tau\beta e^{n+1} + 2\tau\gamma\left(U^n\right)^{.2}e^{n+1} - \tau\epsilon^2 D_h e^{n+1}\right\|_\infty \ge (1-\tau)\left\|e^{n+1}\right\|_\infty. \tag{5.4}$$

Now, let us estimate the right-hand side terms of (5.3). Under the conditions in Theorem 1, we can estimate Q_1–Q_8 as

$$\|Q_1\|_\infty + \|Q_2\|_\infty \le \left(\frac{1}{2}+\tau\right)\left\|e^{n-1}\right\|_\infty + 2\tau\left(\beta+|\gamma|\right)\left\|e^{n-1}\right\|_\infty, \tag{5.5}$$

$$\|Q_3\|_\infty \le \left\|\frac{1}{2}I + \tau\epsilon^2 D_h\right\|_\infty \left\|e^{n-1}\right\|_\infty \le \frac{1}{2}\left\|e^{n-1}\right\|_\infty, \tag{5.6}$$

$$\|Q_4\|_\infty \le 4\tau\left(\beta+|\gamma|\right)\left\|e^n\right\|_\infty, \tag{5.7}$$

$$\|Q_5\|_\infty = 2\tau\left\|e^n\left(\left(\boldsymbol{u}^n\right)^{.2} + \boldsymbol{u}^n U^n + \left(U^n\right)^{.2}\right)\right\|_\infty \le 6\tau\left\|e^n\right\|_\infty, \tag{5.8}$$

$$\|Q_6\|_\infty + \|Q_7\|_\infty + \|Q_8\|_\infty \le C(\epsilon,\gamma,\beta)\tau\left(\tau^2+h^2\right). \tag{5.9}$$

It follows from (5.4)–(5.9) that

$$(1-\tau)\|e^{n+1}\|_\infty \le \big(1+\tau+2\tau\beta+2\tau|\gamma|\big)\|e^{n-1}\|_\infty + \big(6\tau+4\tau(\beta+|\gamma|)\big)\|e^n\|_\infty \\ + C(\epsilon,\gamma,\beta)\tau\big(\tau^2+h^2\big),$$

namely,

$$\|e^{n+1}\|_\infty \le \left(1+\frac{2\tau+2\tau\beta+2\tau|\gamma|}{1-\tau}\right)\|e^{n-1}\|_\infty + \frac{6\tau+4\tau(\beta+|\gamma|)}{1-\tau}\|e^n\|_\infty \\ + C(\epsilon,\gamma,\beta)\tau\big(\tau^2+h^2\big).$$

Then, we have

$$\big(\|e^{n+1}\|_\infty+\|e^n\|_\infty\big)-\big(\|e^n\|_\infty+\|e^{n-1}\|_\infty\big) \le \big(12\tau+8\tau(\beta+|\gamma|)\big)\big(\|e^n\|_\infty+\|e^{n-1}\|_\infty\big) \\ + C(\epsilon,\gamma,\beta)\tau\big(\tau^2+h^2\big). \tag{5.10}$$

Summing over n from 1 to $l-1$ ($2 \le l \le T/\tau$) at both sides of (5.10), we conclude that

$$\|e^l\|_\infty+\|e^{l-1}\|_\infty \le \big(12\tau+8\tau(\beta+|\gamma|)\big)\sum_{n=1}^{l-1}\big(\|e^n\|_\infty+\|e^{n-1}\|_\infty\big) \\ + \|e^1\|_\infty+\|e^0\|_\infty + C(\epsilon,\gamma,\beta,T)\big(\tau^2+h^2\big). \tag{5.11}$$

Notice that $\|e^0\|_\infty = 0$, and $\|e^1\|_\infty \le C(\epsilon,T)(\tau^2+h^2)$. Thus, apply the discrete Gronwall inequality to (5.11) to get estimate (5.1). □

6 Numerical results

In this section, we present two numerical examples to verify the theoretical results obtained in the previous sections. We consider 1D problem in the first example and 2D problem in the second example.

Example 1 We consider the 1D space-fractional Allen–Cahn equation with a force term. We add a force term on the equation to guarantee the exact solution have enough regularity. With the help of the following force term

$$\begin{aligned} f(x,t) &= e^{-3t}x^{12}(1-x)^{12} - 2e^{-t}x^4(1-x)^4 \\ &\quad + \frac{\epsilon^2}{2\cos(0.5\alpha\times\pi)}e^{-t}\bigg[\frac{\Gamma(5)}{\Gamma(5-\alpha)}\big(x^{4-\alpha}+(1-x)^{4-\alpha}\big) \\ &\quad - \frac{4\Gamma(6)}{\Gamma(6-\alpha)}\big(x^{5-\alpha}+(1-x)^{5-\alpha}\big) + \frac{6\Gamma(7)}{\Gamma(7-\alpha)}\big(x^{6-\alpha}+(1-x)^{6-\alpha}\big) \\ &\quad - \frac{4\Gamma(8)}{\Gamma(8-\alpha)}\big(x^{7-\alpha}+(1-x)^{7-\alpha}\big) + \frac{\Gamma(9)}{\Gamma(9-\alpha)}\big(x^{8-\alpha}+(1-x)^{8-\alpha}\big)\bigg], \end{aligned}$$

we set exact solution as

$$u(x,t) = e^{-t}x^4(1-x)^4, \quad x\in[0,1], t\in[0,T].$$

Table 1 The error of $\|\boldsymbol{u}^r - U^r\|$ and $\|\boldsymbol{u}^r - U^r\|_\infty$ with $r = 500$

h	$\|\boldsymbol{u}^r - U^r\|$	Rate	$\|\boldsymbol{u}^r - U^r\|_\infty$	Rate
1/30	3.3505e–06	–	6.7561e–06	–
1/60	8.3413e–07	2.0060	1.6918e–06	1.9976
1/120	1.8260e–07	2.1916	3.6293e–07	2.2208
1/240	4.7483e–08	1.9432	7.6760e–08	2.2413

Table 2 The error of $\|\boldsymbol{u}^r - U^r\|$ and $\|\boldsymbol{u}^r - U^r\|_\infty$ with $r = \frac{0.8}{\tau}$

τ	$\|\boldsymbol{u}^r - U^r\|$	Rate	$\|\boldsymbol{u}^r - U^r\|_\infty$	Rate
1/50	2.3890e–06	–	4.3654e–06	–
1/100	6.1717e–07	1.9527	1.1288e–06	1.9513
1/200	1.5818e–07	1.9641	2.9042e–07	1.9586
1/400	4.1494e–08	1.9306	7.7267e–08	1.9102

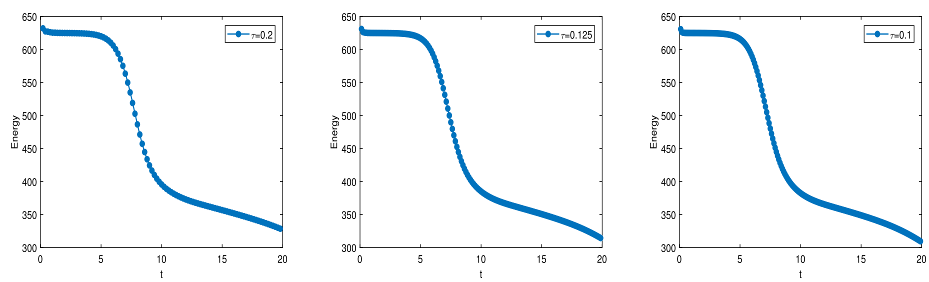

Figure 1 The discrete energy with $h_x = h_y = 0.02$ and different τ

Using scheme (2.7), we solve the problem with extra force term model. For the first step, we use the Newton method to solve the nonlinear Crank–Nicolson scheme (2.8). We set $\alpha = 1.4$, $\beta = 2$, $\gamma = 0.2$, $\epsilon = 0.5$, $T = 4$ and $\tau = 0.004$. Then, we test the convergence rate for spatial discretization in Table 1. Next, we set $\alpha = 1.6$, $\beta = 3$, $\gamma = 0.4$, $h = 0.002$, $\epsilon = 0.01$ and $T = 1$. We test the convergence rate for temporal discretization in Table 2.

Example 2 We consider the 2D space-fractional Allen–Cahn equation with initial value

$$u_0(x,y) = 0.1 \times \text{rand}(x,y) - 0.05, \quad (x,y) \in (0,1)^2.$$

We set the boundary value of $u_0(x,y)$ to zero.

Let $\alpha = 1.8$, $\beta = 4$, $T = 20$, $\gamma = 0.5$ and $\epsilon = 0.05$. We check the maximum values of the numerical solutions and the discrete energy with different τ in Fig. 1 and Fig. 2, respectively. These results validate Theorems 1 and 2.

Next, we observe the influence of fractional diffusion on the phase separation and coarsening process. We choose $\beta = 3.5$, $\gamma = 0.1$, $\epsilon = 0.02$, $h_x = h_y = 0.01$, $T = 50$ and $\tau = 0.125$. The snapshots of the numerical solutions at $t = 5, 20, 45$ with different α are shown in Fig. 3 that start from random initial values. We can see from the figure that reducing the fractional power results in a thinner interface, allowing for smaller bulk regions and a more heterogeneous phase structure. At the same time, a smaller fractional diffusion power has a slower phase coarsening process.

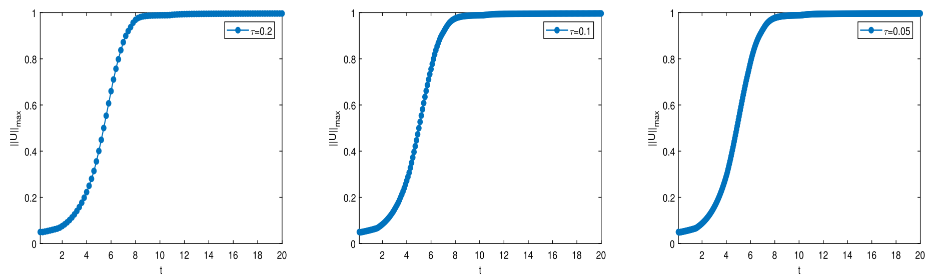

Figure 2 The maximum values of the numerical solution with $h_x = h_y = 0.05$ and different τ

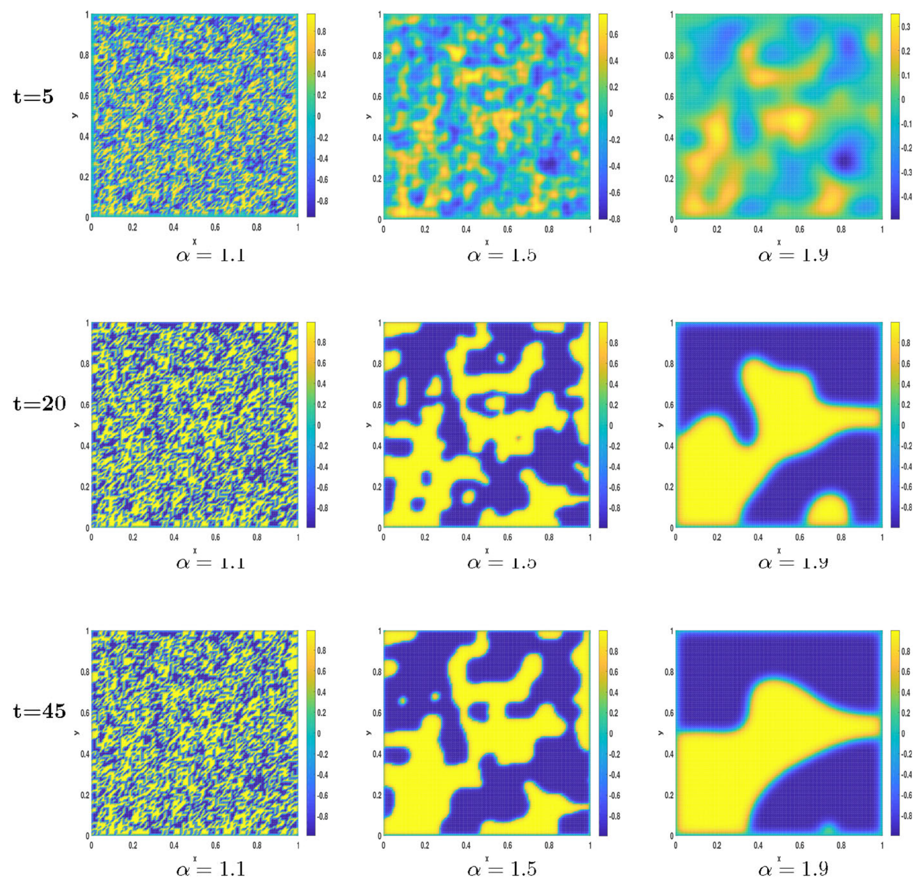

Figure 3 Numerical dynamics with different fractional derivatives $\alpha = 1.1, 1.5, 1.9$

7 Conclusions

In this paper, a new linear second-order finite difference scheme with two stabilized terms for space-fractional Allen–Cahn equations is presented. The discrete maximum principle, the maximum-norm error, and the discrete energy stability are discussed. A similar numerical design can be applied to the space-fractional Cahn–Hilliard equation; the energy stability analysis could be theoretically justified. However, the optimal maximum-norm error estimate cannot be established because the discrete maximal principle does not hold at this time. In the next work, we will discuss a higher-order finite difference scheme in space or time for space-fractional Allen–Cahn equations.

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Declarations

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The authors declare that they have no competing interests.

Author contribution
All authors contributed equally to the writing of this paper. Furthermore, all authors also read carefully and approved the final manuscript.

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